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# Accretion of Planetesimals within a Gaseous Ring

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*Abstract*

An analytical study is made of the accretion of planetesimals by a planetary embryo within the framework of a modern Laplacian theory for the formation of the planetary system. The equation of motion of the particle, which is initially comoving ahead of (or behind) the growing planet on the same circular Keplerian orbit about the Sun, is examined both in the presence and absence of a gaseous torus which is also centred on the same mean orbit. The gas density in the torus is taken to be uniform and the drag exerted on the particle is assumed to vary as the square of the relative velocity, corresponding to motion at high Reynolds number. It is found that the gas acts as a damper to the coriolis acceleration due to the Sun in the rotating frame of reference of the embryo, which tends to pull the particle off the mean circular orbit, thus preventing accretion. In the absence of the gaseous drag, less than 1% of particles lying well inside the so-called sphere of gravitational influence of the embryo are accreted, whilst if the gas drag is included nearly all of these particles are captured. In all instances the accreting particles impart a spin angular momentum to the embryo which is prograde with the orbital motion. The actual spin rate decreases with increasing gas drag and is found to be lowest for the innermost planets Mercury and Venus, where the gas density is greatest. A more detailed numerical study is probably required to determine the rotational period of larger planets and planetary cores which possess an outer atmosphere, not included in the present study, and where nonlinear effects in the particle's equation of motion cannot be ignored.

## Introduction

In 1796 the celebrated French mathematician P. S. de Laplace proposed that the solar system had formed through condensation from a concentric system of fluid rings which were shed by the primordial Sun. The Sun was originally supposed to have been a huge diffuse cloud having dimensions exceeding the orbit of Neptune, as pictorially illustrated in Fig. 1. As the cloud cooled off it contracted inwards and proceeded to spin faster on its axis of rotation owing to the conservation of total angular momentum. Eventually the stage was reached when the centrifugal force at the equator overcame the gravitational force there and a ring of matter was shed at the present orbit of Neptune. The process of successive contraction and ring shedding repeated itself at the orbits of each of the planets until the Sun reached its present size, still spinning quite rapidly. Laplace did not specify the physical mechanism whereby the nebula discretely abandoned its fluid rings, nor did he attempt to explain how the planets formed from these rings. Nonetheless, the hypothesis was conceptually simple and attractive and it did account for the broad physical structure of the planetary system, including the near circularity of the planetary orbits, as well as their common motion around the Sun.

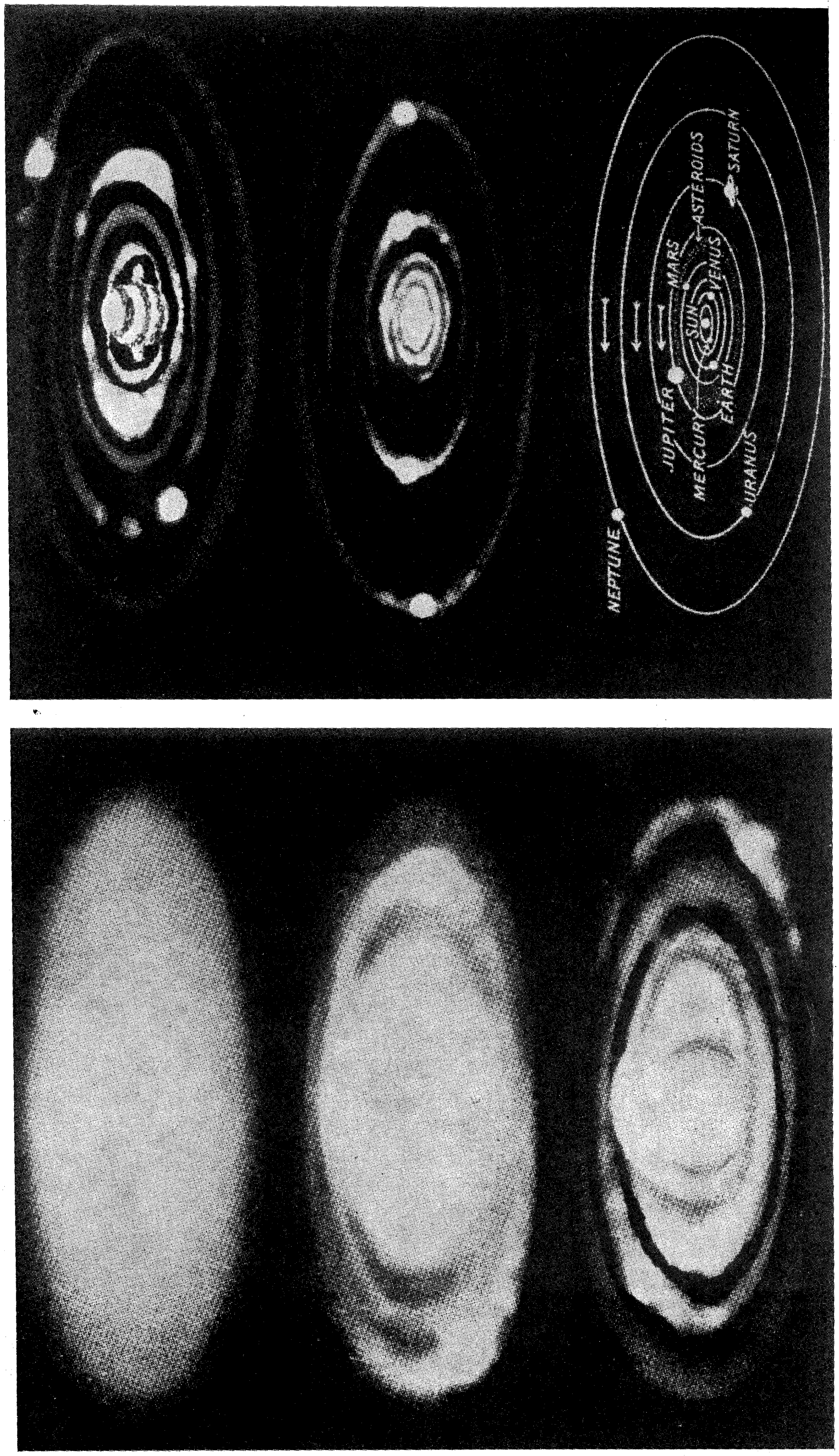


**Fig. 1.** Visual impression of the original Laplacian nebula hypothesis. The young contracting and rotating protosun sheds a concentric system of gaseous rings from which the planets later condense. [From drawings by Scriven Bolton F.R.A.S., Fig. 158 of Whipple 1963.]

During the second half of the nineteenth century, the various unexplained features of Laplace's theory attracted the criticism which eventually led to its downfall. Babinet (1861) drew attention to the great discrepancy which existed between the observed distributions of mass and angular momentum in the solar system and the values which should exist if the Sun had behaved in the manner proposed by Laplace. In addition, Fouché (1884) pointed out that the Sun would have been spinning with a period of only a few hours and not the 25 days which is presently observed. Various other weaknesses were also found in the hypothesis, as have been recently reviewed by Brush (1978). The final overthrow of the nebular hypothesis appears to be largely due to the efforts of the Chicago scientists Chamberlin (1900) and Moulton (1900). In order to account for the direct rotation of the known planetary spins, Laplace had assumed that the rings were liquid in composition and rotated with a uniform angular velocity, just like a solid body. Chamberlin, who was a geologist, correctly pointed out that the Earth could not have formed from such material, whilst his mathematical colleague Moulton proved that it would not be possible for a uniformly rotating ring to coalesce into a single body. These two scientists proceeded to develop a planetesimal hypothesis in place of the nebula hypotheses. In fact, Laplace was aware (cf. Maxwell 1859) that a uniformly rotating ring was dynamically unstable and would tend to break up by precipitating itself onto the surface of the Sun.

*A Modern Laplacian Theory with Supersonic Convective Turbulence*

Recently I have attempted to construct a modern Laplacian theory taking into account new astronomical data relating to star formation and the properties of young forming suns (Prentice 1978*a*, 1978*b*). The most important of these data concern the T Tauri stars and the inference from the violent surface activity observed in these stars that there may exist a large supersonic turbulent stress $\langle \rho_t v_t^2 \rangle$ in their interiors which has a value many times the local gas pressure $\rho \mathscr{R} T/\mu$. By incorporating this additional stress into the structure equations one can understand how a collapsing gas cloud can shed a discrete system of concentric gaseous rings, each of about the same mass $m$, at orbital radii $R_n$ ($n = 0, 1, 2, \ldots$) which form a geometric sequence given by

$$R_n/R_{n+1} = (1 + m/Mf)^2 . \qquad (1)$$

Here $M$ and $f$ denote the mass and the coefficient of the moment of inertia of the contracting cloud, given by $f = I/MR_e^2$ where $I$ is the axial moment of inertia and $R_e$ the equatorial radius. For a strongly turbulent cloud, the interior mass distribution is very centrally condensed, with $f = 0{\cdot}01$–$0{\cdot}02$, whilst the mass $m$ of the disposed rings is very small compared with the envelope mass $M$. Typically we find $m \approx 0{\cdot}003\,M = 10^3 M_\oplus$ ($M_\oplus$ being the Earth mass) on setting $M = M_\odot$ (the solar mass). This result alone allows us to resolve the angular momentum difficulty raised by Babinet (1861).

As regards Fouché's (1884) objection, the present slow rotation of the Sun is by no means indicative of the initial rotation that would have been present at the time of the Sun's formation. Observations of T Tauri stars suggest that these objects have very high equatorial velocities, up to 100 km s$^{-1}$, which imply that these bodies are rapidly rotating (Herbig 1962). This fact, together with the evidence for the existence of a strong large-scale magnetic field in the early Sun, suggests that the Sun may have given up any initial rapid spin through an interaction between the magnetic field and a strong solar wind during the final stages of the protosun's contraction from radius $10\,R_\odot$ to $R_\odot$ (Freeman 1978). This electromagnetic interaction and the subsequent transfer of angular momentum would become important only during the final stages of the contraction from radius $\sim 10\,R_\odot$ to $R_\odot$ when the temperature at the photosurface was high enough for the gas to ionize ($T \gtrsim 3500$ K).

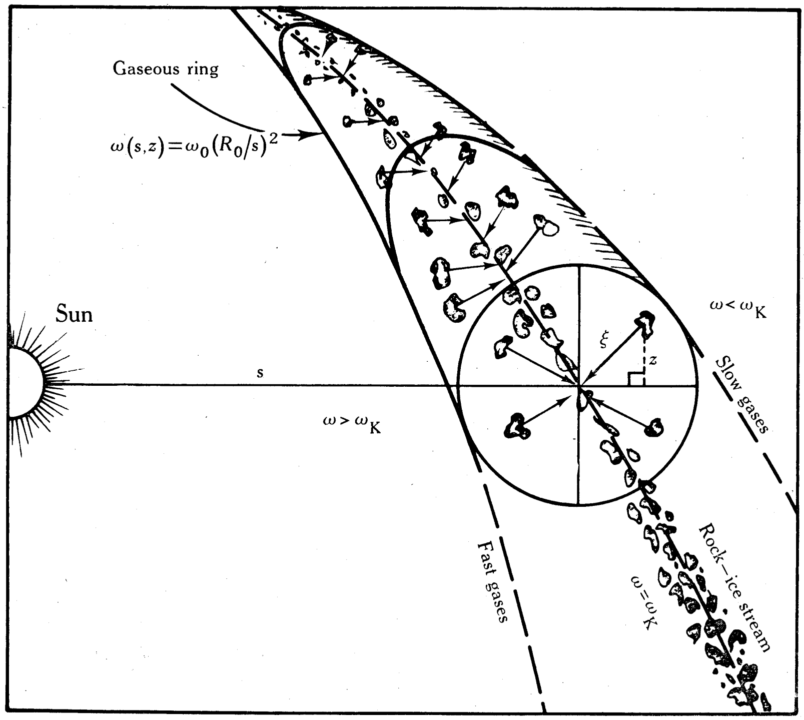


**Fig. 2.** Schematic illustration of the gravitational settling of condensed particles onto the mean circular Keplerian orbit of the gaseous ring to form a concentrated orbiting stream of planetesimals. [Reprinted with permission from Prentice 1974.]

In this paper, attention is concentrated on the other principal objection which was cast at the Laplacian hypothesis, namely how the planets managed to accrete from the system of gaseous rings. In the modern Laplacian theory (Prentice 1978*a*, 1978*b*) the rings are not uniformly rotating liquids, as Laplace had envisaged, but instead are gaseous and have differential orbital angular velocity ($\omega$) and density ($\rho_g$) distributions given by the equations

$$\omega(s,z) = \omega_n R_n^2/s^2, \qquad \rho_g(s,z) \approx \rho_n \exp(-\tfrac{1}{2}\alpha\xi^2/R_n^2). \tag{2}$$

Here $(s, \phi, z)$ denote cylindrical polar coordinates defined relative to the axis of the $n$th ring and $\xi$ is the distance measured off the mean orbit $s = R_n$, $z = 0$ of this ring where the angular velocity $\omega_n$ equals the local Kepler value and the gas density $\rho_n$ is a local maximum; that is,

$$\omega_n = (GM/R_n^3)^{\frac{1}{2}}, \qquad \rho_n = \alpha m/4\pi^2 R_n^3, \tag{3}$$

and $\alpha = \mu GM/\mathscr{R}T_n R_n \approx O(400)$, where $T_n$ is the temperature of the ring which is assumed to be nearly uniform. These gaseous rings are stable against fragmentation or coalescence since their mass is so much smaller than that of the central governing Sun (cf. Polyachenko and Fridman 1972).

After a gaseous ring has been shed by the protosun, the various chemical species whose condensation point lies above that determined by the prevailing temperature and pressure proceed to condense out of the gas forming fine solid grains. These grains then migrate under the influence of the Sun's gravitational force and the orbital centrifugal force as well as gaseous drag onto the mean circular orbit $R_n$ of the gas ring where the gas and grains have a common orbital angular velocity. That is, the differential velocity distribution of the gas, coupled with gaseous drag, focuses the condensate material into a single well-defined circular Keplerian orbit to form a concentrated stream of rocks or ices, as schematically illustrated in Fig. 2 (Prentice 1974, 1978*a*; Hourigan 1977). Alfvén and Arrhenius (1976) have also drawn attention to the importance of concentrating the planetary material into 'jet streams'. They have argued on dynamical grounds that subsequent accretion may take place in the absence of gas drag.

Next, Hourigan and Prentice (1979) have shown that, as soon as the mass density of the condensate stream exceeds a critical value, it will become gravitationally unstable and break up to form isolated groups of self-gravitationally bound masses called planetesimals. That is, we have a Jeans-type instability mechanism operating where the self-gravitational energy of the condensate stream first exceeds its total 'thermal' or random kinetic motion. Gaseous drag serves to dampen out this random excess particle motion as well as to re-direct wandering particles and planetesimals back onto the mean Keplerian orbit of the ring (Prentice 1978*a*).

Consider now the evolution of the fragmenting stream. Since particles are preferentially attracted towards regions of higher mass density, we have a situation where a statistical runaway may occur in the growth of the largest planetesimal group. Thus if one group is initially slightly more massive than any others as a result of spatial inhomogeneities in the line density distribution of the condensate stream it will compete for the remaining planetesimals at a faster rate, resulting perhaps in the emergence of a single embrionic planetary body at each orbital radius $R_n$. As long as the gaseous ring remains intact and so acts to confine the reservoir of condensate material on the mean orbit of the ring, as well as supplying fresh grain material, this embryo may eventually swallow up all of the other slower growing planetesimals in each ring. Hourigan (1977) has shown that, as soon as the gaseous ring disperses, the stream of planetesimals also disperses both directly through being dragged away by the gas, as is the case for the smaller particles, and through collisional interactions with one another, leading to a broadened stream of scattered planetesimals similar to the asteroidal belt. The gaseous rings therefore play a crucial role in the planetary aggregation process.

In this paper we analytically examine in detail the possible capture by a planetary embryo of mass $M_E$ and radius $R_E$ of a particle of mass $m$ which is initially comoving on the same mean circular orbit of the gas ring within the so-called sphere of gravitational influence of the embryo, given by

$$R_I = R_0(M_E/3M)^{\frac{1}{3}}, \tag{4}$$

where $M$ is the mass of the central body (sun). For the Earth we find $R_{\mathrm{I}} = 0\cdot01\,R_0 = 235\,R_\oplus$ whilst, for an icy embryo of mass $10\,M_\oplus$ and density $\rho_{\mathrm{E}} = 1\ \mathrm{g\,cm^{-3}}$ at Jupiter's orbit, $R_{\mathrm{I}} = 0\cdot0216\,R_0 = 690\,R_{\mathrm{E}}$.

We assume that we have

$$m \ll M_{\mathrm{E}} \ll M, \tag{5}$$

so that the sun dominates the motion of the embryo which moves on the circular orbit of the gas ring undisturbed by the motion of the particle. We also compute the rotational period of the embryo on the basis of the spin angular momentum imparted to it by the accreting planetesimal material.

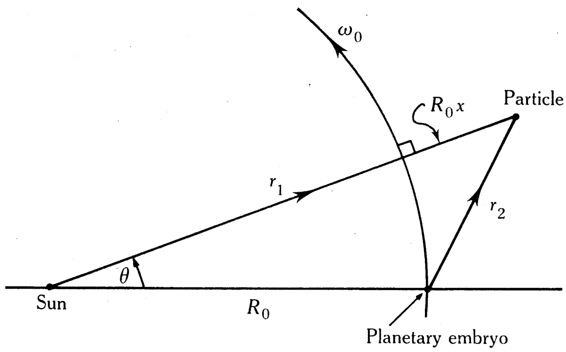


**Fig. 3.** Rotating coordinate system defined in the orbital plane of the planet. The dimensionless coordinates $(x, \theta)$ of the particle moving in this plane are indicated.

## Equations of Motion

Fig. 3 shows the geometry of a particle P in a coordinate system which is centred on a planetary embryo and rotating about a sun with a Keplerian angular velocity $\omega_0$ appropriate to the distance $R_0$. If $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ denote the vector positions of P relative to the sun and the embryo, the absolute equation of motion of the particle reads

$$\ddot{\boldsymbol{r}}_1 = -(GM/r_1^3)\boldsymbol{r}_1 - (GM_{\mathrm{E}}/r_2^3)\boldsymbol{r}_2 + \boldsymbol{f}_{\mathrm{D}}, \tag{6}$$

where $\boldsymbol{f}_{\mathrm{D}}$ is the acceleration due to gas drag. We suppose the particle to be much larger in radius $a$ than the mean free path length $\lambda_{\mathrm{g}}$ of the gas, which is given by

$$\lambda_{\mathrm{g}} = 2\times10^{-9}\rho_{\mathrm{g}}^{-1}\ \mathrm{cm} = 0\cdot06\ \mathrm{cm}\quad \text{(Earth's orbit)}, \tag{7a}$$

$$= 20\ \mathrm{cm}\quad \text{(Jupiter's orbit)}, \tag{7b}$$

for a gas of solar composition, and also suppose the particle to have a Reynolds number $Re \gtrsim 10$ so that the flow is turbulent. In this case

$$\boldsymbol{f}_{\mathrm{D}} = \gamma_2\,|\boldsymbol{v}_{\mathrm{g}} - \dot{\boldsymbol{r}}_1|\,(\boldsymbol{v}_{\mathrm{g}} - \dot{\boldsymbol{r}}_1), \tag{8a}$$

where

$$\gamma_2 = \tfrac{3}{8}C_{\mathrm{D}}\,\rho_{\mathrm{g}}/\rho_s a \tag{8b}$$

is the inverse gas drag length, $C_{\mathrm{D}} \sim 0\cdot3$ being the drag coefficient (Goldreich and Ward 1973). As long as the particle remains fairly close to the mean orbit $R_0$ of the gas ring, $\rho_g$ is nearly constant and $\gamma_2$ may be regarded as a constant too. Equation (8a) is also valid for small particles ($a \lesssim \lambda_g$) which are moving supersonically, though the form of $\gamma_2$ is somewhat different. In equation (8a) the local orbital velocity $\boldsymbol{v}_g$ of the gas is given by

$$\boldsymbol{v}_g = s\,\omega(s,z)\,\hat{\boldsymbol{\phi}} = (\omega_0 R_0^2/s)\hat{\boldsymbol{\phi}}, \tag{9}$$

where $\hat{\boldsymbol{\phi}}$ is the unit vector tangential to the circular orbit passing through P.

We next introduce the dimensionless coordinates $(x, \theta)$, where $R_0 x$ denotes the deviation in plane polar radius of the particle from the mean orbit $s = R_0$, and $\theta$ is the difference in polar angle $\phi$ between the particle and the embryo subtended at the sun (see Fig. 3). We further assume that the particle is initially moving in the orbital plane of the embryo ($z = 0$). In this case, equations (6), (8) and (9) ensure that the particle always remains in that plane and the planar equations of motion separate into the dimensionless forms

$$\ddot{x} - (1+x)(\omega_0+\dot{\theta})^2 = -\omega_0^2/(1+x)^2 - \varepsilon\omega_0^2(x+2\sin^2\tfrac{1}{2}\theta)/d^3 - k\dot{x}(\dot{x}^2+u^2)^{\frac{1}{2}}, \tag{10a}$$

$$(1+x)\ddot{\theta} + 2\dot{x}(\omega_0+\dot{\theta}) = -\varepsilon\omega_0^2(\sin\theta)/d^3 - ku(\dot{x}^2+u^2)^{\frac{1}{2}}, \tag{10b}$$

where

$$d = \{x^2+\sin^2\theta + 4\sin^2\tfrac{1}{2}\theta\,(x+\sin^2\tfrac{1}{2}\theta)\}^{\frac{1}{2}}, \tag{11a}$$

$$u = (\omega_0+\dot{\theta})(1+x) - \omega_0/(1+x), \tag{11b}$$

$$\varepsilon = M_E/M, \qquad k = \gamma_2 R_0. \tag{11c}$$

Equations (10a) and (10b) are too nonlinear to admit any analytic solution. If, however, we assume

$$x \ll 1, \qquad \theta \ll 1, \qquad \dot{\theta} \ll \omega_0, \tag{12}$$

the equations may be linearized, yielding

$$\ddot{x} - 2\omega_0\dot{\theta} = \omega_0^2 x\{3-\varepsilon(x^2+\theta^2)^{-3/2}\} - k\dot{x}\{\dot{x}^2+(\dot{\theta}+2\omega_0 x)^2\}^{\frac{1}{2}}, \tag{13a}$$

$$\ddot{\theta} + 2\omega_0\dot{x} = -\varepsilon\omega_0^2\theta(x^2+\theta^2)^{-3/2} - k(\dot{\theta}+2\omega_0 x)\{\dot{x}^2+(\dot{\theta}+2\omega_0 x)^2\}^{\frac{1}{2}}. \tag{13b}$$

It follows from equation (13a) that the gravitational attraction of the embryo exceeds the difference between the sun's attraction and the orbital centrifugal acceleration provided the particle lies within the radius

$$d_I = (\tfrac{1}{3}\varepsilon)^{\frac{1}{3}}, \tag{14}$$

the same as given earlier (equation 4). Having an initial distance $d_0 < d_I$ is not, however, a sufficient condition for capture, as we shall see below.

**Solution for Aximuthal Motion $\theta(t)$**

We now examine the analytic solution of the linearized equations of motion (13) for the case of physical interest where the particle initially lies an angular distance

$\theta_0$ ahead of (or behind) the embryo on the same circular Keplerian orbit; that is, at time $t = 0$,

$$x(0) = 0, \qquad \dot{x}(0) = 0, \qquad \theta(0) = \theta_0, \qquad \dot{\theta}(0) = 0, \tag{15a}$$

and

$$\theta_0^3 \ll \tfrac{1}{3}\varepsilon, \tag{15b}$$

in view of equation (14). Suppose now we make the further assumptions

$$x^2 \ll \theta^2, \qquad \dot{x}^2 \ll \dot{\theta}^2, \qquad 2\omega_0 x \ll \dot{\theta}. \tag{16}$$

In this case, we find that equation (13b) simplifies to

$$\ddot{\theta} = -\varepsilon\omega_0^2\theta^{-2} + k\dot{\theta}^2, \tag{17}$$

noting that $\dot{\theta}$ is negative during the first approach to the embryo. This equation may be readily integrated to yield the solution

$$\dot{\theta}(t) = -\omega_0(2\varepsilon)^{\frac{1}{2}}\exp(k\theta)\left(\int_\theta^{\theta_0} \theta^{-2}\exp(-2k\theta)\,\mathrm{d}\theta\right)^{\frac{1}{2}}. \tag{18}$$

This solution may be integrated only in the regimes of very low and high drag determined by the dimensionless parameter $k\theta_0$.

*Case* (*i*), $k\theta_0 \ll 1$

In this case we have from equation (18)

$$\dot{\theta}(t) \approx -\omega_0(2\varepsilon)^{\frac{1}{2}}(\theta^{-1} - \theta_0^{-1})^{\frac{1}{2}}, \tag{19}$$

which gives

$$\omega_0 t = (\theta_0^3/2\varepsilon)^{\frac{1}{2}}[\tfrac{1}{2}\pi - \arcsin(\theta/\theta_0)^{\frac{1}{2}} + \{(\theta/\theta_0)(1-\theta/\theta_0)\}^{\frac{1}{2}}]. \tag{20}$$

This is a free-fall solution. The particle reaches the point $\theta = 0$ after a time

$$t_0 = (\pi/2\omega_0)(\theta_0^3/2\varepsilon)^{\frac{1}{2}} \ll 2\pi/\omega_0 = 1 \text{ yr} \quad \text{(at Earth's orbit)}. \tag{21}$$

In view of the assumptions (12), the above solution is only valid provided we have

$$(2\varepsilon)^{\frac{1}{2}} \ll \theta_0^{\frac{1}{2}}, \qquad \theta_0 \ll 1, \tag{22}$$

that is, provided the particle does not lie too close to the embryo, or otherwise the nonlinear terms cannot be ignored.

*Case* (*ii*), $k\theta_0 \gg 1$

In this case equation (18) simplifies to

$$\dot{\theta} \approx -\omega_0(\varepsilon/k)^{\frac{1}{2}}\theta^{-1} \tag{23}$$

which has solution

$$\theta(t) = \theta_0\{1-(2\omega_0/\theta_0^2)(\varepsilon/k)^{\frac{1}{2}}t\}^{\frac{1}{2}}. \tag{24}$$

This yields a 'capture' time

$$t_0 = \theta_0^2(k/\varepsilon)^{\frac{1}{2}}/2\omega_0. \tag{25}$$

The solution is valid provided we have

$$(\varepsilon/\theta_0)^{\frac{1}{2}} \ll (k\theta_0)^{\frac{1}{2}}, \qquad \theta_0 \ll 1, \tag{26}$$

which, in view of the fact that $k\theta_0 \gg 1$, is a condition which can be met for a far larger range of initial starting positions $\theta_0$ than those that are valid in the zero-drag regime.

**Solution for Trajectory** $x(\theta)$

Consider now the solution for $x$. If we incorporate the conditions (12) and (16) and further suppose that the particle lies initially well inside the sphere of influence of the planetary embryo, that is, $\theta_0^3 \ll \frac{1}{3}\varepsilon$ as before, then equation (13a) simplifies to

$$\ddot{x} - 2\omega_0\dot{\theta} = -\varepsilon\omega_0^2 x\theta^{-3} + k\dot{x}\dot{\theta}. \tag{27}$$

Dividing through by $\dot{\theta}$ and introducing the integrating factor $\exp(k\theta)$, we find that this equation may be integrated exactly once to yield

$$\dot{x}(t) = -\frac{2\omega_0}{k}\Big(1-\exp(k\theta-k\theta_0)\Big) + \varepsilon\omega_0^2\exp(k\theta)\int_\theta^{\theta_0}\frac{x\exp(-k\theta)\,\mathrm{d}\theta}{\theta^3\dot{\theta}(t)}. \tag{28}$$

Now dividing this equation by equation (18), we obtain the differential equation for the time-independent trajectory $x = x(\theta)$, namely

$$\left(\int_\theta^{\theta_0}\frac{\exp(-2k\theta)\,\mathrm{d}\theta}{\theta^2}\right)^{\frac{1}{2}}\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{1}{k}\left(\frac{2}{\varepsilon}\right)^{\frac{1}{2}}\Big(\exp(-k\theta)-\exp(-k\theta_0)\Big)$$

$$-\omega_0\left(\frac{\varepsilon}{2}\right)^{\frac{1}{2}}\int_\theta^{\theta_0}\frac{x\exp(-k\theta)\,\mathrm{d}\theta}{\theta^3\dot{\theta}(t)}. \tag{29}$$

To solve equation (29) it is convenient to define

$$I(\theta) = \int_\theta^{\theta_0}\theta^{-2}\exp(-2k\theta)\,\mathrm{d}\theta. \tag{30}$$

In terms of $I(\theta)$, we have $\dot{\theta} = -\omega_0(2\varepsilon)^{\frac{1}{2}}\exp(k\theta)\,I^{\frac{1}{2}}$ and equation (29) becomes, after some rearrangement,

$$\theta I^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{x}{\theta}\right) = \frac{1}{k}\left(\frac{2}{\varepsilon}\right)^{\frac{1}{2}}\Big(\exp(-k\theta)-\exp(-k\theta_0)\Big) + \int_\theta^{\theta_0} I^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{x}{\theta}\right)\mathrm{d}\theta. \tag{31}$$

Next we put

$$v = I^{\frac{1}{2}}\,\mathrm{d}(x/\theta)/\mathrm{d}\theta \tag{32}$$

and differentiate equation (31) to obtain

$$\mathrm{d}(\theta^2 v)/\mathrm{d}\theta = -(2/\varepsilon)^{\frac{1}{2}}\theta\exp(-k\theta), \tag{33}$$

which can be readily integrated to yield

$$v(\theta) = \frac{1}{k^2\theta^2}\left(\frac{2}{\varepsilon}\right)^{\frac{1}{2}}\Big((1+k\theta)\exp(-k\theta) - (1+k\theta_0)\exp(-k\theta_0)\Big). \tag{34}$$

Returning then to equation (32) we finally recover the desired solution

$$x(\theta) = -\left(\frac{2}{\varepsilon}\right)^{\frac{1}{2}} \frac{\theta}{k^2} \int_{\theta}^{\theta_0} \frac{(1+k\theta)\exp(-k\theta) - (1+k\theta_0)\exp(-k\theta_0)}{\theta^2\{I(\theta)\}^{\frac{1}{2}}} \, \mathrm{d}\theta$$

$$\equiv -(2\theta_0^3/\varepsilon)^{\frac{1}{2}}\theta_0 F(\theta/\theta_0, k\theta_0). \tag{35}$$

Equation (35) describes the trajectory of the accreting planetesimal as seen in the rotating frame of the embryo. We evaluate this integral explicitly in the very low and very high gas-drag regimes.

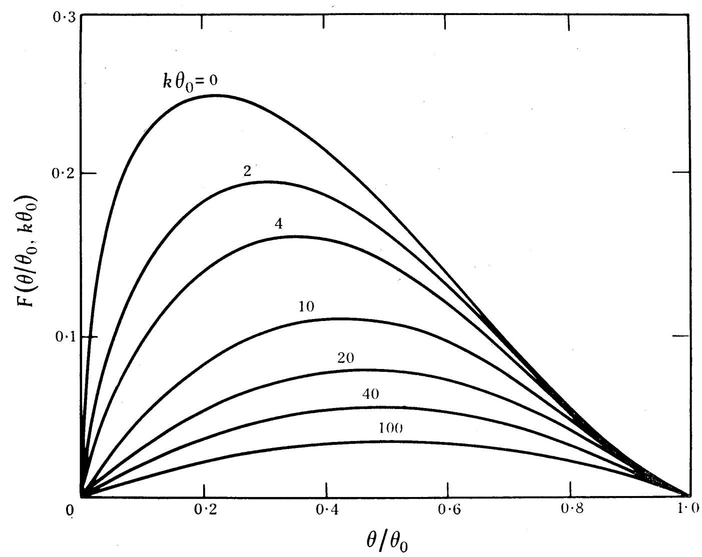


**Fig. 4.** Dimensionless trajectory function $F(\theta/\theta_0, k\theta_0)$ of the particle plotted against the azimuthal angle $\theta$ ahead of the planet, which is standardized against the starting angle $\theta_0$. This function is essentially the radial distance $-x(\theta)$ at which the particle moves towards the sun from the circular Keplerian orbit of the planet, starting from the position $\theta = \theta_0$ where $x(\theta_0) = 0$. The trajectories are plotted for different degrees of gaseous drag, which is measured in terms of the dimensionless parameter $k\theta_0$. For very high drag ($k\theta_0 \gg 1$) the particle hardly ventures from the mean circular orbit of the planet before being accreted at the point $\theta = 0$.

*Case (i)*, $k\theta_0 \ll 1$.

Here it follows from equation (30) that $I(\theta) = \theta^{-1} - \theta_0^{-1}$ and the integrand in equation (35) can be expanded about the point $k = 0$ to yield

$$x(\theta) = -\left(\frac{\theta_0}{2\varepsilon}\right)^{\frac{1}{2}} \theta \int_{\theta}^{\theta_0} (\theta_0 + \theta)(\theta_0 - \theta)^{\frac{1}{2}} \theta^{-3/2} \, \mathrm{d}\theta$$

$$= -\left(\frac{\theta_0^3}{2\varepsilon}\right)^{\frac{1}{2}} \theta \left[ 2\left(\frac{\theta_0}{\theta} - 1\right)^{\frac{1}{2}} + \arcsin\left(\frac{\theta}{\theta_0}\right)^{\frac{1}{2}} - \tfrac{1}{2}\pi - \left\{\frac{\theta}{\theta_0}\left(1 - \frac{\theta}{\theta_0}\right)\right\}^{\frac{1}{2}} \right]. \tag{36}$$

A sketch of this trajectory, scaled in units of $(2\theta_0^5/\varepsilon)^{\frac{1}{2}}$, is shown in Fig. 4 as the branch marked $k\theta_0 = 0$ of the family of curves of the function $F(\theta/\theta_0, k\theta_0)$ defined in (35).

In order for the solution given by equation (36) to be valid it is necessary to see if the conditions (12) and (16) are satisfied. We find that these assumptions are valid over all of the orbit except where $\theta \to 0$, provided we have

$$\theta_0^3/\varepsilon \ll 2, \tag{37}$$

that is, provided the particle lies well inside the sphere of influence of the embryo, as already expressed in the earlier assumption $\theta_0^3 \ll \frac{1}{3}\varepsilon$. As $\theta \to 0$, however, the ratio $x(\theta)/\theta$ diverges as $\theta \to 0$ and it is no longer true that $x^2 \ll \theta^2$. In order for the solution to remain valid right up to the point of impact with the planetary surface, whose angular radius subtended at the sun is

$$\theta_E = R_E/R_0 = (\varepsilon\rho_\odot/\rho_E)^{\frac{1}{3}} R_\odot/R_0, \tag{38}$$

where $\rho_E$ and $\rho_\odot$ are the mean mass densities of the embryo and sun, we shall require $x_E^2 \ll \theta_E^2$. From equation (36), this implies

$$\theta_0^3/\varepsilon \ll \tfrac{1}{2}(2\rho_\odot/\rho_E)^{\frac{1}{4}}(R_\odot/R_0)^{\frac{3}{4}} = 8{\cdot}8\times10^{-3} \quad \text{(Earth's orbit)}, \tag{39a}$$

$$= 3{\cdot}3\times10^{-3} \quad \text{(Jupiter's orbit)}, \tag{39b}$$

where we have evaluated the expression for the case of rock and ice embryos at the orbits of Earth and Jupiter having densities of 3 and 1 $\mathrm{g\,cm^{-3}}$ respectively.

It is clear that the condition (39) is a very much harder one to meet than (37) which states merely that the particle lie well inside the sphere of gravitational influence of the embryo. Unless the condition (39) is satisfied, it is not necessarily true that $x(0) \leqslant \theta_E$, implying that the particle directly impacts with the embryo. Instead if $x(0) > \theta_E$ the particle flies by the embryo and may completely escape its influence. Detailed numerical integration of the complete nonlinear equations of motion are then required to determine if the particle remains bound in the gravitational field of the embryo and is later accreted or if it subsequently escapes from its sphere of influence. Such calculations are being performed by K. Hourigan (personal communication).

*Case (ii)*, $k\theta_0 \gg 1$.

Here we find

$$I(\theta) \approx \exp(-2k\theta)/2k\theta^2 \tag{40}$$

and the integral in equation (35) asymptotes to the solution

$$x(\theta) = 2\theta(\theta_0-\theta)(\varepsilon k)^{-\frac{1}{2}}. \tag{41}$$

This solution is shown in Fig. 4 as the curve marked $k\theta_0 = 100$. The other curves in the diagram correspond to the exact solution of equation (35). The parabolic section defined in equation (41) closely approximates the exact solutions for $k\theta_0 \gtrsim 10$ over all of the range of $\theta$ except for $\theta$ near $\theta_0$. A careful inspection of equation (35) shows that $x$ vanishes like $(\theta_0-\theta)^{3/2}$ as $\theta \to \theta_0$ whilst equation (41) admits only a linear decline to zero. This deviation corresponds to the initial portion of the trajectory where the velocity is very low and the terminal-drag regime, which is represented by equation (41), has not been attained.

The large-drag solution in equation (41) satisfies all of the assumptions given in equation (16) provided that we have

$$\theta_0^3/\varepsilon \ll \min\left(\tfrac{27}{16}, \tfrac{1}{4}k\theta_0\right). \tag{42}$$

In view of the condition $\theta_0^3 \ll \frac{1}{3}\varepsilon$ and the fact that we have supposed $k\theta_0 \gg 1$, the inequality (42) is automatically satisfied, that is, the solution given by equation (41) is valid for each starting position $\theta_0$ which satisfies the condition $\theta_0^3 \ll \frac{1}{3}\varepsilon$. Thus as $\theta \to 0$ it follows that $x \to 0$ and hence the particle is directly accreted by the embryo. Gaseous drag therefore secures the accretion of planetesimals which lie well inside the sphere of influence of the embryo by resisting the transverse component of Coriolis acceleration in the rotating frame which tends to pull the particle away from the mean circular orbit of the embryo.

**Planetary Spins**

There is one further general feature of the integral solution for the particle trajectory which is worth noting here. When the particle impinges on the embryo it imparts a spin angular momentum whose sign depends on the angle between the incoming velocity vector and the outward radial vector at the planet's surface. The net transfer of angular momentum imparted by a mass d$m$ is seen to be, with the help of equations (18), (30) and (35),

$$\begin{aligned} \mathrm{d}\boldsymbol{h} &= (\boldsymbol{r}-\boldsymbol{r}_\mathrm{E})\times(\dot{\boldsymbol{r}}-\boldsymbol{v}_\mathrm{E})\mathrm{d}m = -R_0^2\,\dot{\theta}\,\theta^2\{\mathrm{d}(x/\theta)/\mathrm{d}\theta\}\mathrm{d}m\,\hat{\boldsymbol{\omega}}_0 \\ &= +(2\omega_0 R_0^2/k^2)\{1+k\theta_\mathrm{E}-(1+k\theta_0)\exp(k\theta_\mathrm{E}-k\theta_0)\}\mathrm{d}m\,\hat{\boldsymbol{\omega}}_0, \end{aligned} \tag{43}$$

where $\hat{\boldsymbol{\omega}}_0$ is the unit normal vector to the orbital plane of the embryo having the same sense as the orbital angular momentum; that is, since $\theta_0 > \theta_\mathrm{E}$, the particle always imparts a positive spin angular momentum to the planet irrespective of the initial starting position $\theta_0$.

Hence if planetary formation takes place through accretion of material from the mean orbit of the gas ring, the resulting spin moment of the planet is prograde with the orbital motion around the sun. This is indeed observed to be the case for all of the planets except Venus, which has a very low retrograde spin with a period of 243 days. Mercury's spin period of 59 days is exactly 2/3 of its orbital period, suggesting that it has become locked into a spin–orbit resonance by the solar tide (Goldreich and Peale 1968). Radar observations of the asteroids show that most of these bodies also have a positive spin (Hansen 1977).

Let us now compute the expected planetary rotation rate on the basis of equation (43).

*Case* (*i*), $k\theta_0 \to 0$.

Here equation (43) simplifies to

$$\mathrm{d}h = \omega_0\,R_0^2(\theta_0^2-\theta_\mathrm{E}^2)\,\mathrm{d}m, \tag{44}$$

which is simply the initial angular momentum of the particle relative to the embryo due to their common motion around the sun.

We now suppose that the planet results from the accretion of all matter originally in its orbital path. We also assume that this material is fed in at the outer limit of the embryo's sphere of capture, which is a fixed multiple $N_0$ of embryo radii $R_E$, so that $\theta_0 = N_0\,\theta_E$ and $\rho_E$ is constant. The total spin angular momentum acquired by the embryo during growth from radius 0 to $R_E$ is then

$$H = \tfrac{3}{5}(N_0^2-1)MR_E^2\,\omega_0. \tag{45}$$

This leads to a final absolute rotation rate

$$\omega_E = \omega_0 + H/\tfrac{2}{5}MR_E^2 = \tfrac{1}{2}(3N_0^2-1)\omega_0. \tag{46}$$

Since accretion occurs only for starting distances $\theta_0$ which satisfy the condition (39), it follows that

$$N_0 \lesssim N_{max} = 2^{-7/12}(\rho_E/\rho_\odot)^{1/4}(R_0/R_\odot)^{3/4} \sim 50 \quad \text{(Earth's orbit)}. \tag{47}$$

Choosing $N_0 = N_{max}$, corresponding to the furthermost starting point from which accretion can safely take place, we obtain a spin period of order only 2 h. This period is much shorter than that observed amongst the terrestrial planets, indicating that we cannot overlook the role played by gaseous drag.

*Case (ii)*, $k\theta_0 \gg 1$.

Here we find

$$dh = (2\omega_0\,R_0^2/k^2)(1+k\theta_E)\,dm, \tag{48}$$

which is independent of the starting position $\theta_0$; that is, all planetesimals impart a constant angular momentum per unit mass to the planet, independent of their point of origin on the mean orbit of the gas ring, provided of course that the particle lies initially well within the sphere of influence of the embryo (condition 37). Again assuming that the density $\rho_E$ of the embryo is constant throughout accretion, so that $\theta_E \propto M_E^{\frac{1}{3}}$, we find that the resulting rotational period of the planet is

$$T = \frac{2\pi}{\omega_0}\left(1+\frac{5(1+\frac{3}{4}\gamma_2 R_E)}{\gamma_2^2 R_E^2}\right)^{-1} \approx 1200\,a^{-2}\ \text{h} \quad \text{(Earth's orbit)}, \tag{49}$$

where $a$ denotes the incoming particle radius, as in equation (8b), and we note that $\gamma_2 R_E \ll 1$. An assumed value for $a$ of 15 cm leads to a period of 5 h, whilst for $a = 10$ cm we find a period of 12 h; these would be the rotational periods if the early Earth formed from rocks of such size and a mean density $\rho_s$ of 3 g cm$^{-3}$. For Mercury the corresponding periods are found to be 90 and 170 h respectively. These latter periods are much longer owing to the higher density of the gas ring at Mercury's orbit, which increases the amount of drag, measured by $\gamma_2$. In fact in the case of very large drag ($\gamma_2 R_E \gg 1$), the accreting particles strike the planet normal to the surface leading to a rotational period equal to the orbital period $2\pi/\omega_0$ about the Sun. For Mercury the observed period does match the orbital one but this is due to the present tidal lock of the Sun. It is probably fair to say that the tidal locking of Mercury and possibly Venus could probably not have been achieved if these two planets had not been born with a very low rotational period. Such an initial state of affairs can be understood in terms of equation (49). We should further note that tidal action between the Earth and Moon has also appreciably lengthened the Earth's rotational period since the time of its formation. Finally, in deriving equation (49) we have assumed that the gas density $\rho_g$ and hence inverse drag length $\gamma_2$ were constant during the flight of the particle. In fact the planet or planetary embryo is likely to accrete a substantial atmosphere of gas during its formation so that $\gamma_2$ will increase sharply as the particle approaches the planet. This means that whilst our analysis for the initial stages of the trajectory is valid, the closing stage is probably far more complicated than we have assumed and is well beyond the scope of this paper especially in the case of the major planets. An increase in gas density as the particle approaches the planet is likely to reduce the relative velocity between these bodies and hence lead to a lower spin angular momentum during accretion.

## Conclusions

We have seen how gaseous drag can lead to the accretion of particles lying on the same mean orbit of a planetary embryo in the modified Laplacian model for the formation of the planetary and satellite systems. If the drag is sufficiently strong, accretion occurs for all particles lying well inside the sphere of gravitational influence of the embryo, whilst in the absence of the gas less than 1% of such particles are accreted. The gas serves to dampen out the Coriolis acceleration due to the sun, as seen in the frame of the embryo, which tends to pull the particle off the mean circular Keplerian orbit, thereby preventing accretion. We have also found that this process of accretion results in a spin angular momentum for the embryo which is prograde with, or in the same sense as, the orbital motion. The spin rate decreases with increased gaseous drag and is least for the innermost planets where the gas density is highest; that is, we can understand why Mercury and Venus should have a very low primordial spin in terms of this model. Nonetheless, the results presented here must still be regarded as being only approximate since they were obtained on the basis of linearizing the equations of motion and so are valid only in narrow regimes of the various parameters. In particular we assumed the gas density in each gaseous ring to be a constant and ignored the possible existence of a dense atmosphere of gas which would have surrounded the planetary embryo during the final stages of its growth. Fortunately the presence of such an atmosphere does not alter the conclusion regarding the likelihood of particle accretion since this depends only on the initial stage of the trajectory, far from the embryo. It does, however, appreciably alter the final stage of the orbit where the velocities are higher and the linear analysis is probably no longer valid. A far more detailed and ambitious program would be required to examine this latter stage beyond what has been presented here. Such a program is being undertaken by K. Hourigan (personal communication).

Finally it should be emphasized that at the start of this paper we explicitly set out to consider the case of accretion within a gaseous ring having an orbital angular velocity distribution of the form $\omega(s) = \omega(R_0)\, R_0^2/s^2$, where $\omega(R_0) = (GM/R_0^3)^{\frac{1}{2}} \equiv \omega_K$ is the circular Keplerian value at an orbital radius $R_0$, the same as that of the embryo. The results we have obtained, however, are in fact generally true for any angular velocity distribution which satisfies the central condition $\omega(R_0) = \omega_K$. The chosen distribution arises naturally in the theory of ring formation (Prentice 1978*b*). The importance of this distribution in the present work is that it automatically focuses fresh grain material, as well as wandering planetesimals, back onto the mean circular orbit of the growing embryo, where accretion may proceed in the manner described.

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