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Particle Motion in Longitudinal Waves. II* Superluminal and Luminal Waves

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Abstract

The motion of charged particles in superluminal and luminal longitudinal waves of arbitrary amplitude is considered in detail, including relativistic effects. In particular we discuss the ability of these waves to accelerate particles. Solutions for the particle orbits are given in both closed form and in terms of relevant expansions. The drift velocity of the particles, which describes the motion of the guiding centre, is identified. Two interesting effects are discovered: (i) the ability of large amplitude superluminal waves to drag particles along at a velocity conjugate to the wave phase speed and (ii) the existence of 'phase locking' particle orbits in the luminal case, in which particles can be accelerated to arbitrary energy.

1. Introduction

In this paper we extend the treatment of particle motion in longitudinal waves to include superluminal (phase speed $v_\phi > c$) and luminal ($v_\phi = c$) waves. Subluminal waves were treated in Part I (Rowe 1992; present issue p. 1). The motivations for the calculations presented here were also detailed in Part I.

Superluminal and luminal longitudinal waves are found in both non-relativistic and relativistic plasmas (e.g. Buti 1962; Silin 1960). In particular, the dispersion relation for longitudinal waves in a non-relativistic, isotropic, homogeneous, thermal plasma is $\omega^2 = \omega_p^2 + 3k^2V_e^2$ where ω_p is the plasma frequency and V_e is the thermal speed of the distribution. The phase speed is luminal or superluminal for $\omega \leq \omega_p/(1 - 3V_e^2/c^2)^{\frac{1}{2}}$, provided that $V_e^2 \leq c^2/3$. It is not possible for such waves to be damped by Cerenkov (or Landau) damping, for the resonance condition $\omega = \mathbf{k}\,.\,\mathbf{v}$ cannot be satisfied. Consequently it is necessary to treat the particle motion in such waves in order to determine the damping or growth due to higher order interactions with the wave (we do not treat the damping here). In the superluminal case there is a reference frame in which the electric field is time varying only, and the phase speed of the wave v_ϕ is infinite (we call this frame the uniform field frame since the electric field is independent of the space coordinate). In contrast, in the subluminal case there is a frame in which the electric field is independent of time and the wave phase speed is zero. We show in this paper that there are important differences between the behaviour of particles in superluminal waves and the behaviour of particles in subluminal

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waves. These differences are due to the different nature of the waves as just described and the fact that particle speeds can only be subluminal. Particle motion in luminal waves must be treated separately from both the subluminal and superluminal cases.

In Section 2 we consider the motion of a particle in a superluminal or luminal wave and in particular the energy gained by a particle during its motion. We include relativistic effects and treat the electric field exactly, but without the effects of radiation reaction. In the superluminal case, particles which are strongly accelerated by the wave drift at a velocity c/v_ϕ where v_ϕ is the wave phase speed. This effect is the analogue of particle trapping in the subluminal case. One also finds that a uniform distribution of particles injected into the wave becomes a distribution in drift momentum which is bunched at the value corresponding to a drift velocity of c/v_ϕ. This is important because it is the drift velocity that characterises the particle motion and thus its emission. In the luminal case particles with a high enough velocity at the minimum of the electric potential have orbits which are non-oscillatory. They experience an almost constant electric field and are accelerated indefinitely, as they become closer to being locked into a particular phase of the wave. This effect does not occur for motion in plane transverse waves.

Section 3 deals with exact solutions for the particle orbits from which the particle drift velocity is determined. The luminal case has three separate solutions, two of them non-oscillatory. We also consider the time taken by a particle to attain a given energy if it is injected into a 'phase locking' orbit in a luminal wave (these orbits are phase locking rather than phase locked since a particle can never truly be phase locked in a luminal wave). In Section 4 we develop two expansions of the oscillatory particle orbits. One is an expansion in harmonics of the wave motion, valid for all electric field stengths and the second is essentially an expansion in the ratio of wave to particle energy. The second expansion provides approximations to the coefficients in the first if the electric field is weak. We use natural units with $\hbar = c = 1$.

2. Particle Motion

(a) Particles in Superluminal Waves

The equation of motion for a particle moving in a longitudinal wave was given in Part I along with its first integral (equations 9 and 14 of Part I). From this one has

$$v = \frac{\gamma_\phi^{*2}/v_\phi + (b_\infty + r_\infty \cos\psi)\{(b_\infty + r_\infty \cos\psi)^2 + 1\}^{\frac{1}{2}}}{\gamma_\phi^{*2} + (b_\infty + r_\infty \cos\psi)^2}, \tag{1}$$

where

$$r_\infty = \frac{qE_0}{m\Omega_\infty} \tag{2}$$

is a dimensionless constant which determines the strength of the wave. In SI units $r_\infty = e(A_\infty c)/mc^2$ for an electron or positron where A_∞ is the amplitude of the vector potential of the electric field in the uniform field frame (in this case $\mathbf{E} = -\partial\mathbf{A}/\partial t$ and $\nabla\times\mathbf{A} = 0$). Electromagnetic theory implies that $A_\infty c \approx \phi_0$ (with ϕ_0 as defined in Part I), so that the strength parameter r_∞ of a superluminal wave is analogous to the strength parameter r_0 (defined in Part I) of a subluminal wave.

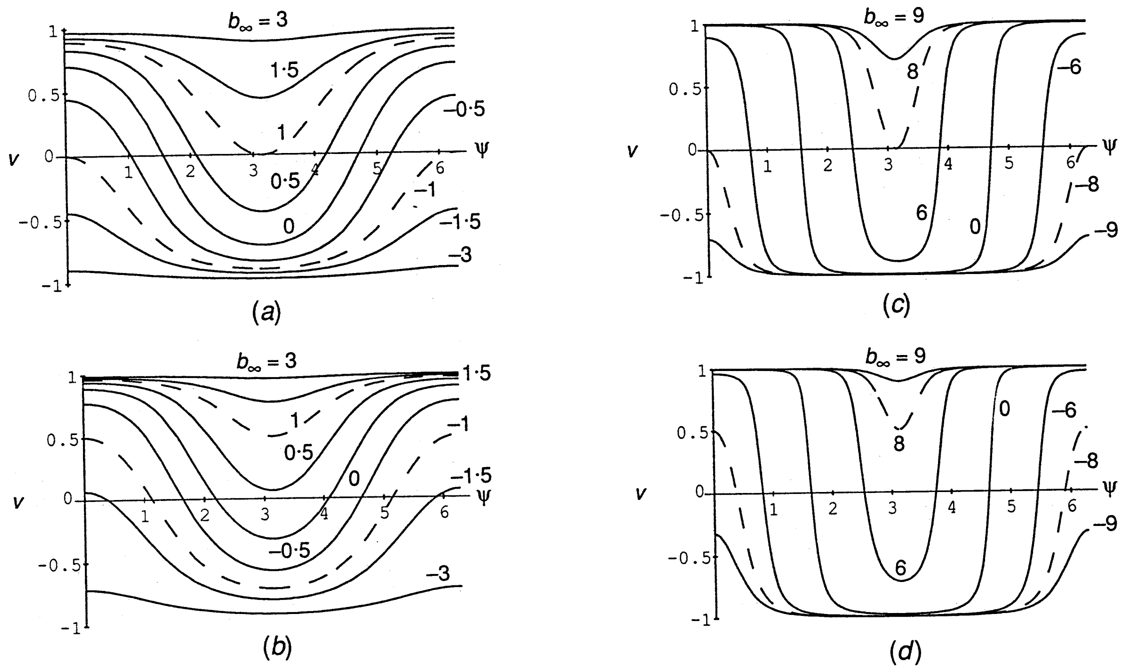

Fig. 1. Phase diagrams for particle motion in a superluminal longitudinal wave. In (a) and (c) v_ϕ is infinite and in (b) and (d) $v_\phi = 2$. The electric field strength is $r_\infty = 1$ in (a) and (b) and $r_\infty = 8$ in (c) and (d). Plotted is particle velocity as a function of wave phase (in radians) for various values of the particle canonical momentum in the uniform field frame, b_∞. Dashed lines represent $|b_\infty| = |r_\infty|$ and trajectories between these lines are those of particles which experience a strong electric field.

The solution (1) is valid for all values of ψ and b_∞ since no particle trapping is possible. The particle velocity is plotted as a function of phase ψ for various values of b_∞, r_∞ and v_ϕ in Fig. 1. Although trapping is not possible, the phase plots in the uniform field frame illustrate an analogous effect by which some particles have near zero average velocities (this is quantified in Section 3*a*). For particles with $b_\infty = 0$, there is no asymmetry in the motion with respect to the direction of the particle velocity in the uniform field frame and thus their average velocity is zero (note that ψ is essentially t in this frame). However, as $|b_\infty|$ increases so does the asymmetry in the motion and the average speed. If r_∞ is large this asymmetry is negligible unless $|b_\infty| > |r_\infty|$. Thus the essence of the effect is that for high electric fields more particles (if we assume a uniform distribution in b_∞) have near zero average velocity. In an arbitrary frame the average velocity for these particles is near $1/v_\phi$ (due to the nature of the Lorentz transform in the superluminal case) and thus these particles are effectively dragged by the wave at a velocity conjugate to the wave phase speed.

Table 1. Energy gain factor for a particle in a superluminal wave for various ranges of the particle canonical momentum in the uniform field frame

The second column describes the particle orbit. The results hold for all values of r_∞

Range of b_∞	Comments	$\gamma_{\max}/\gamma_{\min}$
$b_\infty < -r_\infty - \gamma_\phi^*/v_\phi$	$v < 0$	f_b/f_a
$-r_\infty - \gamma_\phi^*/v_\phi \le b_\infty \le -(\gamma_\phi^{*2} + r_\infty^2)^{\frac{1}{2}}/v_\phi$	$v < 0$ mostly	f_b
$-(\gamma_\phi^{*2} + r_\infty^2)^{\frac{1}{2}}/v_\phi \le b_\infty \le r_\infty - \gamma_\phi^*/v_\phi$	$v > 0$ mostly	f_a
$b_\infty \ge r_\infty - \gamma_\phi^*/v_\phi$	$v > 0$	f_a/f_b

We can calculate the energy gain of a particle in the wave, defined by $R = \gamma_{\max}/\gamma_{\min}$ where $\gamma_{\max}$ and $\gamma_{\min}$ are respectively the maximum and minimum γ-factors attained by the particle throughout its motion. In the uniform field frame, the momenta per unit mass achieved at the opposite ends of the orbit (i.e. $\psi = 0$ and $\psi = \pi$) are $p_a = b_\infty + r_\infty$ and $p_b = b_\infty - r_\infty$, though these do not necessarily correspond to the maximum and minimum values of particle kinetic energy per unit mass attained during the motion. After a Lorentz transform to a general frame these quantities correspond to gamma factors

$$f_a = \frac{\gamma_\phi^*}{v_\phi}\{p_a + v_\phi(1 + p_a^2)^{\frac{1}{2}}\}, \tag{3}$$

$$f_b = \frac{\gamma_\phi^*}{v_\phi}\{p_b + v_\phi(1 + p_b^2)^{\frac{1}{2}}\}, \tag{4}$$

where γ_ϕ^* was defined in Part I. The energy gain in the general frame involves f_a and f_b in various ways depending upon the value of the Lorentz boost (Table 1).

The energy gain R is depicted in Fig. 2 and its qualitative behaviour as a function of b_∞ is as follows. For $b_\infty \ll -r_\infty - \gamma_\phi^*/v_\phi$, R is close to unity. It increases with b_∞ until a local maximum is attained with b_∞ slightly less than $-r_\infty - \gamma_\phi^*/v_\phi$. This maximum gives the largest energy gain attainable for a particle which has negative velocity throughout its motion. The energy gain then decreases with b_∞ until $b_\infty = -(\gamma_\phi^{*2} + r_\infty^2)^{\frac{1}{2}}/v_\phi$ at which point particles

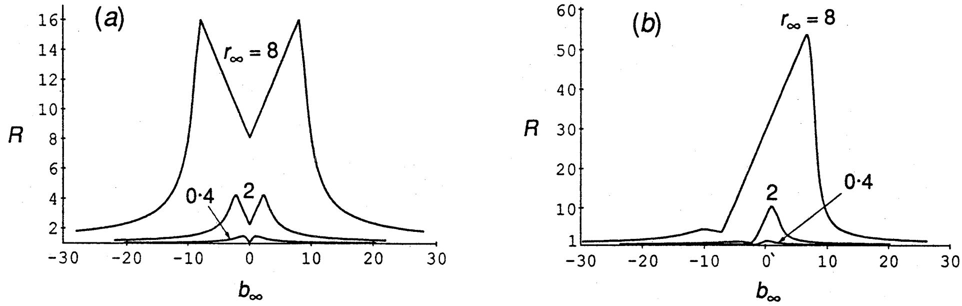

Fig. 2. Energy gain for particles in a superluminal wave as a function of the particle canonical momentum in the uniform field frame, b_∞. In (a) $v_\phi = \infty$ and in (b) $v_\phi^2 = \frac{4}{3}$.

attain a maximum velocity during their motion which is equal in magnitude to the minimum velocity. For b_∞ greater than this value, R is again an increasing function until another local maximum is reached for b_∞ slightly greater than $r_\infty - \gamma_\phi^*/v_\phi$. This is the maximum energy gain attainable by particles which have positive velocity throughout their motion. Increasing b_∞ still further decreases R rapidly to unity. Of most interest is the absolute maximum value for the energy gain. If v_ϕ is taken to be positive this maximum is the one obtained by particles with positive velocity in the uniform field frame. The closer v_ϕ is to unity the greater the ratio of the energy gain for positively moving particles to the energy gain for negatively moving particles.

The absolute maximum gain is

$$R_{\max} = \frac{b_\infty + r_\infty + v_\phi\{1 + (b_\infty + r_\infty)^2\}^{\frac{1}{2}}}{b_\infty - r_\infty + v_\phi\{1 + (b_\infty - r_\infty)^2\}^{\frac{1}{2}}}\,, \tag{5}$$

where b_∞ is the single solution of

$$\begin{aligned} 0 = &\left[1 + \frac{v_\phi(b_\infty + r_\infty)}{\{1 + (b_\infty + r_\infty)^2\}^{\frac{1}{2}}}\right][b_\infty - r_\infty + v_\phi\{1 + (b_\infty - r_\infty)^2\}^{\frac{1}{2}}] \\ &-\left[1 + \frac{v_\phi(b_\infty - r_\infty)}{\{1 + (b_\infty - r_\infty)^2\}^{\frac{1}{2}}}\right][b_\infty + r_\infty + v_\phi\{1 + (b_\infty + r_\infty)^2\}^{\frac{1}{2}}]\,, \end{aligned} \tag{6}$$

greater than $r_\infty - \gamma_\phi^*/v_\phi$ (as an example, for nominal values of $r_\infty = 20$ and $1/v_\phi = 0{\cdot}99$, $R_{\max} \approx 500$ at $b_\infty \approx 14$). For $v_\phi = \infty$, the solution to (6) is $b_\infty = (1 + r_\infty^2)^{\frac{1}{2}}$. The maximum energy gain is shown in Fig. 3, together with the solution to (6). For large r_∞ the curves for b_∞ approximate straight lines with slope decreasing with $1/v_\phi$ while the curves for $R_{\max}$ become straight lines with slope increasing with $1/v_\phi$. On physical grounds it is clear that $R_{\max}$ increases with both r_∞ and $1/v_\phi$. It increases with the electric field strength r_∞ because particles are accelerated to higher velocity if the field is stronger. The maximum energy gain $R_{\max}$ also increases with $1/v_\phi$ because particle velocities are shifted with respect to velocities in the uniform field frame (zero particle velocity in the uniform field frame corresponds to $1/v_\phi$ in the observer's frame). Any quantity defined by a ratio of two gamma factors is naturally increased if

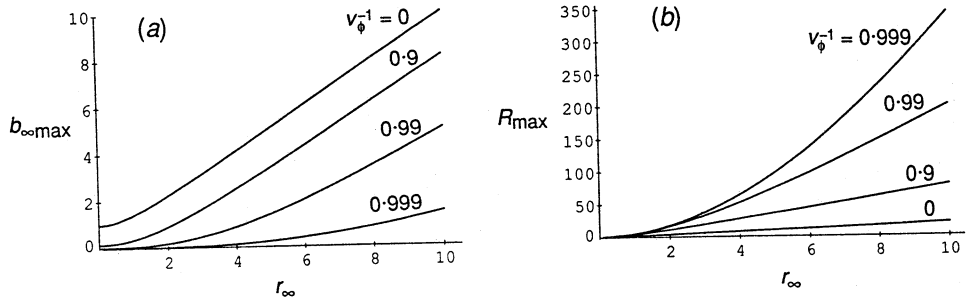

Fig. 3. Maximum energy gain for particles in a superluminal wave (b) together with the corresponding particle injection momentum (a) as functions of the wave strength and for four values of wave phase speed.

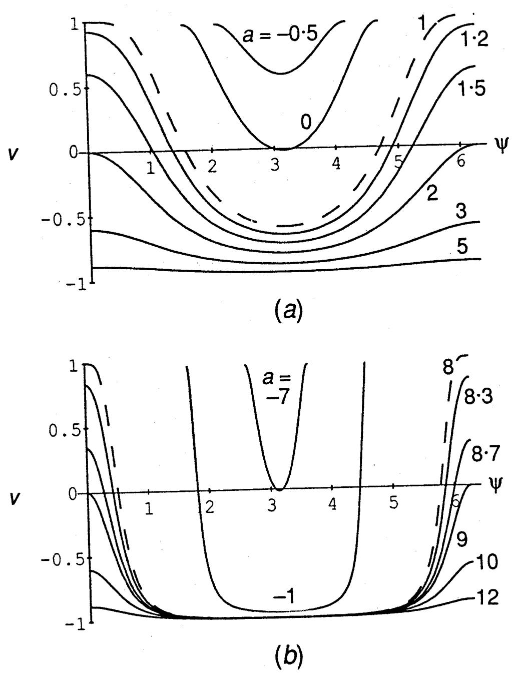

Fig. 4. Phase diagrams for particle motion in a luminal longitudinal wave. In (a) $r = 1$ and in (b) $r = 8$. Plotted is particle velocity as a function of wave phase (in radians) for various values of a. Dashed lines are boundaries between oscillatory and non-oscillatory motion.

a Lorentz transform is made to a frame in which the gamma factors are both larger. The maximum energy gain for particles in a superluminal wave is of the same order of magnitude as for trapped particles in a subluminal wave if $r_0 \approx r_\infty$ and $\gamma_\phi \approx \gamma_\phi^*$.

(b) Particles in Luminal Waves

The velocity of a particle under the influence of a wave with phase speed unity is, from equation (16) of Part I,

$$v = \frac{1-(a-r\cos\psi)^2}{1+(a-r\cos\psi)^2}, \tag{7}$$

where

$$r = \frac{qE_0}{m\Omega} \tag{8}$$

is a dimensionless strength parameter for the wave, taking the same values as r_0 and r_∞ in the subluminal and superluminal cases respectively. The following points are of interest: (i) if $a > r$ the particle velocity is physical for all values of ψ; (ii) if $a \leq r$ the particle velocity becomes unphysical for $\cos\psi \geq a/r$; (iii) if $a < -r$ the particle velocity is always unphysical. The domain $|a| \leq r$ represents particles somewhat analogous to the particles trapped in a subluminal wave or those with $|b_\infty| \leq |r_\infty|$ in a superluminal wave. The fact that the particle velocity becomes unphysical if ψ passes a particular value suggests that ψ is bounded. Equation (5) of Part I shows that $d\psi/dt$ approaches zero as the particle velocity approaches unity while the equation of motion (equation 2 of Part I) verifies that the particle acceleration decreases to zero simultaneously. The particle thus accelerates indefinitely to the speed of light as it approaches a phase $\psi_c = \cos^{-1}(a/r)$, its acceleration decreasing as it does so. This 'phase locking' particle motion has no counterpart in the case of particle motion in a plane transverse wave. The particle velocity is shown in Fig. 4 as a function of ψ for various values of a and r.

Table 2. Energy gain factor for a particle in an oscillatory orbit in a luminal longitudinal wave

The first two columns give the range of r and a for which the result in the last column is valid and the third column describes the particle orbit

Range of r	Range of a	Comments	$\gamma_{\max}/\gamma_{\min}$
All values	$(1+r^2)^{\frac{1}{2}} \leq a \leq 1+r$	$v<0$ mostly	γ_a
	$a \geq 1+r$	$v<0$	γ_a/γ_b
$0 \leq r \leq \frac{1}{2}$	$r < a \leq 1-r$	$v>0$	γ_b/γ_a
	$1-r \leq a \leq (1+r^2)^{\frac{1}{2}}$	$v>0$ mostly	γ_b
$r > \frac{1}{2}$	$r \leq a \leq (1+r^2)^{\frac{1}{2}}$	$v>0$ mostly	γ_b

Let us now consider the energy gain for particles in oscillatory orbits in a luminal wave (i.e. $a > r$). The particle velocity is zero if $\cos\psi = (a-1)/r$ and this is only possible if $1-r \leq a \leq 1+r$. If $a > 1+r$ then the particle velocity is always negative, whereas if $a < 1-r$ it is always positive. The gamma factor of the particle at any point of the orbit is

$$\gamma = \frac{1+(a-r\cos\psi)^2}{2(a-r\cos\psi)}, \tag{9}$$

and at either end of the orbit one has

$$\gamma_a = \frac{1+(a+r)^2}{2(a+r)}, \tag{10}$$

$$\gamma_b = \frac{1+(a-r)^2}{2(a-r)}. \tag{11}$$

The energy gain involves either or both of γ_a and γ_b depending upon whether the particle velocity goes through zero or not at some value of ψ (Table 2).

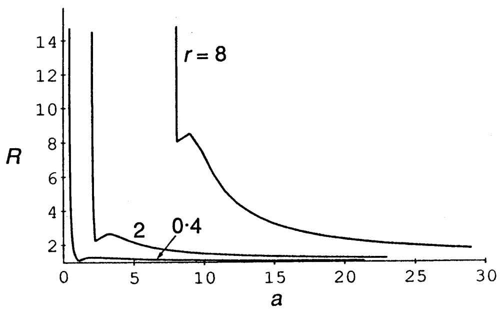

Fig. 5. Energy gain for non-phase locking particles in a luminal wave as a function of the constant of the motion a for three electric field strengths.

The energy gain is shown in Fig. 5 and its main features are as follows. Values of a arbitrarily close to r give arbitrarily large gain as particles are accelerated arbitrarily close to the speed of light. The gain drops as a is increased up to $a = (1+r^2)^{\frac{1}{2}}$. Beyond this value particles are moving predominantly in the negative direction and this results in an increase in the energy gain. There is a maximum for negatively moving particles at $a = \{2+r^2+(5+4r^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$ with

$$R_{\max} = \frac{(a-r)\,\{1+(a+r)^2\}}{(a+r)\,\{1+(a-r)^2\}}, \tag{12}$$

which is an increasing function of r (as an example, for $r = 20$, $R_{\max} \approx 20$ at $a \approx 21$). For larger values of a, R decreases towards unity. For particles in non-oscillatory orbits it is not possible to define an energy gain as for oscillatory orbits because the particle energy is unbounded. Instead let us write down the phase of the wave at which a particle has a given γ-factor;

$$\cos\psi = \frac{a-\gamma+(\gamma^2-1)^{\frac{1}{2}}}{r}. \tag{13}$$

The time taken by a particle in a non-oscillatory orbit to accelerate to a given energy is calculated in Section 3b below, with the aid of (13).

3. Exact Solutions for Particle Orbits

(a) Particles in Superluminal Waves

In the uniform field frame the equation of motion (equation 6 of Part I) reduces to

$$\frac{dz}{d\psi} = \frac{1}{\Omega_\infty}\frac{b_\infty + r_\infty \cos\psi}{\{(b_\infty + r_\infty \cos\psi)^2 + 1\}^{\frac{1}{2}}}\,. \tag{14}$$

As in the subluminal untrapped case, treated in Part I, a change of variable puts the equation into the standard elliptic integral form. The sign of the jacobian must be chosen differently for $\sin\psi$ positive and for $\sin\psi$ negative giving a piecewise solution. We define the following parameters

$$\zeta_\pm = r_\infty^2 \pm h_\infty^2\,, \tag{15}$$

$$\Delta_\infty = (\zeta_-^2 + 4r_\infty^2)^{\frac{1}{2}}\,, \tag{16}$$

$$\alpha_\infty = \left(\frac{2r_\infty^2}{\zeta_+ + \Delta_\infty}\right)^{\frac{1}{2}}, \tag{17}$$

$$k_\infty = \left(\frac{\zeta_- + \Delta_\infty}{2\Delta_\infty}\right)^{\frac{1}{2}}, \tag{18}$$

$$\eta_\infty = \frac{1}{2b_\infty\alpha_\infty}\frac{\zeta_+ - \Delta_\infty + 2b_\infty r_\infty \cos\psi}{\{(r_\infty \cos\psi + b_\infty)^2 + 1\}^{\frac{1}{2}}}\,, \tag{19}$$

$$p_\infty = \frac{2r_\infty^2 \cos^2\psi + 2b_\infty r_\infty \cos\psi - \zeta_-}{2b_\infty r_\infty \cos\psi + \zeta_+}\,, \tag{20}$$

with $h_\infty = (b_\infty^2 + 1)^{\frac{1}{2}}$, and the functions

$$\Pi_\infty(\eta_\infty) = \Pi(\alpha_\infty^2, k_\infty) \mp \Pi(\sin^{-1}\eta_\infty, \alpha_\infty^2, k_\infty)\,, \tag{21}$$

$$S_\infty(p_\infty) = \mp\left(\sin^{-1} p_\infty - \frac{\pi}{2}\right), \tag{22}$$

with the upper sign for $\psi \in [2N\pi, (2N+1)\pi]$ and the lower sign for $\psi \in [(2N+1)\pi, 2(N+1)\pi]$. The subscript infinity is used to indicate the superluminal wave case.

The particle orbit is

$$z_\infty = \frac{N\pi\beta_{D\infty}}{\Omega_\infty} + \frac{1}{\Omega_\infty}\left\{\frac{\Delta_\infty - \zeta_- - 2}{2b_\infty(\Delta_\infty)^{\frac{1}{2}}}\Pi_\infty(\eta_\infty) + \tfrac{1}{2}S_\infty(p_\infty)\right\} - \bar{z}_\infty\,, \tag{23}$$

where the integer N is the number of *half* periods already completed by the particle, $\bar{z}_\infty$ is an arbitrary constant and $\beta_{D\infty}$ is interpreted as the particle drift velocity (defined by $\beta_{D\infty} = \Delta z/\Delta t$ where Δz is the distance travelled in an integer multiple of half periods Δt). A Lorentz transform yields the result in an arbitrary frame

$$z = \gamma_\phi^*\left(z_\infty + \frac{\gamma_\phi^*}{v_\phi\Omega}\psi\right). \tag{24}$$

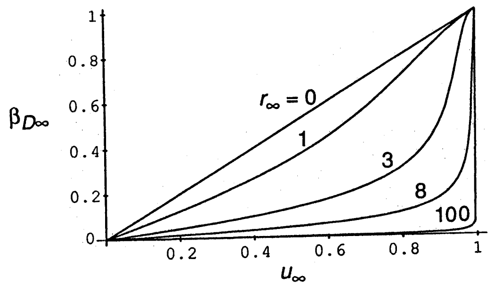

Fig. 6. Drift velocity of a particle in a superluminal wave shown as a function of zero order velocity u_∞ for five wave strengths.

The drift velocity, corresponding to the quantity β_{D0} in the subluminal case (Part I), is

$$\beta_{D\infty} = \frac{\Delta_\infty - \zeta_- - 2}{\pi b_\infty (\Delta_\infty)^{\frac{1}{2}}} \Pi_\infty(\alpha_\infty^2, k_\infty)\,, \tag{25}$$

which is shown in Fig. 6 as a function of the velocity u_∞ (related in the natural way to the momentum b_∞) for various values of r_∞. The dragging effect described in Section 2a is obvious in the strong field cases in which the drift velocity can remain quite low for large values of u_∞.

(b) Particles in Luminal Waves

The equation of motion in this case is equation (6) of Part I with $\Omega = K$ and the particle velocity is given by (7). One has

$$\frac{dz}{d\psi} = \frac{1}{2K}\left\{\frac{1}{(a - r\cos\psi)^2} - 1\right\}. \tag{26}$$

The integral in this case can be done with the aid of equations (2.554.3), (2.553.3) and (2.555.4) of Gradshteyn and Ryzhik (1980). There are three separate solutions for the cases $a > r$, $a < r$ and $a = r$. Defining the function

$$\begin{aligned} Z(\psi) &= \frac{1}{a^2 - r^2}\left[\frac{r\sin\psi}{a - r\cos\psi} + \frac{2a}{(a^2 - r^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{(a^2 - r^2)^{\frac{1}{2}}\tan\psi/2}{a - r}\right\}\right] \\ &\quad + \begin{cases} 0 & \psi \in [2N\pi,\ (2N+1)\pi] \\ \dfrac{\pi a}{(a^2 - r^2)^{\frac{3}{2}}} & \psi \in [(2N+1)\pi,\ 2(N+1)\pi]\,, \end{cases} \end{aligned} \tag{27}$$

one has the oscillatory solution in the $a > r$ case

$$z(\psi) = \frac{1}{2K}\left\{N\pi\frac{1 + \beta_{Dc}}{1 - \beta_{Dc}} + Z(\psi) - \psi\right\} + \bar{z}\,, \tag{28}$$

where N is the number of completed half periods, $\bar{z}$ is an arbitrary constant and β_{Dc} is the particle drift velocity. The last term is included in the definition of $Z(\psi)$ so that $Z(\psi)$ is continuous at $\psi = N\pi$. As in the subluminal and superluminal cases one can calculate β_{Dc} exactly by taking $\Delta z/\Delta t$ where Δz is the distance travelled after any integral multiple of a half period and Δt is the time taken for that part of the particle orbit. One then has

$$\beta_{Dc} = \frac{a-(a^2-r^2)^{\frac{3}{2}}}{a+(a^2-r^2)^{\frac{3}{2}}}\,. \tag{29}$$

The drift velocity is shown in Fig. 7 as a function of the injection velocity u_c [related to the constant of the motion a through the equation $a=\gamma_c(1-u_c)$] for various values of r. The value of a for which β_{Dc} is zero is given by

$$a^2 = \begin{cases} \left(\frac{4}{3}\right)^{\frac{1}{2}}\cos\frac{\theta}{3}+r^2 & r^4\leq\frac{4}{27} \\ \left\{\frac{r^2}{2}+\left(\frac{r^4}{4}-\frac{1}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}+\left\{\frac{r^2}{2}+\left(\frac{r^4}{4}-\frac{1}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}+r^2 & r^4\geq\frac{4}{27}\,, \end{cases} \tag{30}$$

where $\theta=\cos^{-1}\{(27)^{\frac{1}{2}}r^2/2\}$. For very low r, the drift velocity is equal to u_c and increasing the electric field decreases the value of u_c for which the particle can attain drift velocities close to the velocity of light.

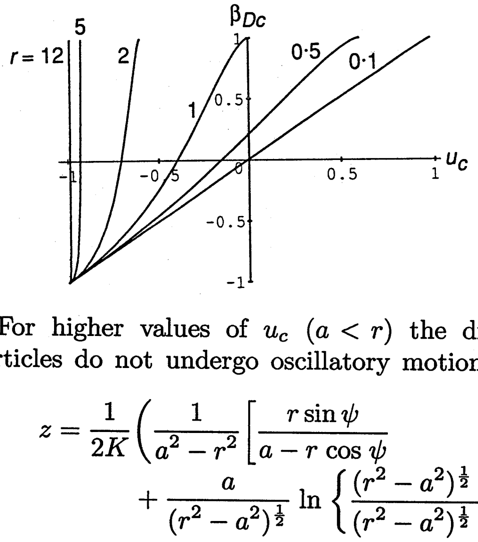

Fig. 7. Drift velocity of a particle in a luminal wave shown as a function of zero order velocity u_c for six wave strengths.

For higher values of u_c $(a<r)$ the drift velocity is not valid since these particles do not undergo oscillatory motion. The orbit for these particles is

$$z=\frac{1}{2K}\left(\frac{1}{a^2-r^2}\left[\frac{r\sin\psi}{a-r\cos\psi}\right.\right.$$
$$\left.\left.+\frac{a}{(r^2-a^2)^{\frac{1}{2}}}\ln\left\{\frac{(r^2-a^2)^{\frac{1}{2}}\tan(\psi/2)+a-r}{(r^2-a^2)^{\frac{1}{2}}\tan(\psi/2)-a+r}\right\}\right]-\psi\right)+\bar{z}\,, \tag{31}$$

which has a singularity at $\cos\psi=a/r$ due to the first term in square brackets. This solution is well defined at $\psi=\pi$ due to the presence of $\tan(\psi/2)$ in both the numerator and denominator of the fraction appearing in the logarithm. Particles with this orbit either take an infinite time to accelerate from minus unity and move away from $\psi = \cos^{-1}(a/r)$ or to accelerate to unity and move towards $\psi=2\pi-\cos^{-1}(a/r)$. A third solution (valid for $a=r$) is also non-oscillatory,

$$z=\frac{1}{2K}\left\{-\frac{1}{2r^2}\tan\left(\frac{\pi}{2}-\frac{\psi}{2}\right)-\frac{1}{6r^2}\tan^3\left(\frac{\pi}{2}-\frac{\psi}{2}\right)-\psi\right\}+\bar{z}\,, \tag{32}$$

and particles with this solution behave in much the same way as those with the previous solution.

The orbit (31) can be used to calculate the time taken by a particle in a non-oscillatory orbit to attain a given energy. The definition of phase, $\psi = \Omega t - Kz$, together with (31) gives time as a function of phase. Equation (13) implies that for $\gamma \gg 1$, $a - r \cos \psi \approx (2\gamma)^{-1}$. If ψ is sufficiently close to ψ_c, the first term in square brackets in (31) dominates and one finds

$$t \approx \frac{1}{2K}\left(\frac{1}{a^2 - r^2}\frac{r \sin \psi}{a - r \cos \psi} + \psi\right). \tag{33}$$

One then has

$$\gamma \approx 2\pi(r^2 - a^2)^{\frac{1}{2}}\left(\frac{t}{T}\right) + \gamma_0 , \tag{34}$$

for particles which gain energy from the wave, where T is the period of the wave and γ_0 is the energy at $t = 0$ (the sign of the first term is negative for particles which give up energy to the wave). Thus after an initial period of time, during which ψ is near a value of π, the particle gains energy linearly with time (for example, a particle with $a = 0$ has energy $\gamma = 2\pi|r| + \gamma_0$ after one wave period). An important point is that both electrons and positrons are accelerated in the same direction, at different phases of the wave. The energy gained after one wave period increases approximately linearly with $|r|$ and decreases with $|a|$.

4. Orbit Expansions

(a) Particles in Superluminal Waves

In the uniform field frame expanding the orbit in a Fourier series gives

$$z_\infty = \sum_{p=1}^{\infty} S_p \sin(p\psi) + \frac{\beta_{D\infty}\psi}{\Omega_\infty} + \bar{z}_\infty , \tag{35}$$

where the S_p are constants to be determined and $\bar{z}_\infty$ is an arbitrary constant. Differentiating (35) leads to

$$\frac{dz_\infty}{d\psi} = \sum_{p=1}^{\infty} pS_p \cos(p\psi) + \frac{\beta_{D\infty}}{\Omega_\infty} , \tag{36}$$

and the left hand side is determined by equation (6) of Part I. The order of magnitude of the S_p varies with b_∞ and r_∞ (however S_1 is the dominant coefficient since it corresponds to a term with the same period as that of the particle motion). The coefficients S_p and $\beta_{D\infty}$ are

$$S_p = \frac{2}{p\pi\Omega_\infty}\int_0^\pi \frac{b_\infty + r_\infty \cos \psi}{\{(b_\infty + r_\infty \cos \psi)^2 + 1\}^{\frac{1}{2}}} \cos(p\psi) d\psi , \tag{37}$$

$$\beta_{D\infty} = \frac{1}{\pi}\int_0^\pi \frac{b_\infty + r_\infty \cos \psi}{\{(b_\infty + r_\infty \cos \psi)^2 + 1\}^{\frac{1}{2}}} d\psi , \tag{38}$$

with, for example

$$S_1 = -\frac{2}{\pi\Omega_\infty}\frac{\Delta_\infty - \zeta_-}{r_\infty(\Delta_\infty)^{\frac{1}{2}}}\left\{K(k_\infty) + \frac{2\Delta_\infty}{\zeta_- - \Delta_\infty}E(k_\infty)\right\}. \tag{39}$$

An appropriate Lorentz transform (see Appendix 1) yields the result in an arbitrary frame

$$z = \frac{\psi\beta_D}{K(v_\phi - \beta_D)} + \sum_{p=1}^{\infty}\gamma_\phi^* S_p \sin(p\psi) + \gamma_\phi^{*2}(\bar{z} - \bar{t}/v_\phi)\,, \tag{40}$$

where $\bar{z}$ and $\bar{t}$ are arbitrary constants.

For very strong electric fields, $r_\infty \gg b_\infty$, one can approximate the value of S_p by taking the particle velocity in the uniform field frame to be

$$v = \begin{cases} 1 & 0 < \psi < \pi/2 \\ -1 & \pi/2 < \psi < \pi\,, \end{cases} \tag{41}$$

and then one has

$$S_p \approx \frac{4\sin(p\pi/2)}{p^2\pi\Omega}\,, \tag{42}$$

and $\beta_{D\infty} \approx 0$.

In the opposite limit, $r_\infty < b_\infty$, one can perform an alternative expansion of (14) as in the subluminal case in Part I. To third order in r_∞/b_∞ one has

$$\begin{aligned} z = -\frac{\gamma_\phi^* u_\infty^3}{\Omega_\infty b_\infty^2}\left(\frac{r_\infty}{b_\infty}\right)\Bigg[&\left\{1 + \frac{3u_\infty^2}{8}(5u_\infty^2 - 1)\left(\frac{r_\infty}{b_\infty}\right)^2\right\}\sin\psi - \frac{3u_\infty^2}{8}\left(\frac{r_\infty}{b_\infty}\right)\sin 2\psi \\ &+ \frac{u_\infty^2}{24}(5u_\infty^2 - 1)\left(\frac{r_\infty}{b_\infty}\right)^2 \sin 3\psi\Bigg] + \frac{\psi\beta_D}{K(v_\phi - \beta_D)} + \text{const}\,, \end{aligned} \tag{43}$$

and the drift velocity is

$$\beta_{D\infty} = u_\infty\left\{1 - \frac{3u_\infty^4}{4b_\infty^2}\left(\frac{r_\infty}{b_\infty}\right)^2\right\}. \tag{44}$$

It is interesting to note that the second order contribution to the drift velocity in the superluminal case is smaller than the contribution in the subluminal case, by a factor of u_∞^4 and this reflects the fact that for weak fields superluminal waves have less effect on particles than subluminal waves since the particle velocity cannot be arbitrarily close to the wave velocity. This of course presupposes that the particle has a high enough velocity to be untrapped in the subluminal case. The more interesting results of this paper (e.g. low drift velocity of particles in a strong superluminal wave) cannot be treated by such approximations. Full expansions in the strong and weak field limits are discussed in Appendix 2.

In the strong field case, $S_p \approx p^{-2}$ for p odd and is zero otherwise. This dependence on p is stronger than in the corresponding subluminal case and thus fewer terms need to be retained in the strong field expansion for the superluminal wave case. It is shown in Appendix 2 that in the weak field case, $S_p \approx$ const $^{-p}$.

(b) Particles in Luminal Waves

An expansion of the particle orbits in the luminal case is restricted to the oscillating orbits ($a > r$) which we can expand in a Fourier series

$$z = \sum_{p=1}^{\infty} Z_p \sin(p\psi) + \frac{\beta_{Dc}}{1-\beta_{Dc}} \frac{\psi}{K} + \bar{z}, \tag{45}$$

where the Z_p are constants and $\bar{z}$ is an arbitrary constant. The form of the linear term in ψ is such that defining Δz and Δt as the change in z and t respectively over an integral multiple of half periods of the motion gives $\Delta z/\Delta t = \beta_{Dc}$. Differentiating (45) gives

$$\frac{dz}{d\psi} = \sum_{p=1}^{\infty} pZ_p \cos(p\psi) + \frac{\beta_{Dc}}{1-\beta_{Dc}} \frac{1}{K}, \tag{46}$$

with

$$Z_p = \frac{1}{p\pi K} \int_0^{\pi} \left\{ \frac{1}{(a - r\cos\psi)^2} - 1 \right\} \cos(p\psi) d\psi, \tag{47}$$

$$\frac{\beta_{Dc}}{1-\beta_{Dc}} = \frac{1}{2\pi} \int_0^{\pi} \left\{ \frac{1}{(a - r\cos\psi)^2} - 1 \right\} d\psi. \tag{48}$$

The second integral is effectively given by (29) and the remaining ones involve elementary functions. The first two constants Z_1 and Z_2 are

$$Z_1 = \frac{1}{K} \frac{r}{(a^2 - r^2)^{\frac{3}{2}}}, \tag{49}$$

$$Z_2 = \frac{1}{2r^2 K} \left\{ 2 + \frac{a(3r^2 - 2a^2)}{(a^2 - r^2)^{\frac{3}{2}}} \right\}. \tag{50}$$

All of the constants in this case have singularities at $a = r$ when the second non-oscillatory solution is valid. For small $z = a/r - 1$, the Z_p and $\beta_{Dc}/(1-\beta_{Dc})$ vary as

$$Z_p \approx \frac{1}{pKr^2} \frac{1}{(2z)^{\frac{3}{2}}}, \tag{51}$$

$$\frac{\beta_{Dc}}{1-\beta_{Dc}} \approx \frac{1}{2r^2} \frac{1}{(2z)^{\frac{3}{2}}}. \tag{52}$$

An expansion of the orbit in powers of r/a is possible in the weak field case and this is detailed in Appendix 2. To third order in r/a one has

$$z = \frac{1}{Ka^2}\left(\frac{r}{a}\right)\left[\left\{1+\tfrac{3}{2}\left(\frac{r}{a}\right)^2\right\}\sin\psi + \tfrac{3}{8}\left(\frac{r}{a}\right)\sin 2\psi + \tfrac{1}{6}\left(\frac{r}{a}\right)^2\sin 3\psi\right] + \frac{\psi\beta_{Dc}}{K(1-\beta_{Dc})} + \text{const}, \tag{53}$$

$$\frac{\beta_{Dc}}{1-\beta_{Dc}} = \frac{1}{2a^2}\left\{1-a^2+\tfrac{3}{2}\left(\frac{r}{a}\right)^2\right\}. \tag{54}$$

If the electric field is strong, the Z_p vary as p^{-1} and if the electric field is weak they vary as const^{-p}.

5. Conclusions

In this paper the treatment of the motion of a particle in a longitudinal wave is extended to the case of superluminal and luminal waves. Particle orbits are given in both closed and expanded forms. The results given here and in Part I are used extensively in the treatment of emission by particles in longitudinal waves which is to be given in a later paper. The treatment of the orbit presented here is important in that no approximation is made and so emission from particles in very strong plasma waves can be explored.

The wave strength for a subluminal wave r_0 in the case of pulsars was briefly considered in Part I where it was found that it is between unity and 10^6. The parameter r_∞, defined in Section 2, takes similar values. The development of a large amplitude coherent plasma wave in a pulsar magnetosphere, probably during the breakdown of the polar gap, needs to be considered in detail if this range of values is to be narrowed.

Two major results of the work presented here are as follows. The first involves the superluminal wave case. A distribution of particles which is uniform in particle momentum before injection into the wave becomes a non-uniform distribution in drift momentum after injection. The peak occurs at a drift momentum corresponding to a particle drift velocity of $1/v_\phi$, where v_ϕ is the wave phase speed (particles tend to be dragged along at a drift velocity of $1/v_\phi$). This effect occurs because particles with low injection momentum in the uniform field frame b_∞ are accelerated to approximately the same speed in either direction, during any wave cycle. If the electric field is strong, particles are accelerated to essentially the speed of light in either direction (unless they have highly relativistic injection momenta). The average or drift velocity is thus very close to zero compared with the injection velocity. For strong electric fields the distribution of particles in drift momentum is relevant (drift momentum is a constant of the motion) and thus this result has consequences as far as the absorption or growth of radiation is concerned. The second result concerns the luminal wave case. The existence of non-oscillatory particle orbits in which particles gain energy indefinitely (assuming the wave amplitude is held constant) provides a plasma version of a linear accelerator which accelerates positively and negatively charged particles in the same direction. These 'phase locking' orbits describe particles which slowly 'catch up' with the wave. Such particles are continuously accelerated in the wave direction as they gradually approach a critical wave phase. The acceleration of the particle is such that the approach to the critical phase takes an infinitely long time, during which particle energy increases without bound. Particles in these orbits experience an essentially constant electric field. These results may have interesting astrophysical implications. The maximum energy that a particle can attain depends primarily upon the time spent in the wave field. This time can be affected by various factors such as the angle between the particle velocity and the wave propagation (in the case assumed here, this angle is zero), collisions of particles with one another and the growth or decay of the wave (the acceleration of particles itself causes the wave to decay). We do not go into these details here.

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Appendix 1: Lorentz Transformations

Consider a Lorentz transform from the uniform field frame $\mathcal{K}_\infty$ to an arbitrary reference frame $\mathcal{K}$. Let quantities defined in $\mathcal{K}_\infty$ have subscript ∞ (except E_0 which is the same in all frames moving parallel to $\mathcal{K}_\infty$). The main quantities are $E_0, \Omega_\infty, v_\infty, \psi_\infty, \gamma_\infty, r_\infty$ and $\beta_{D\infty}$ where E_0 is an electric field, Ω_∞ is a frequency, v_∞ and $\beta_{D\infty}$ are velocities, ψ_∞ is a phase, γ_∞ is an energy and r_∞ is a dimensionless parameter defined in Section 2*a*. The wavevector and frequency in an arbitrary frame moving parallel to E_0 are

$$K = \gamma_R(K_\infty - v_R\Omega_\infty), \qquad \Omega = \gamma_R(\Omega_\infty - v_R K_\infty)\,,$$

where v_R and γ_R are the relative velocity and the corresponding γ-factor, respectively, between $\mathcal{K}$ and $\mathcal{K}_\infty$. By definition, $K_\infty = 0$ and thus $v_R = -1/v_\phi$ where $v_\phi = \Omega/K$ is the wave phase speed in $\mathcal{K}$. We adopt the notation γ_ϕ^* for $\gamma_R = 1/(1 - 1/v_\phi^2)^{\frac{1}{2}}$.

Using the inverse transform one finds the quantities in $\mathcal{K}_\infty$ in terms of those in $\mathcal{K}$:

$$E_0 = E\,, \tag{A1}$$

$$\psi_\infty = \psi\,, \tag{A2}$$

$$\Omega_\infty = \gamma_\phi^*(\Omega - K/v_\phi) = \Omega/\gamma_\phi^*\,, \tag{A3}$$

$$v_\infty = \frac{v - 1/v_\phi}{1 - v/v_\phi}\,, \tag{A4}$$

$$\beta_{D\infty} = \frac{\beta_D - 1/v_\phi}{1 - \beta_D/v_\phi}\,, \tag{A5}$$

$$\gamma_\infty = \gamma_\phi^* \gamma (1 - v/v_\phi), \tag{A6}$$

$$r_\infty = \gamma_\phi^* r. \tag{A7}$$

Appendix 2: Complete Strong and Weak Field Expansions

(a) Superluminal Case

The strong field expansion is developed as follows. Defining

$$I(y) = \frac{(y+\cos\psi)r_\infty}{\{(y+\cos\psi)^2 r_\infty^2 + 1\}^{\frac{1}{2}}}, \tag{A8}$$

one has

$$S_p = \frac{2}{p\pi\Omega_\infty} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{b_\infty}{r_\infty}\right)^n \int_0^\pi I^{(n)}(0) \cos(p\psi) d\psi, \tag{A9}$$

where

$$I^{(l)}(0) = \frac{(l+1)!}{2^l} \sum_{n=[l/2]}^{l} G_{l,n} \frac{(\cos\psi)^{2n-l+1}}{\{1/r_\infty^2 + \cos^2\psi\}^{n+\frac{1}{2}}}, \tag{A10}$$

with $G_{l,n}$ defined by (A11) of Part I. The coefficients in the expansion are sums of elliptic integrals.

The weak field expansion of S_p in r_∞/b_∞ is obtained from the subluminal case (Part I, Appendix 2) by means of the following substitutions: $\pm K_0 \to \Omega_\infty$, $r_0/h_0 \to -r_\infty/b_\infty$, $C_p \to -S_p$, $u_0 \to 1/u_\infty$. In the weak field limit then, $S_p \approx$ const $^{-p}$ as in the subluminal case.

(b) Luminal Case

In the luminal case only a weak field expansion can be found. Defining

$$\mathcal{Z}(y) = \frac{1}{2a^2(1-y)^2} - \tfrac{1}{2}, \tag{A11}$$

one finds that Z_p and $\beta_{Dc}/(1-\beta_{Dc})$ are given by the expansions for C_p and $1/\beta_{D0}$ (A9 and A12 of Part I) with the following replacements: $K_0 \to K$, $r_0/h_0 \to r/a$ and $F^{(l)}(0) \to \mathcal{Z}^{(l)}(0)$ where

$$\mathcal{Z}^{(l)}(0) = \frac{(l+1)!}{2a^2} - \begin{cases} \frac{1}{2} & l = 0 \\ 0 & l > 0 \,. \end{cases} \tag{A12}$$

Retaining only the highest order terms gives

$$Z_p \approx \frac{p+1}{p}\frac{1}{Ka^2}\left(\frac{r}{2a}\right)^p, \tag{A13}$$

which has a p dependence of form const^{-p} for large p.

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