Aust. J. Phys., 1978, 31, 137–49

Component Minimization of the Bargmann–Wigner Wavefunction

E. A. Jeffery

Division of Applied Organic Chemistry, CSIRO, P.O. Box 4331, Melbourne, Vic. 3001.

Abstract

The Bargmann–Wigner equations are used to derive relativistic field equations with only $2(2j+1)$ components of the original wavefunction. The other components of the Bargmann–Wigner wavefunction are superfluous and can be defined in terms of the $2(2j+1)$ components. The results are compared with various $2(2j+1)$ theories in the literature. Sylvester's theorem and some properties of induced matrices give simple relationships between the operator matrices of the field equations and the arbitrary spin operator matrices.

1. Introduction

The relativistic equations of Bargmann and Wigner (1948) describe particles of arbitrary spin. They have their foundations in the earlier work of Dirac (1936), Fierz (1939), Fierz and Pauli (1939) and many others, all of whose contributions are well summarized and discussed by Corson (1955). They generalize the Dirac spin 1/2 equation, retaining the linear differential operator but replacing the 4-spinor wavefunction by a $2j$th-rank 4-spinor (j being the spin quantum number) as follows:

$$\left.\begin{aligned}(\gamma^\mu\partial_\mu+m)_{\alpha\alpha'}\psi_{\alpha'\beta\gamma\ldots\tau}(x) &= 0\,,\\ (\gamma^\mu\partial_\mu+m)_{\beta\beta'}\psi_{\alpha\beta'\gamma\ldots\tau}(x) &= 0\,,\\ \vdots\qquad\qquad &\quad\ \vdots\end{aligned}\right\}\tag{1}$$

The γ^μ, with $\mu = 1\ldots4$, are the Dirac matrices and ∂_μ is the 4-vector operator $(\nabla, -\mathrm{i}\,\partial/\partial t)$. Since the operator matrix is contracted with one index of the wavefunction at a time, some of the results for the spin 1/2 equation can be carried over directly into the Bargmann–Wigner equations; notably, manifest covariance and that the components of the wavefunctions obey the Klein–Gordon equation. The simplicity of these equations, however, breaks down as soon as attempts are made to subject them to second quantization. In particular, a Lagrangian is difficult to determine directly. Kamefuchi and Takahashi (1966) found a systematic approach to a Lagrangian formalism by initially expanding the wavefunction in terms of all 4×4 matrices with a desired symmetry. Substitution of the expanded wavefunction back into the Bargmann–Wigner equations yields dynamical field equations together with supplementary conditions. A suitable Lagrangian can then be found. The text book by Lurié (1968) describes the procedure well for second-rank symmetric spinor fields (yielding a vector boson field) and third-rank symmetric spinor fields (yielding the Rarita–Schwinger equations) and his is the nomenclature adopted here (Appendix 1). A similar but somewhat more general approach to a Lagrangian has been described recently by Doria (1977) who makes use of the properties of spinors embedded in a Clifford algebra (Sauter spinors).

By contrast, arbitrary spin fields can also be described by equations containing $2(2j+1)$ components in their wavefunctions (the minimum number of components is either $2j+1$ or $2(2j+1)$ depending on the form of the equation). Foldy (1956) described such a theory but he sacrificed manifest covariance. Now manifest covariance is not a necessity but it obviously simplifies the handling of transformations. Furthermore, how does one include minimal coupling into a noncovariant set of equations? Later work by Joos (1962), Weinberg (1964), Weaver *et al.* (1964), Mathews (1966), Williams *et al.* (1966), Hammer *et al.* (1968), Guertin (1974) and Weaver (1975) established $2(2j+1)$ component wave equations which were either manifestly covariant or had a well-defined covariance.

The equation of Joos (1962) and Weinberg (1964), namely

$$(\gamma^{\mu_1\mu_2\dots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{2j}}+m^{2j})\psi(x)=0\,, \tag{2}$$

where the $\gamma^{\mu_1\mu_2\dots\mu_{2j}}$ are $2(2j+1)\times 2(2j+1)$ matrices, is a generalization of the *covariant* form of the Dirac spin 1/2 equation. Like the spin 1/2 equation each component of $\psi(x)$ obeys the Klein–Gordon equation. Weaver *et al.* (1964), Mathews (1966), Williams *et al.* (1966) and Hammer *et al.* (1968) generalized the *Hamiltonian* form of the Dirac spin 1/2 equation by considering a generalized Foldy–Wouthuysen transformation. Relevant here is the more recent work of Krajcik and Nieto (1977) who have considered Foldy–Wouthuysen transformations, and their relationships to Lorentz transformations, for linear arbitrary spin wavefunctions in a space with an indefinite metric (as opposed to a positive-definite metric) with particular emphasis on Bhabba equations. Such systems, however, are not high spin $2(2j+1)$ component theories.

Second quantization is straightforward (Joos 1962; Weinberg 1964, 1969; Mathews and Ramakrishnan 1967; Nelson and Good 1968; Weaver 1968). However, the approach has been to develop propagators, S-matrices and Feynman rules without the intermediate machinery of a Lagrangian. Hurley (1971, 1972, 1974) has developed arbitrary spin equations which are amenable to a simple Lagrangian formalism and, indeed, are causal in the presence of minimal coupling and are readily second-quantized. He succeeds by introducing wavefunctions with $12j+2$ components, $4(2j+1)$ of which are independent. Ideally, one would like similar results for a $2(2j+1)$ component theory.

In this paper the Bargmann–Wigner equations, with arbitrary spin *symmetric* $2j$th-rank spinor wavefunctions, will be shown to yield a $2(2j+1)$ wave component equation (an antisymmetric second-rank spinor is used for spin 0). The extra components in the Bargmann–Wigner wavefunction are superfluous and can be defined (nondynamically) in terms of the $2(2j+1)$ components. Thus the Bargmann–Wigner and the $2(2j+1)$ theories are interrelated: a result which is not surprising since they purport to describe the same physics.

A relationship between rotation matrices and spin matrices is obtained using Sylvester's theorem, and is probably much simpler than relationships already in the literature.

There are high spin field theories other than those discussed above; for example, the 16-component theory for spin 1 developed largely by Durand (1975), and generalized to arbitrary spin. However, these are not of direct concern in this paper and will not be considered. Likewise, the many problems often found in high spin interacting fields, such as acausality, imaginary eigenenergies and loss of constraints, will not be discussed; for recent summaries the reader is referred to the papers by Babu Joseph and Sabir (1977) and Cox (1977).

The general spin case will be discussed first and then the results for spin 1 will be examined in detail.

2. Arbitrary Spin Particles

Consider the Dirac spin 1/2 equations

$$(-E+\mu)\psi_{1,2} = (-\boldsymbol{P}.\boldsymbol{\sigma})\psi_{3,4}, \tag{3a}$$

$$(E+\mu)\psi_{3,4} = (\boldsymbol{P}.\boldsymbol{\sigma})\psi_{1,2}, \tag{3b}$$

where $\psi_{1,2}$ and $\psi_{3,4}$ are two-component spinors (ψ_1, ψ_2) and (ψ_3, ψ_4), $\boldsymbol{\sigma}$ is the Pauli spin matrix vector $(\sigma_1, \sigma_2, \sigma_3)$ and E and $\boldsymbol{P}$ are the differential operators $\mathrm{i}\,\partial/\partial t$ and $-\mathrm{i}\nabla$ respectively.

In this section, the Bargmann–Wigner equations will be used to generalize equations (3) to

$$(-E+\mu)^{2j}\psi_{1,2}^{[2j]} = (-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}\psi_{3,4}^{[2j]}, \tag{4a}$$

$$(E+\mu)^{2j}\psi_{3,4}^{[2j]} = (\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}\psi_{1,2}^{[2j]} \tag{4b}$$

for spin j, where the symbol $[2j]$ denotes the $2j$th induced matrix or spinor, the definitions and derivations of which are discussed in Appendix 2. For example, the second and third induced matrices and corresponding spinors of $\boldsymbol{P}.\boldsymbol{\sigma}$ and $\psi_{1,2}$ are

$$\begin{bmatrix} P_z^2 & \sqrt{2}\,P_z P_- & P_-^2 \\ \sqrt{2}\,P_z P_+ & (-P_z^2+P_+P_-) & -\sqrt{2}\,P_z P_- \\ P_+^2 & -\sqrt{2}\,P_z P_+ & P_z^2 \end{bmatrix}, \qquad \begin{bmatrix} \psi_{11} \\ \sqrt{2}\,\psi_{12} \\ \psi_{22} \end{bmatrix} \tag{5}$$

and

$$\begin{bmatrix} P_z^3 & \sqrt{3}\,P_z^2 P_- & \sqrt{3}\,P_z P_-^2 & P_-^3 \\ \sqrt{3}\,P_z^2 P_+ & (-P_z^3+2P_zP_+P_-) & (P_-^2P_+-2P_z^2P_-) & -\sqrt{3}\,P_zP_-^2 \\ \sqrt{3}\,P_zP_+^2 & (-2P_z^2P_++P_+^2P_-) & (-2P_zP_+P_-+P_z^3) & \sqrt{3}\,P_z^2P_- \\ P_+^3 & -\sqrt{3}\,P_+^2P_z & \sqrt{3}\,P_z^2P_+ & -P_z^3 \end{bmatrix}, \qquad \begin{bmatrix} \psi_{111} \\ \sqrt{3}\,\psi_{112} \\ \sqrt{3}\,\psi_{122} \\ \psi_{222} \end{bmatrix}. \tag{6}$$

Here $P_\pm = P_x \pm \mathrm{i}P_y$, and the spinor components are completely symetric in their indices and are in general multiplied by the square root of $(2j)!/(j+m)!(j-m)!$, where m ranges from j to $-j$ down the column.

The induced matrices of equations (4) are related to the more familiar rotation matrices. The matrices $-\mathrm{i}\boldsymbol{n}.\boldsymbol{\sigma}$, $(-\mathrm{i}\boldsymbol{n}.\boldsymbol{\sigma})^{[2]}$ and $(-\mathrm{i}\boldsymbol{n}.\boldsymbol{\sigma})^{[3]}$ are those for rotation of spin 1/2, 1 and 3/2 particles by an angle π about the unit vector $\boldsymbol{n}$, a point discussed further in Section 4.

Equations (4) can be derived from the Bargmann–Wigner equations by eliminating all spinor components except those of $\psi_{1,2}^{[2j]}$ and $\psi_{3,4}^{[2j]}$. The process will be carried out explicitly in Section 3 for spin 1. However, for arbitrary spin the proof is by the method of induction. Thus equations (4) are assumed to be derivable from the Bargmann–Wigner equations for spin j and with this assumption are proved, via the Bargmann–Wigner equations, to hold for spin $j+1/2$. Obviously the results (4) are derivable for $j = 1/2$, so by induction must be correct for all spins.

For spin j the Bargmann–Wigner equations with a totally symmetric wavefunction are

$$\begin{bmatrix} -E+\mu & 0 & P_z & P_- \\ 0 & -E+\mu & P_+ & -P_z \\ -P_z & -P_- & E+\mu & 0 \\ -P_+ & P_z & 0 & E+\mu \end{bmatrix} \begin{bmatrix} \psi_{1\alpha\beta\ldots\tau(v)} \\ \psi_{2\alpha\beta\ldots\tau(v)} \\ \psi_{3\alpha\beta\ldots\tau(v)} \\ \psi_{4\alpha\beta\ldots\tau(v)} \end{bmatrix} = 0, \qquad (7)$$

where there are a total of $2j$ suffixes on each component such that $\alpha < \beta < \ldots < \tau$ and the suffix v is excluded. For spin $j+1/2$ the suffix v is included and is allowed to range from 1 to 4. Now, the Bargmann–Wigner equations for spin $j+1/2$ can be manipulated in exactly the same way as the equations for spin j provided the floating index v is held constant. As a result, if equations (4) hold for spin j then they will also hold for spin $j+1/2$, provided the extra constant suffix v is added to the wavefunction components.

The expanded top row of equations (4) with v included is

$$(-E+\mu)^{2j}\psi_{11\ldots1(v)} + P_z^{2j}(-)^{2j+1}\psi_{33\ldots3(v)} + 2jP_z^{2j-1}P_-(-)^{2j+1}\psi_{33\ldots4(v)} + \ldots = 0. \qquad (8)$$

When $v = 3$ or 4, equation (8) contains the non-essential terms $\psi_{11\ldots13}$ or $\psi_{11\ldots14}$. However, these terms can be eliminated by using the relationships

$$-P_z\psi_{11\ldots11} - P_-\psi_{21\ldots11} + (E+\mu)\psi_{31\ldots11} = 0, \qquad (9a)$$

$$-P_+\psi_{11\ldots11} + P_z\psi_{21\ldots11} + (E+\mu)\psi_{41\ldots11} = 0 \qquad (9b)$$

derived from equation (7) with $\alpha, \beta, \ldots, \tau, v = 1, 1, \ldots, 1, 1$. The results are

$$(-E+\mu)^{2j}(P_z\psi_{11\ldots11} + P_-\psi_{21\ldots11}) + (E+\mu)P_z^{2j}(-)^{2j+1}\psi_{33\ldots33}$$
$$+ (E+\mu)2jP_z^{2j-1}P_-(-)^{2j+1}\psi_{33\ldots34} + \ldots = 0 \qquad (10a)$$

and

$$(-E+\mu)^{2j}(P_+\psi_{11\ldots11} - P_z\psi_{21\ldots11}) + (E+\mu)P_z^{2j}(-)^{2j+1}\psi_{33\ldots34}$$
$$+ (E+\mu)2jP_z^{2j-1}P_-(-)^{2j+1}\psi_{33\ldots44} + \ldots = 0. \qquad (10b)$$

Eliminating first $\psi_{21\ldots11}$ and then $\psi_{11\ldots11}$ from equations (10a) and (10b), multiplying the results throughout by $(-E+\mu)$ and dividing by $\boldsymbol{P}^2$, we obtain

$$(-E+\mu)^{2j+1}\psi_{11\ldots11} + (-)^{2j+2}P_z^{2j+1}\psi_{33\ldots33}$$
$$+ (-)^{2j+2}(2j+1)P_z^{2j}P_-\psi_{33\ldots34} + \ldots = 0, \qquad (11a)$$

$$(-E+\mu)^{2j+1}\psi_{21\ldots11}+(-)^{2j+2}P_z^{2j}P_+\psi_{33\ldots34}$$
$$+(-)^{2j+2}(2jP_z^{2j-1}P_+P_- -P_z^{2j+1})\psi_{33\ldots44}+\ldots=0. \tag{11b}$$

These correspond to the expanded first two rows of the $j+1/2$ equations analogous to (4) (i.e. replace the superscripts $2j$ by $2j+1$). The other rows of the $j+1/2$ equations may be derived in a similar way. The set of equations (4) is known to be correct for $j=1/2$, so that by induction it is correct for all spins.

Since $(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}=\{(\boldsymbol{P}.\boldsymbol{\sigma})(\boldsymbol{P}.\boldsymbol{\sigma})\}^{[2j]}=(\boldsymbol{P}^2)^{2j}$, equations (4) result in

$$[(E^2-\mu^2)^{2j}-(\boldsymbol{P}^2)^{2j}]\begin{Bmatrix}\psi_{1,2}^{[2j]}\\ \psi_{3,4}^{[2j]}\end{Bmatrix}=0, \tag{12}$$

which factorizes to

$$(E^2-\mu^2-\boldsymbol{P}^2)[(E^2-\mu^2)^{2j-1}+(E^2-\mu^2)^{2j-2}\boldsymbol{P}^2+\ldots+(\boldsymbol{P}^2)^{2j-1}]\begin{Bmatrix}\psi_{1,2}^{[2j]}\\ \psi_{3,4}^{[2j]}\end{Bmatrix}=0. \tag{13}$$

The Klein–Gordon equation is therefore always a solution but is not unique. In order to obtain a unique solution, E^2 is replaced by $\boldsymbol{P}^2+\mu^2$ in the expansions of $(-E+\mu)^{2j}$ and $(E+\mu)^{2j}$; a process that is valid because the Bargmann–Wigner equations obey the Klein–Gordon condition uniquely (see Lurié 1968, p. 27). Thus we have

$$(-E+\mu)^{2j}=-ES+R,\qquad (E+\mu)^{2j}=ES+R, \tag{14}$$

where

$$R=\sum_{l=0}^{\infty}\frac{(2j)!\,\mu^{2j}}{(2j-2l)!(2l)!}\left(1+\frac{\boldsymbol{P}^2}{\mu^2}\right)^l,\qquad S=\sum_{l=0}^{\infty}\frac{(2j)!\,\mu^{2j-1}}{(2j-2l-1)!(2l+1)!}\left(1+\frac{\boldsymbol{P}^2}{\mu^2}\right)^l.$$

Substitution from equations (14) into (4) then gives

$$(-ES+R)\psi_{1,2}^{[2j]}=(-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}\psi_{3,4}^{[2j]}, \tag{15a}$$

$$(ES+R)\psi_{3,4}^{[2j]}=(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}\psi_{1,2}^{[2j]}. \tag{15b}$$

Equations (15) obey the Klein–Gordon condition uniquely, as can be shown by using the easily derivable relationship

$$R^2-(\boldsymbol{P}^2+\mu^2)S^2=(-\boldsymbol{P}^2)^{2j}.$$

Equations (4) and (15) should be compared with equations 7(17) and 7(18) given in the paper by Weinberg (1964). His operators $\pi(-i\partial)$ and $\bar{\pi}(-i\partial)$ are equivalent to $[E-\boldsymbol{P}.\boldsymbol{\sigma}]^{[2j]}$ and $[E+\boldsymbol{P}.\boldsymbol{\sigma}]^{[2j]}$ in our notation. Thus equations (4) and (15) and the Joos–Weinberg equations (2) are simply *alternative ways of writing the Bargmann–Wigner equations.*

In the next section the particular case for spin 1 will be derived in detail rather than using equations (4) directly. This allows a comparison of our results and derivations with those commonly found in text-book decriptions of vector boson field equations. Spin 0 is also considered.

3. Spin 1 and Spin 0 Particles

The Bargmann–Wigner equations for spin 1 are,

$$\begin{bmatrix} -E+\mu & \boldsymbol{P}.\boldsymbol{\sigma} \\ -\boldsymbol{P}.\boldsymbol{\sigma} & E+\mu \end{bmatrix} \begin{bmatrix} \psi_{11} & \cdots & \psi_{14} \\ \vdots & & \vdots \\ \psi_{41} & \cdots & \psi_{44} \end{bmatrix} = 0, \tag{16}$$

where $\psi_{\mu\nu} = \psi_{\nu\mu}$. They can be expanded to give

$$\begin{bmatrix} -E+\mu & 0 \\ 0 & -E+\mu \end{bmatrix}\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} + \begin{bmatrix} P_z & P_- \\ P_+ & -P_z \end{bmatrix}\begin{bmatrix} \psi_{13} & \psi_{23} \\ \psi_{14} & \psi_{24} \end{bmatrix} = 0, \tag{17a}$$

$$-\begin{bmatrix} P_z & P_- \\ P_+ & -P_z \end{bmatrix}\begin{bmatrix} \psi_{13} & \psi_{14} \\ \psi_{23} & \psi_{24} \end{bmatrix} + \begin{bmatrix} E+\mu & 0 \\ 0 & E+\mu \end{bmatrix}\begin{bmatrix} \psi_{33} & \psi_{34} \\ \psi_{43} & \psi_{44} \end{bmatrix} = 0, \tag{17b}$$

$$\begin{bmatrix} -E+\mu & 0 \\ 0 & -E+\mu \end{bmatrix}\begin{bmatrix} \psi_{13} & \psi_{14} \\ \psi_{23} & \psi_{24} \end{bmatrix} + \begin{bmatrix} P_z & P_- \\ P_+ & -P_z \end{bmatrix}\begin{bmatrix} \psi_{33} & \psi_{34} \\ \psi_{43} & \psi_{44} \end{bmatrix} = 0, \tag{17c}$$

$$-\begin{bmatrix} P_z & P_- \\ P_+ & -P_z \end{bmatrix}\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} + \begin{bmatrix} E+\mu & 0 \\ 0 & E+\mu \end{bmatrix}\begin{bmatrix} \psi_{13} & \psi_{23} \\ \psi_{14} & \psi_{24} \end{bmatrix} = 0. \tag{17d}$$

Equation (17c) transposes to

$$\begin{bmatrix} -E+\mu & 0 \\ 0 & -E+\mu \end{bmatrix}\begin{bmatrix} \psi_{13} & \psi_{23} \\ \psi_{14} & \psi_{24} \end{bmatrix} + \begin{bmatrix} \psi_{33} & \psi_{34} \\ \psi_{43} & \psi_{44} \end{bmatrix}\overleftarrow{\begin{bmatrix} P_z & P_+ \\ P_- & -P_z \end{bmatrix}} = 0, \tag{18}$$

where the arrow indicates that the operator matrix acts to the left on the wavefunction matrix.

Multiplying equation (17d) by $-E+\mu$ and (18) by $E+\mu$ and then subtracting both sides of the resulting equations, we obtain

$$(-E+\mu)\begin{bmatrix} P_z & P_- \\ P_+ & -P_z \end{bmatrix}\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} + (E+\mu)\begin{bmatrix} \psi_{33} & \psi_{34} \\ \psi_{43} & \psi_{44} \end{bmatrix}\overleftarrow{\begin{bmatrix} P_z & P_+ \\ P_- & -P_z \end{bmatrix}} = 0 \tag{19}$$

and hence

$$(-E+\mu)^2\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} - \begin{bmatrix} P_z & P_- \\ P_+ & -P_z \end{bmatrix}\begin{bmatrix} \psi_{33} & \psi_{34} \\ \psi_{43} & \psi_{44} \end{bmatrix}\overleftarrow{\begin{bmatrix} P_z & P_+ \\ P_- & -P_z \end{bmatrix}} = 0, \tag{20a}$$

$$-\begin{bmatrix} P_z & P_- \\ P_+ & -P_z \end{bmatrix}\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}\overleftarrow{\begin{bmatrix} P_z & P_+ \\ P_- & -P_z \end{bmatrix}} + (E+\mu)^2\begin{bmatrix} \psi_{33} & \psi_{34} \\ \psi_{43} & \psi_{44} \end{bmatrix} = 0, \tag{20b}$$

where the fact that the components of the Bargmann–Wigner wavefunction obey the Klein–Gordon equation (Lurié 1968, p. 27) has been used.

Expansion of equations (20a) and (20b) shows that they can be put in the alternative forms

$$(-E+\mu)^2\begin{bmatrix}\psi_{11}\\ \sqrt{2}\psi_{12}\\ \psi_{22}\end{bmatrix}-\begin{bmatrix}P_z^2 & \sqrt{2}P_zP_- & P_-^2\\ \sqrt{2}P_zP_+ & (-P_z^2+P_+P_-) & -\sqrt{2}P_zP_-\\ P_+^2 & -\sqrt{2}P_zP_+ & P_z^2\end{bmatrix}\begin{bmatrix}\psi_{33}\\ \sqrt{2}\psi_{34}\\ \psi_{44}\end{bmatrix}=0, \quad (21a)$$

$$-\begin{bmatrix}P_z^2 & \sqrt{2}P_zP_- & P_-^2\\ \sqrt{2}P_zP_+ & (-P_z^2+P_+P_-) & -\sqrt{2}P_zP_-\\ P_+^2 & -\sqrt{2}P_zP_+ & P_z^2\end{bmatrix}\begin{bmatrix}\psi_{11}\\ \sqrt{2}\psi_{12}\\ \psi_{22}\end{bmatrix}+(E+\mu)^2\begin{bmatrix}\psi_{33}\\ \sqrt{2}\psi_{34}\\ \psi_{44}\end{bmatrix}=0. \quad (21b)$$

The 3×3 matrix, containing components quadratic in P_z, P_+ and P_-, is the second induced matrix of $\boldsymbol{P}.\boldsymbol{\sigma}$ (Appendix 2). The factor of $\sqrt{2}$ in the wavefunction conforms with the usual convention (Landau and Lifshitz 1959). Obviously equations (21a) and (21b) can be combined and result in a $2(2j+1)$ component wave equation

$$\begin{bmatrix}(-E+\mu)^2 & \pi\\ \pi & (E+\mu)^2\end{bmatrix}\begin{bmatrix}\psi_A\\ \psi_B\end{bmatrix}=0, \quad (22)$$

with

$$\pi=\begin{bmatrix}P_z^2 & \sqrt{2}P_zP_- & P_-^2\\ \sqrt{2}P_zP_+ & (-P_z^2+P_+P_-) & -\sqrt{2}P_zP_-\\ P_+^2 & -\sqrt{2}P_zP_+ & P_z^2\end{bmatrix},$$

$$\psi_A=\begin{bmatrix}\psi_{11}\\ \sqrt{2}\psi_{12}\\ \psi_{22}\end{bmatrix}, \qquad \psi_3=-\begin{bmatrix}\psi_{33}\\ \sqrt{2}\psi_{34}\\ \psi_{44}\end{bmatrix}.$$

Since $\pi^2=\boldsymbol{P}^4$, equations (21a) and (21b) can also be reduced to

$$[(E^2-\mu^2)^2-\boldsymbol{P}^4]\psi_A=0, \quad (23)$$

which contains the Klein–Gordon equation as a solution together with the 'unphysical' equation $(E^2-\mu^2+\boldsymbol{P}^2)\psi_A=0$. However, if ψ_A can be expanded in terms of plane waves with real energies and momenta and travelling at less than the speed of light, then the algebraic relationship resulting after operating with $E^2-\mu^2+\boldsymbol{P}^2$ is inadmissible if relativity holds, so that only the Klein–Gordon equation can be obeyed. Similar arguments apply equally well to the arbitrary spin equations derived in Section 2 and to the arbitrary spin equation (2) of Joos and Weinberg considered in the Introduction. Hammer *et al.* (1968) have shown that the hyperplane formalism of Fleming (1966) can be used to ensure that the wavefunction components of equation (2) always obey the Klein–Gordon equation. We considered an alternative procedure in Section 2.

With the unitary matrix

$$T=\begin{bmatrix}1/\sqrt{2} & -i/\sqrt{2} & 0\\ 0 & 0 & -1\\ -1/\sqrt{2} & -i/\sqrt{2} & 0\end{bmatrix}, \quad (24)$$

equation (21a) can be transformed to

$$(-E+\mu)^2 T^{-1}\begin{bmatrix}\psi_{11}\\ \sqrt{2}\,\psi_{12}\\ \psi_{22}\end{bmatrix}$$

$$-T^{-1}\begin{bmatrix}P_z^2 & \sqrt{2}\,P_z P_- & P_-^2\\ \sqrt{2}\,P_z P_+ & (-P_z^2+P_+P_-) & -\sqrt{2}\,P_z P_-\\ P_+^2 & -\sqrt{2}\,P_z P_+ & P_z^2\end{bmatrix} TT^{-1}\begin{bmatrix}\psi_{33}\\ \sqrt{2}\,\psi_{34}\\ \psi_{44}\end{bmatrix} = 0 \tag{25a}$$

or

$$(-E+\mu)^2\begin{bmatrix}(\psi_{11}-\psi_{22})/\sqrt{2}\\ \mathrm{i}(\psi_{11}+\psi_{22})/\sqrt{2}\\ -\sqrt{2}\,\psi_{12}\end{bmatrix}$$

$$-\begin{bmatrix}(P_z^2-P_x^2+P_y^2) & -2P_xP_y & -2P_xP_z\\ -2P_yP_x & (P_z^2+P_x^2-P_y^2) & -2P_yP_z\\ -2P_zP_x & -2P_zP_y & (-P_z^2+P_x^2+P_y^2)\end{bmatrix}\begin{bmatrix}(\psi_{33}-\psi_{44})/\sqrt{2}\\ \mathrm{i}(\psi_{33}+\psi_{44})/\sqrt{2}\\ -\sqrt{2}\,\psi_{34}\end{bmatrix} = 0. \tag{25b}$$

The 3×3 matrix in equation (25b) is equivalent to $2\,\mathrm{graddiv}-\nabla^2$, when it operates on a vector. Since

$$[(\psi_{11}-\psi_{22})/\sqrt{2},\ \mathrm{i}(\psi_{11}+\psi_{22})/\sqrt{2},\ -\sqrt{2}\,\psi_{12}]$$

and

$$[(\psi_{33}-\psi_{44})/\sqrt{2},\ \mathrm{i}(\psi_{33}+\psi_{44})/\sqrt{2},\ -\sqrt{2}\,\psi_{34}]$$

transform like two vectors (Cartan 1966), equation (25b) becomes

$$\mathrm{i}\,\partial_t\psi_1 = -\nabla^2(\psi_1+\psi_2)/2\mu + \mu\psi_1 + \mathrm{graddiv}\,\psi_2/\mu, \tag{26}$$

where

$$\psi_1 = \begin{bmatrix}(\psi_{11}-\psi_{22})/\sqrt{2}\\ \mathrm{i}(\psi_{11}+\psi_{22})/\sqrt{2}\\ -\sqrt{2}\,\psi_{12}\end{bmatrix}, \qquad \psi_2 = -\begin{bmatrix}(\psi_{33}-\psi_{44})/\sqrt{2}\\ \mathrm{i}(\psi_{33}+\psi_{44})/\sqrt{2}\\ -\sqrt{2}\,\psi_{34}\end{bmatrix}$$

and $(-E+\mu)^2$ has been replaced by $-\nabla^2+2\mu^2-2\mathrm{i}\mu\partial_t$. Similarly, equation (21b) reduces to

$$\mathrm{i}\,\partial_t\psi_2 = \nabla^2(\psi_1+\psi_2)/2\mu - \mu\psi_2 - \mathrm{graddiv}\,\psi_1/\mu. \tag{27}$$

Equations (26) and (27) are identical with equations 1(124a) and 1(124b) given by Lurié (1968), and describe a massive vector field. The notation used by Lurié is $\psi_1 = (A-\mathrm{i}E/\mu)/\sqrt{2}$ and $\psi_2 = -(A+\mathrm{i}E/\mu)/\sqrt{2}$, where A and E reduce to the vector potential and electric field when the mass becomes zero.

The nondynamical relationship

$$2\mu\begin{bmatrix}\psi_{13} & \psi_{23}\\ \psi_{14} & \psi_{24}\end{bmatrix} - \begin{bmatrix}P_z & P_-\\ P_+ & -P_z\end{bmatrix}\begin{bmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{bmatrix} + \begin{bmatrix}\psi_{33} & \psi_{34}\\ \psi_{43} & \psi_{44}\end{bmatrix}\overleftarrow{\begin{bmatrix}P_z & P_+\\ P_- & -P_z\end{bmatrix}} = 0, \tag{28}$$

formed by adding together both sides of equations (17d) and (18), serves to define ψ_{13}, ψ_{23}, ψ_{14} and ψ_{24} in terms of the other $\psi_{\mu\nu}$; hence making these four bispinor components non-essential. By substituting for the $\psi_{\mu\nu}$ from the definitions

$$A_\mu = \mathrm{i}(C\gamma_\mu)^{\alpha\beta}\psi_{\beta\alpha}/4\mu, \qquad F_{\mu\nu} = -(C\Sigma_{\mu\nu})^{\alpha\beta}\psi_{\beta\alpha}/4, \qquad \Sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2\mathrm{i},$$

where $F_{ij} = \varepsilon_{ijk}B_k$ and $F_{k4} = -\mathrm{i}E_k$ (see Lurié 1968), equation (28) is found to be equivalent to $\boldsymbol{B} = \mathrm{curl}\boldsymbol{A}$ and $\nabla.\boldsymbol{E} = -\mu^2 A_0$ (i.e. equations 1(122a) and 1(122d) in the book by Lurié 1968).

When $\psi_{\alpha\beta}$ is antisymmetric the Bargmann–Wigner equations describe spin 0 particles because all ψ_{ii} are zero and ψ_{12} and ψ_{34} each have zero spin components along the z axis. Thus consider the 16 equations contained in (16) with $\psi_{\alpha\beta} = -\psi_{\beta\alpha}$. Relationships between the various components are easily derived; for example,

$$(-E+\mu)\psi_{12} = (E+\mu)\psi_{34} \qquad (\psi \text{ antisymmetric}). \tag{29}$$

Since ψ_{12} and ψ_{34} both obey the Klein–Gordon equation, the relationship (29) can be recaste into

$$E\psi_{12} = \tfrac{1}{2}\mu^{-1}\boldsymbol{P}^2(\psi_{12}+\psi_{34})+\mu\psi_{12}, \tag{30a}$$

$$E\psi_{34} = -\tfrac{1}{2}\mu^{-1}\boldsymbol{P}^2(\psi_{12}+\psi_{34})-\mu\psi_{34} \tag{30b}$$

by multiplying it throughout by $(-E+\mu)$ or $(E+\mu)$ and replacing E^2 by $\boldsymbol{P}^2+\mu^2$. Then ψ_{12} and ψ_{34} are equivalent to ψ_1 and ψ_2 in the spin 0 field equations 1(19a) and 1(19b) of the text by Lurié (1968).

If no conditions are imposed on the symmetry of $\psi_{\alpha\beta}$ then the Bargmann–Wigner equations give relationships between $\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}$ and $\psi_{33}, \psi_{34}, \psi_{43}, \psi_{44}$ which are identical with equations (21a) and (21b) with $\sqrt{2}\psi_{12}$ and $\sqrt{2}\psi_{34}$ replaced by $(\psi_{12}+\psi_{21})/\sqrt{2}$ and $(\psi_{34}+\psi_{43})/\sqrt{2}$ respectively, together with equation (29) with ψ_{12} and ψ_{34} replaced by $\frac{1}{2}(\psi_{12}-\psi_{21})$ and $\frac{1}{2}(\psi_{34}-\psi_{43})$ respectively.

Just as the spin 1 equations (21) can be derived by taking second *induced* matrices of the 2×2 matrices in the Dirac spin 1/2 equation, so the spin 0 equations (30) can be derived by taking the second *compound* matrices. In fact, the decomposition of the direct product of a 2×2 matrix with itself into its second induced matrix and second compound matrix (Littlewood 1958) is directly related to the representation decomposition $D(\frac{1}{2})\times D(\frac{1}{2}) = D(1)\oplus D(0)$ (see Appendix 2).

4. Sylvester's Theorem and Induced Matrices

The $2j$th induced matrices that occur in equations (4) can be expanded in terms of spin matrices. To any rotation by an angle θ about a unit vector $\boldsymbol{n}$ in three-space there is associated a two-component spinor transformation given by the matrix $\cos\frac{1}{2}\theta - \mathrm{i}\boldsymbol{n}.\boldsymbol{\sigma}\sin\frac{1}{2}\theta$. The associated $(2j+1)$ component spinor, having the form described in Section 2 above, undergoes a transformation given by the matrix $[\cos\frac{1}{2}\theta - \mathrm{i}\boldsymbol{n}.\boldsymbol{\sigma}\sin\frac{1}{2}\theta]^{[2j]}$ (see e.g. Appendix II of Rose 1957). Also, the $(2j+1)$th dimensional rotation matrix is well known to be equivalent to $\exp(-\mathrm{i}\theta\boldsymbol{J}.\boldsymbol{n})$, where $\boldsymbol{J}$ denotes the three spin-j matrices that have elements given by

$$\langle jm\pm 1|J_\pm|jm\rangle = \{(j\mp m)(j\pm m+1)\}^{\frac{1}{2}} \qquad \text{and} \qquad \langle jm|J_z|jm\rangle = m,$$

where $J_{\pm} = J_x \pm iJ_y$ and all other elements are zero (cf. Ch. IV of Rose 1957). Thus the identity

$$[\cos\tfrac{1}{2}\theta - i\boldsymbol{n}.\boldsymbol{\sigma}\sin\tfrac{1}{2}\theta]^{[2j]} = \exp(-i\theta\boldsymbol{J}.\boldsymbol{n}) \tag{31}$$

follows and has a form somewhat akin to De Moivre's theorem. The matrix function $\exp(-i\theta\boldsymbol{J}.\boldsymbol{n})$ can be written

$$\sum_r \exp(-i\theta m_r)P_r, \qquad \text{where} \qquad P_r = \prod_{m_s=-j}^{+j}(m_s - \boldsymbol{J}.\boldsymbol{n}) \Big/ \prod_{m_s}(m_s - m_r), \tag{32}$$

as a direct consequence of Sylvester's theorem (Gantmacher 1960). Here m_r and m_s are eigenvalues of $\boldsymbol{J}.\boldsymbol{n}$ (i.e. spin components along the Z axis) such that $m_r \neq m_s$. The factor P_r is the projection operator for the spin component m_r, so that the expression (32) amounts to a suitably weighted sum of projection operators, i.e. (32) is the spectral decomposition of the rotation operator. Similarly, the formula

$$[\cosh\tfrac{1}{2}\phi + \boldsymbol{n}.\boldsymbol{\sigma}\sinh\tfrac{1}{2}\phi]^{[2j]} = \exp(\phi\boldsymbol{J}.\boldsymbol{n}) \tag{33}$$

holds, as can be seen by substituting $i\phi$ for θ in equation (31) and noting that $\cos i\phi = \cosh\phi$ and $\sin i\phi = i\sinh\phi$.

All the formulae that relate rotation matrices to spin matrices cannot be fully reviewed here; each author seems to have his own preference. For example, Torruella (1975) obtains separate formulae for integer and half-integer spin by inverting the power series for $\exp(i\theta\boldsymbol{J}.\boldsymbol{n})$. Weber and Williams (1965) express $\exp(iZ\theta)$ in terms of complex polynomials in Z and then place $Z = \boldsymbol{J}.\boldsymbol{n}$. Weinberg (1964) proceeds by expressing the hyperbolic functions $\cosh Z\theta$ and $\sinh Z\theta$ as power series in $\sinh\theta$ to obtain a relationship between his $\pi^{(j)}(q)$ matrix, for a Lorentz 'boost', and $\boldsymbol{J}.\boldsymbol{q}$. The set of values for $\pi^{(j)}(q)$ given in his Table I are equal to $[q_0 - \boldsymbol{q}.\boldsymbol{\sigma}]^{[2j]}$, that is, $[\cosh\theta - \boldsymbol{q}.\boldsymbol{\sigma}\sinh\theta]^{[2j]}$ where $\cosh\theta = q_0/m$. In a similar manner, Weaver *et al.* (1964) also consider Lorentz 'boosts' and their quantity $S_t = \exp[(s\varepsilon\alpha.\boldsymbol{P}/P)\text{arctanh}(P/E)]$, where ε is the operator $i\partial_t/|i\partial_t|$, is given by

$$S_t = [\cosh\tfrac{1}{2}\theta + (\varepsilon\alpha.\boldsymbol{P}/P)\sinh\tfrac{1}{2}\theta]^{[2j]},$$

where $\cosh\theta = E/m$ and the superfix $[2j]$ now means taking the $2j$th induced matrices of $\boldsymbol{\sigma}.\boldsymbol{P}/P$ into which $\alpha.\boldsymbol{P}/P$ is decomposable. The values of $S(\boldsymbol{P})$ given in their Table I can then be obtained by replacing ε by their β.

The $2j$th induced matrix of $\boldsymbol{P}.\boldsymbol{\sigma}$ that occurs in the general equations (4) can be calculated in terms of $\boldsymbol{J}.\boldsymbol{P}$ by using (31) and (32). Thus, putting $\theta = \pi$ and $\boldsymbol{n} = \boldsymbol{P}/|\boldsymbol{P}|$, we have

$$[\boldsymbol{P}.\boldsymbol{\sigma}]^{[2j]} = (i|\boldsymbol{P}|)^{2j}\exp(-i\pi\boldsymbol{J}.\boldsymbol{P}/|\boldsymbol{P}|).$$

Hence

$$[\boldsymbol{P}.\boldsymbol{\sigma}]^{[2j]} = (i|\boldsymbol{P}|)^{2j}\sum_r \exp(-i\pi m_r)\prod_{m_s=-j}^{+j}(m_s - \boldsymbol{J}.\boldsymbol{P}/|\boldsymbol{P}|) \Big/ \prod_{m_s}(m_s - m_r), \tag{34}$$

and $\exp(-i\pi m_r)$ can be simplified to $(-1)^{m_r}$ and some $(|\boldsymbol{P}|)^{2j}$ terms cancelled. However, the left-hand side of equation (34) is much easier to calculate than the right-hand side (see Appendix 2), so there is no cause to make explicit use of this equation.

Acknowledgment

Dr R. A. Lee of this Division is thanked for his interest and enlightening discussions.

References

Babu Joseph, K., and Sabir, M. (1977). *J. Phys.* A **10**, 1225.
Bargmann, V., and Wigner, E. P. (1948). *Proc. Nat. Acad. Sci. U.S.A.* **34**, 211.
Cartan, E. (1966). 'The Theory of Spinors' (MIT: Cambridge, Mass.).
Corson, E. M. (1955). 'Introduction to Tensors, Spinors, and Relativistic Wave-Equations' (Blackie: London).
Cox, W. (1977). *J. Phys.* A **10**, 1409.
Dirac, P. A. M. (1936). *Proc. R. Soc. London* A **155**, 447.
Doria, F. A. (1977). *J. Math. Phys.* **18**, 564.
Durand, E. (1975). *Phys. Rev.* D **11**, 3405.
Fierz, M. (1939). *Helv. Phys. Acta* **12**, 3.
Fierz, M., and Pauli, W. (1939). *Helv. Phys. Acta* **12**, 297; *Proc. R. Soc. London* A **173**, 211.
Fleming, G. N. (1966). *J. Math. Phys.* **7**, 1959.
Foldy, L. L. (1956). *Phys. Rev.* **102**, 568.
Gantmacher, F. R. (1960). 'The Theory of Matrices', Vol. 1, p. 101 (Chelsea: New York).
Guertin, R. F. (1974). *Ann. Phys. (New York)* **88**, 504.
Hammer, C. L., McDonald, S. C., and Pursey, D. L. (1968). *Phys. Rev.* **171**, 1349.
Hurley, W. J. (1971). *Phys. Rev.* D **4**, 3605.
Hurley, W. J. (1972). *Phys. Rev. Lett.* **29**, 1475.
Hurley, W. J. (1974). *Phys. Rev.* D **10**, 1185.
Joos, H. (1962). *Fortschr. Phys.* **10**, 65.
Kamefuchi, S., and Takahashi, Y. (1966). *Nuovo Cimento* **44**, 1.
Krajcik, R. A., and Nieto, M. M. (1977). *Phys. Rev.* D **15**, 416, 426, 433, 445.
Landau, L. D., and Lifshitz, E. M. (1959). 'Quantum Mechanics' (Pergamon: London).
Littlewood, D. E. (1958). 'The Theory of Group Characters and Matrix Representations of Groups', Ch. 10 (Oxford: London).
Lurié, D. (1968). 'Particles and Fields' (Interscience: New York).
Mathews, P. M. (1966). *Phys. Rev.* **143**, 978.
Mathews, P. M., and Ramakrishnan, S. (1967). *Nuovo Cimento* **50**, A339.
Nelson, T. J., and Good, R. H., Jr (1968). *Rev. Mod. Phys.* **40**, 508.
Rose, M. E. (1957). 'Elementary Theory of Angular Momentum' (Wiley: New York).
Torruella, A. J. (1975). *J. Math. Phys.* **16**, 1637.
Weaver, D. L. (1968). *Nuovo Cimento* A **53**, 667.
Weaver, D. L. (1975). *Phys. Rev.* D **12**, 2325.
Weaver, D. L., Hammer, C. L., and Good, R. H., Jr (1964). *Phys. Rev.* **135**, B241.
Weber, T. A., and Williams, S. A. (1965). *J. Math. Phys.* **6**, 1980.
Weinberg, S. (1964). *Phys. Rev.* **133**, B1318.
Weinberg, S. (1969). *Phys. Rev.* **181**, 1893.
Williams, S. A., Draayer, J. P., and Weber, T. A. (1966). *Phys. Rev.* **152**, 1207.

Appendix 1

The nomenclature generally used in the text is that given by Lurié (1968):

$$\hbar = c = 1, \qquad \partial_\mu = \partial/\partial x_\mu = (\nabla, -\mathrm{i}\,\partial/\partial_t),$$

$$\gamma_i = \begin{bmatrix} 0 & -\mathrm{i}\sigma_i \\ \mathrm{i}\sigma_i & 0 \end{bmatrix}, \qquad \gamma_4 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

$$\Sigma_{\mu\nu} = (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)/2\mathrm{i}, \qquad C = \begin{bmatrix} 0 & \mathrm{i}\sigma_2 \\ \mathrm{i}\sigma_2 & 0 \end{bmatrix}.$$

Appendix 2

Consider the spinor equation

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}, \tag{A1}$$

where the nonsingular 2×2 matrix transforms (ψ_1, ψ_2) into (χ_1, χ_2), and suppose we wish to determine how the quadratic terms ψ_1^2, $\psi_1\psi_2$ and ψ_2^2 transform. The expanded form of equation (A1) is

$$a\psi_1 + c\psi_2 = \chi_1, \tag{A2a}$$

$$b\psi_1 + d\psi_2 = \chi_2, \tag{A2b}$$

and taking the quadratics of both sides gives

$$(a\psi_1 + c\psi_2)^2 = \chi_1^2, \tag{A3a}$$

$$(a\psi_1 + c\psi_2)(b\psi_1 + d\psi_2) = \chi_1\chi_2, \tag{A3b}$$

$$(b\psi_1 + d\psi_2)^2 = \chi_2^2. \tag{A3c}$$

These equations can be written in matrix form as

$$\begin{bmatrix} a^2 & \sqrt{2}\,ac & c^2 \\ \sqrt{2}\,ab & (ad+bc) & \sqrt{2}\,cd \\ b^2 & \sqrt{2}\,bd & d^2 \end{bmatrix}\begin{bmatrix} \psi_1^2 \\ \sqrt{2}\,\psi_1\psi_2 \\ \psi_2^2 \end{bmatrix} = \begin{bmatrix} \chi_1^2 \\ \sqrt{2}\,\chi_1\chi_2 \\ \chi_2^2 \end{bmatrix}, \tag{A4}$$

provided, of course, that a, b, c and d are numbers and not operators. The 3×3 matrix in equation (A4) is the second induced matrix of the 2×2 matrix in (A1). Notice that the definition given here differs slightly from that given by Littlewood (1958) because of the $\sqrt{2}$ factor in the wavefunction. More generally the $2j$th induced matrix is generated from the set of equations

$$(a\psi_1 + c\psi_2)^{j+m}(b\psi_1 + d\psi_2)^{j-m} = \chi_1^{j+m}\chi_2^{j-m}, \tag{A5}$$

with $+j \geqslant m \geqslant -j$.

With $[2j]$ as the symbol for the $2j$th induced matrix then $(\boldsymbol{\sigma}\,.\,\boldsymbol{P})^{[2j]}$, which occurs in equations (4) of Section 2, can be readily determined from equation (A5). The formula for the mm' element is

$$(\boldsymbol{\sigma}\,.\,\boldsymbol{P})^{[2j]}_{mm'} = (-)^{m'-m}\sum_s \frac{(-)^s\{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!\}^{\frac{1}{2}}}{s!\,(j-s-m')!\,(j+m-s)!\,(m'+s-m)!}$$
$$\times P_z^{j+m-s}(-P_z)^{j-m'-s}(-P_+)^{m'+s-m}P_-^s, \tag{A6}$$

with $+j \geqslant m$ (or m') $\geqslant -j$. The element of the first row and first column is labelled by $m = m' = j$ and the values of m and m' decrease down the columns and along the rows respectively. The sum over s is between $\pm\infty$ but is limited by the usual definition $1/n! = 0$ for $n < 0$. The formula (A6) should be compared with equation (II.15), Appendix II, of the text by Rose (1957), for the elements of the general rotation matrix.

There is often great simplicity in manipulating induced matrices. For example, if the 2×2 matrix A has eigenvalues λ_1 and λ_2 then $(A)^{[2j]}$ has eigenvalues $\lambda_1^{j+m}\lambda_2^{j-m}$, with $+j \geqslant m \geqslant -j$. Thus the eigenvalues of $(\boldsymbol{\sigma}.\boldsymbol{P})^{[2j]}$ are $(|\boldsymbol{P}|)^{2j}$ and $-(|\boldsymbol{P}|)^{2j}$, each being $j+\frac{1}{2}$ degenerate for half-integer spin and being $j+1$ and j degenerate respectively for integer spin. Similarly if Q transforms A to the diagonal form $\tilde{Q}AQ$ then $Q^{[2j]}$ will diagonalize $A^{[2j]}$.

The direct product of a matrix with itself is reducible to the second induced matrix and the second compound matrix (Littlewood 1958). This is related to the decomposition

$$D(\tfrac{1}{2})\times D(\tfrac{1}{2}) = D(1)\oplus D(0). \tag{A7}$$

In order to make the connection more precise consider the equations

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}, \qquad \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \begin{bmatrix} \chi_1' \\ \chi_2' \end{bmatrix}. \tag{A8}$$

Hence

$$(a\psi_1+c\psi_2)(a\psi_1'+c\psi_2') = \chi_1\chi_1', \tag{A9a}$$

$$(a\psi_1+c\psi_2)(b\psi_1'+d\psi_2') = \chi_1\chi_2', \tag{A9b}$$

$$(b\psi_1+d\psi_2)(a\psi_1'+c\psi_2') = \chi_2\chi_1', \tag{A9c}$$

$$(b\psi_1+d\psi_2)(b\psi_1'+d\psi_2') = \chi_2\chi_2', \tag{A9d}$$

or in matrix form

$$\begin{bmatrix} a^2 & ac & ca & c^2 \\ ab & ad & cb & cd \\ ba & bc & da & dc \\ b^2 & bd & db & d^2 \end{bmatrix}\begin{bmatrix} \psi_1\psi_1' \\ \psi_1\psi_2' \\ \psi_2\psi_1' \\ \psi_2\psi_2' \end{bmatrix} = \begin{bmatrix} \chi_1\chi_1' \\ \chi_1\chi_2' \\ \chi_2\chi_1' \\ \chi_2\chi_2' \end{bmatrix}, \tag{A10}$$

where the 4×4 matrix is the direct product of the 2×2 matrix in equations (A8) with itself. By elementary manipulations equation (A10) is equivalent to

$$\begin{bmatrix} a^2 & \sqrt{2}\,ac & c^2 & 0 \\ \sqrt{2}\,ab & (ad+bc) & \sqrt{2}\,cd & 0 \\ b^2 & \sqrt{2}\,bd & d^2 & 0 \\ 0 & 0 & 0 & (ad-bc) \end{bmatrix}\begin{bmatrix} \psi_1\psi_1' \\ (\psi_1\psi_2'+\psi_2\psi_1')/\sqrt{2} \\ \psi_2\psi_2' \\ (\psi_1\psi_2'-\psi_2\psi_1')/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \chi_1\chi_1' \\ (\chi_1\chi_2'+\chi_2\chi_1')/\sqrt{2} \\ \chi_2\chi_2' \\ (\chi_1\chi_2'-\chi_2\chi_1')/\sqrt{2} \end{bmatrix},$$

where the 4×4 matrix has now been reduced to the 3×3 second induced matrix and the 1×1 second compound matrix. The three factors $\psi_1\psi_1'$, $(\psi_1\psi_2'+\psi_2\psi_1')/\sqrt{2}$ and $\psi_2\psi_2'$ transform as symmetrical second-rank spinor components and $(\psi_1\psi_2'-\psi_2\psi_1')/\sqrt{2}$ transforms as an antisymmetrical second-rank spinor. Thus where the 2×2 matrix is the spin 1/2 rotation matrix $D(\frac{1}{2})$ then the decomposition (A7) corresponds to the decomposition of the direct product into the second induced matrix and the second compound matrix.

Manuscript received 2 December 1977