

No Way Back: Maximizing Survival Time Below the Schwarzschild Event Horizon

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Abstract: It has long been known that once you cross the event horizon of a black hole, your destiny lies at the central singularity, irrespective of what you do. Furthermore, your demise will occur in a finite amount of proper time. In this paper, the use of rockets in extending the amount of time before the collision with the central singularity is examined. In general, the use of such rockets can increase your remaining time, but only up to a maximum value; this is at odds with the ‘more you struggle, the less time you have’ statement that is sometimes discussed in relation to black holes. The derived equations are simple to solve numerically and the framework can be employed as a teaching tool for general relativity.

Keywords: black hole physics — relativity — methods: numerical — methods: analytical

1 Introduction

General relativity is one of the pillars of modern physics, providing an accurate mathematical picture of gravitation and cosmology (see Misner et al. 1973, for a superb description). While extremely successful, the theory predicts the existence of black holes, completely collapsed massive objects which possess a one-way membrane (the event horizon) through which objects can pass through from the Universe, but not return. The strange properties of these objects has sparked the public imagination (Thorne 1994; Novikov 1995) and are the staple of most undergraduate courses on general relativity.

In this article, the question of the journey within the event horizon is examined, especially with regards to attempts to prolong, through the use of powerful rockets, the time to the inevitable collision with the central singularity at $r = 0$. While touched upon in many texts, the discussion of their use in the vicinity of black holes is not common. Hence this article is a pedagogical study of the use of coordinates and physical acceleration in general relativity. Furthermore, it aims to clear up a few black hole myths, especially those that appear on authoritative internet websites¹. In Section 2 a little history is presented, while Section 3 outlines the approach taken. The results of this study appear in Section 4 and the conclusions in Section 5.

2 A Little History

It has been ninety years since Schwarzschild presented the first exact solution to the field equations of general

relativity (Schwarzschild 1916). Representing the space-time curvature outside of a spherical mass distribution, the existence of singularities in the solution led to several confusing problems. Importantly, the coordinate time is seen to diverge as an object falling in this spacetime approaches the Schwarzschild radius ($r = 2m$, in units where $G = c = 1$ and where m is the mass of the black hole), with the conclusion that the entire history of the Universe can pass before anything actually falls to this radius. Paradoxically, the proper time as experienced by the falling object is finite through $r = 2m$ and the faller reaches $r = 0$ in finite time.

A reformulation of the Schwarzschild solution in free-falling coordinates revealed the Schwarzschild radius to be an event horizon, a boundary which can only be crossed from $r > 2m$, but not from $r < 2m$, leading to notion of complete gravitational collapse and the formation of black holes (Painlevé 1921). However, even with these advances, the singular state of the Schwarzschild solution at $r = 2m$ led even the most famous relativist to suggest that black holes cannot form (Einstein 1939). The resolution was ultimately provided by Finkelstein (1958) who derived a coordinate transformation of the Schwarzschild solution which made it finite at $r = 2m$; this was, however, a rediscovery of the earlier work by Eddington (1924) who apparently did not realize its significance². With this transformation the true nature of the Schwarzschild radius was revealed, acting as a one-way membrane between the Universe and inner region of the black hole. Surprisingly,

¹ While the authors acknowledges that the Internet is not the ultimate font of knowledge, anyone who has marked a few undergraduate essays will know that many students see it as their only source of knowledge.

² It is more astounding that in his analysis, Eddington (1924) explicitly considered outgoing light rays which, in his transformed coordinates, clearly crossed the event horizon from the inside to the outside. While he did not note it, Eddington had uncovered the white hole Schwarzschild solution.

the analysis of Finkelstein (1958) also possesses a time reversed black hole solution, a white hole in which the one-way membrane is reversed.

As discussed in many texts, the transformation to Eddington–Finkelstein coordinates clearly reveals the ultimate fate of an infalling observer. Now crossing the event horizon in a finite coordinate time, the future light cones for all massive explorers are tilted over such that there is no way back and the future ultimately lies at the central singularity. But after crossing the horizon, how long does the intrepid explorer have until this happens, and what can they do to maximize their survival time? For a free-falling path, the calculation of the proper time experienced by the explorer is a question found in graduate texts (e.g. see problem 12–14 in Hartle 2003) and it is straightforward to show that the maximum time that can be experienced below the event horizon is

$$\tau = \pi m \quad (1)$$

For a stellar mass black hole, this will be a fraction of a second, but for a supermassive black hole, this may be hours. As will be shown later, this maximum time applies to a faller who drops from rest at the event horizon and any one who starts falling from above the event horizon and free falls into the hole will experience less proper time on the journey from the event horizon to the singularity.

3 Setting Up the Problem

In this paper, only purely radial motion will be considered and the faller will be assumed to be impervious to the significant inertial and tidal forces it will suffer on its journey.

3.1 Eddington–Finkelstein Metric

In considering a radial journey across the event horizon, the advanced Eddington–Finkelstein coordinates will be employed. With this, the Schwarzschild solution is represented by the invariant interval of the form³

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{4m}{r}dtdr + \left(1 + \frac{2m}{r}\right)dr^2 + r^2d\Omega^2 \quad (2)$$

As noted previously, in this form the interval is non-singular at the event horizon ($r = 2m$).

3.2 4-Velocity and 4-Acceleration

The majority of texts on general relativity consider free fall motion through spacetime, with no acceleration terms due to non-gravitational forces. Such free fall paths are

³ There is more than one representation of the Eddington–Finkelstein metric for the Schwarzschild solution, and often it is written in terms of an advanced time parameter. However, as this parameter is null, the metric is often recast in terms of a new time-like parameter, resulting in the metric given above (see Chapter 11.5 in Hobson, Efstathiou, & Lasenby 2005). This is explicitly the form of the metric investigated by Eddington (1924) and Finkelstein (1958).

governed by the well-known geodesic equation which parameterizes the coordinates, x^α of a massive object in terms of its proper time, τ ,

$$x^\alpha = (t(\tau), r(\tau), \theta(\tau), \phi(\tau)) \quad (3)$$

From this, it is simple to define a 4-velocity, u^α , of the form

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right) \quad (4)$$

If the massive body undergoes a 4-acceleration, a^α , due to a force, the equation of motion can be written as

$$a^\alpha = \frac{du^\alpha}{d\tau} + \Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma \quad (5)$$

where $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols or affine connections; clearly, if the 4-acceleration is zero, the standard geodesic equation is recovered. The required Christoffel symbols are simply calculated from the Eddington–Finkelstein metric using *GRTensor*⁴ in MATHEMATICA. The non-zero components needed for this study are

$$\begin{aligned} \Gamma_{tt}^t &= \frac{2m^2}{r^3} & \Gamma_{rr}^t &= \frac{2m(m+r)}{r^3} \\ \Gamma_{tt}^r &= \frac{m(r-2m)}{r^3} & \Gamma_{rr}^r &= \frac{-m(2m+r)}{r^3} \\ \Gamma_{tr}^t &= \Gamma_{rt}^t = \frac{m(2m+r)}{r^3} & \Gamma_{tr}^r &= \Gamma_{rt}^r = \frac{-2m^2}{r^3} \end{aligned} \quad (6)$$

The path of an accelerated object is also constrained through the normalization of the 4-velocity of a massive particle

$$\mathbf{u} \cdot \mathbf{u} = g_{\alpha\beta} u^\alpha u^\beta = -1 \quad (7)$$

and its orthogonality with the 4-acceleration

$$\mathbf{a} \cdot \mathbf{u} = g_{\alpha\beta} a^\alpha u^\beta = 0 \quad (8)$$

where $g_{\alpha\beta}$ are the components of the metric (Equation 2). The final constraining equation is the normalization of the 4-acceleration

$$\mathbf{a} \cdot \mathbf{a} = g_{\alpha\beta} a^\alpha a^\beta = a^2 \quad (9)$$

where a is the magnitude of the acceleration. Note that this also represents the magnitude of the acceleration as experienced by our faller due to the presence of the rockets.

3.3 Hyperbolic Motion

In an insightful paper, Rindler (1960) demonstrated that all bodies undergoing constant acceleration undertake hyperbolic motion. While this result is well known in the framework of special relativity, this paper was the first to determine that accelerated bodies execute hyperbolic motion in the curved spacetime of general relativity (see also Gautreau 1969; Karlov & Rindler 1971). Through an examination of the geometry of motion, Rindler (1960)

⁴ <http://grtensor.phy.queensu.ca/>

showed that the components of the 4-velocity and 4-acceleration can be given in terms of two other tensors, M^α and L^α , such that

$$\begin{aligned} u^\alpha &= (\cosh a\tau)L^\alpha + (\sinh a\tau)M^\alpha \\ a^\alpha &= a[(\sinh a\tau)L^\alpha + (\cosh a\tau)M^\alpha] \end{aligned} \quad (10)$$

where a is the magnitude of the acceleration (Equation 9) and τ is the proper time as measured by the accelerated body. The tensors L^α and M^α , are orthogonal unit-vectors, being time-like and space-like respectively. Operationally, these tensors are parallel-propagated along the path of the accelerated motion, such that

$$\begin{aligned} \frac{dL^\alpha}{d\tau} + \Gamma_{\beta\gamma}^\alpha L^\beta u^\gamma &= 0 \\ \frac{dM^\alpha}{d\tau} + \Gamma_{\beta\gamma}^\alpha M^\beta u^\gamma &= 0 \end{aligned} \quad (11)$$

and the initial conditions can be set by noting that at $\tau = 0$, then $L^\alpha = u^\alpha$ and $M^\alpha = a^\alpha/a$. Hence, given a fixed magnitude of acceleration, a , the normalization equations in the previous section can be used to determine the components of the 4-acceleration a^t and a^r . With this, the equations of motion can be derived from Equation (11) and the resulting coupled differential equations were integrated with ODEPACK⁵.

3.4 Killing Vectors and Conserved Quantities

In treating physical problems, conserved quantities are often employed to ease the understanding of the solutions. In general relativity, these are provided by Killing vectors. Simply put, a Killing vector ‘points’ in a direction along which the metric does not change. For a given Killing vector, ξ^α , a conserved quantity can be found for an object that moves along a geodesic to be

$$e = \xi \cdot \mathbf{u} = g_{\alpha\beta} \xi^\alpha u^\beta \quad (12)$$

Clearly, the components of the Eddington–Finkelstein representation of the Schwarzschild solution (Equation 2) are independent of the t coordinate (i.e. a translation in this coordinate leaves the metric the same) and its associated Killing vector is given by $\xi^\alpha = (1, 0, 0, 0)$ and the resultant conserved quantity is

$$e = g_{tt}u^t + g_{tr}u^r \quad (13)$$

It must be remembered that this quantity is conserved along geodesics and so only for freely-falling objects. For objects undergoing acceleration (e.g. due to rockets) this quantity is not conserved. This has significant implications for maximizing the proper time below the event horizon.

With the above definition of the conserved quantity related to the Killing vector, as well as the 4-velocity and 4-acceleration normalization and orthogonality, a little algebra reveals that for an acceleration of magnitude a , then

$$a^r = a \frac{u^r e}{\sqrt{e^2 + g_{tt}}} \quad (14)$$

and

$$a^t = \frac{(1 + u^t e)}{u^r e} a^r \quad (15)$$

It is important to remember that in the presence of a non-zero acceleration, the quantity e is no longer conserved.

4 Results

4.1 Analytic Checks

Before considering the influence of the rocket, it is important to check the computational solutions with comparison to analytic results for freely falling objects. Assuming the faller begins from rest beyond the event horizon at a radius r_s , so $u^r(r_s) = 0$, then the conserved quantity given by the Killing vector (Equation 13) is

$$e = g_{tt}u^t = -\sqrt{1 - \frac{2m}{r_s}} \quad (16)$$

where u^t at r_s is determined from the normalization of the 4-velocity (Equation 7). Clearly, if the faller starts from $r_s = \infty$ then $e = -1$ and, conversely, if the faller drops from rest at the event horizon, ($r_s = 2m$), then $e = 0$; examining Equation (12), it is clear that $e = 0$ corresponds to a 4-velocity which is orthogonal to the spacetime Killing vector. As noted previously, the free fall journey from rest at a particular radius to the central singularity is discussed in many text books and will not be reproduced here, but it can be shown that the proper time as measured by the faller is given by

$$\tau_{\max} = \frac{\pi}{2\sqrt{2m}} r_s^{3/2} \quad (17)$$

(e.g. see problem 12–5 in Hartle 2003). Note this is the proper time for the entire journey. The time spent on the portion of the trip between the event horizon and central singularity is given by

$$\tau = \left\{ \frac{1}{\sqrt{2}} \left[\frac{r_s}{m} \right]^{3/2} \arctan \left[\sqrt{\frac{2m}{r_s - 2m}} \right] - \frac{\sqrt{r_s(r_s - 2m)}}{m} \right\} m \quad (18)$$

For all $r_s > 2m$, the proper time experienced by the faller between the event horizon and the singularity is less than Equation (17). Conversely, the minimum time that can be experienced by a free-faller (found by taking the limit of $r_s \rightarrow \infty$) is

$$\tau_{\min} = \frac{4}{3}m \quad (19)$$

Figure 1 presents the results of the numerical integration of Equation (11), assuming the rockets are not used and so the acceleration terms are zero. For this example, four paths are examined, differing only in the radial coordinate from which they are dropped from rest; these are $3.0m$ (black), $2.5m$ (red), $2.1m$ (green), and $2.00000001m$ (blue). Note, as the normalization of the 4-velocity diverges for an object at rest at the event horizon, it is not possible to numerically integrate these

⁵ <http://www.llnl.gov/CASC/odepack/>

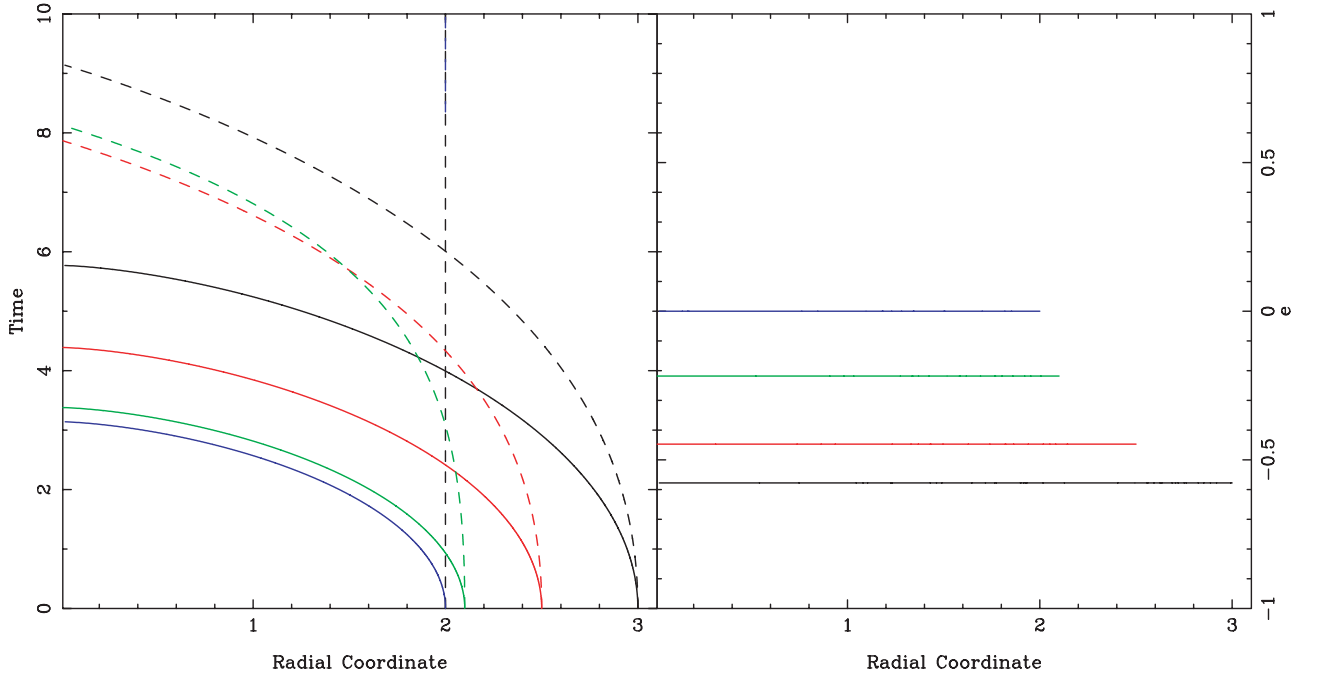


Figure 1 The left hand panel presents several free fall paths into a black hole. The paths begin at $3.0m$ (black), $2.5m$ (red), $2.1m$ (green), and $2.00000001m$ (blue). The solid curves represent the path in terms of the proper time of the faller, while the dashed path is with respect to the coordinate time in Eddington–Finkelstein coordinates. The right hand panel presents the conserved quantity e (Equation 13).

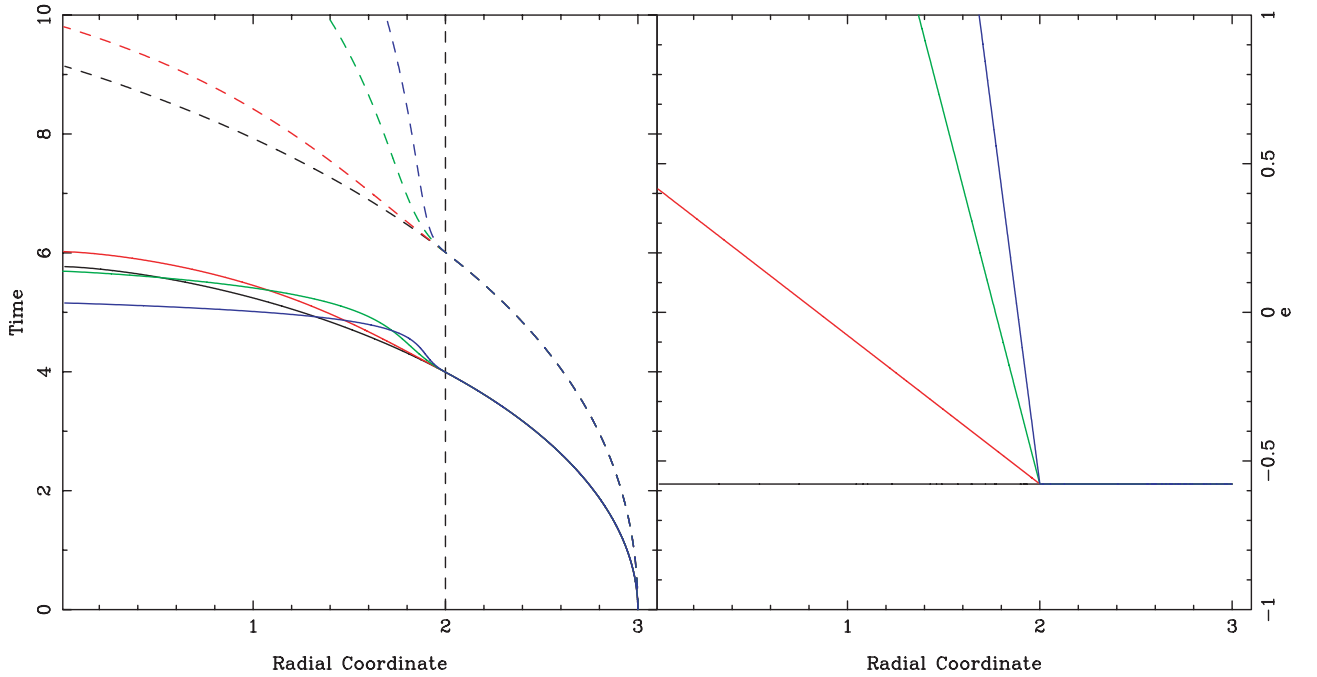


Figure 2 As in Figure 1, for observers falling from $r = 3m$. Here, the black line represents a free faller, while the red, green, and blue represents a rocketeer accelerating outwards at $a = 0.5, 2.5$ and 5.0 respectively.

equations with the initial condition of $r_s = 2m$. In comparing the numerical results for the proper time below the event horizon with the analytic predictions (Equation 18), the maximum fractional error is found to be $\sim 0.005\%$. Similarly, the fractional error in the conserved quantity (Equation 13) is of a similar order over the journey to the singularity.

4.2 Turning On the Rocket

For the purposes of this study, it is assumed that the faller begins from rest at some distance beyond the black hole, free falling to the event horizon. Once across the horizon, the rocket is ignited. Figure 2 presents the case where such an object is dropped from rest at $r = 3m$, with the black curve representing a free falling path (again, the solid line

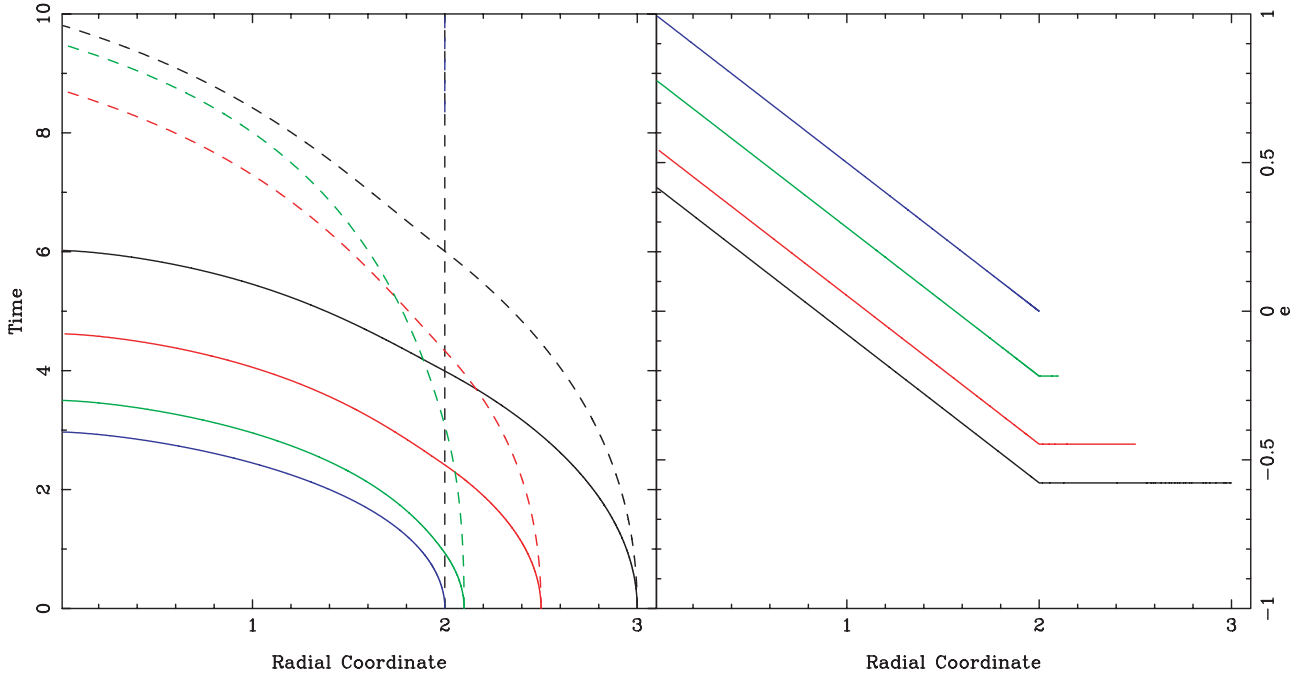


Figure 3 As in Figure 1, except each faller undergoes an outward acceleration of $a = 0.5$ once inside the event horizon.

represents the curve with respect to proper time, while the dotted line is that for coordinate time). For the red curve, the rocketeer ignites the rocket as they pass $r = 2m$ and undergoes a constant, outward acceleration of $a = 0.5$, while the green and blue lines suffer an acceleration of $a = 2.5$ and $a = 5$ respectively. Looking at the left hand panel, it is clear that the use of a rocket can increase the proper time of the faller beyond that expected for a purely free fall path (e.g. the red line). However, it is also apparent there is a limit to the increased proper time through firing the rocket as the more extreme accelerations (green and blue line) experience less proper time than the free falling observer on their journey to the singularity.

An examination of the conserved quantity from the Killing vector, e in the right hand panel tells an interesting story; free falling from rest outside the event horizon, all of the fallers have the same value of e , but once the rocket is fired inside the event horizon, the firing of the rocket increases the value of e , and, moreover, the change appears to be linear. In examining this, it is straightforward to show, through a little algebra, that⁶

$$\frac{de}{dr} = \frac{g_{tt}a^t + g_{tr}a^r}{u^r} = -a \quad (20)$$

Figure 3 shows the free fall paths of observers from several different radii to the event horizon. Once within the horizon, each observer fires their rocket with the same acceleration ($a = 0.25$) and continues their journey to the central singularity. As expected from Equation (20),

the quantity e is conserved along the free fall path, but once the rocket is fired e changes linearly with the radial coordinate.

Armed with this knowledge, what should an observer who has fallen from outside the event horizon do to maximize their survival time below the event horizon, if they have at their disposal a rocket that can produce an acceleration a ? As noted earlier, the longest free fall time below the event horizon occurs for an observer who falls from rest at $r = 2m$ (with $e = 0$) and any attempt at accelerated motion for this observer will only diminish the proper time (this is discussed in more detail in the next section). Hence, if the observer starts from beyond the event horizon with any non-zero value of e , the best they can do is fire their rocket until e equals zero and then turn the rocket off and coast on the $e = 0$ geodesic to the central singularity. This is illustrated in Figure 4 for several observers who falls from rest at $r = 3m$ to the event horizon. Once within the horizon, one rocketeer (black curve) continues their free fall path to the singularity, while the others fire their rockets (with $a = 2$). The red path is that of the observer who continues to fire their rocket all the way down, while the light blue, dark blue, and green cease firing when $e = -0.3$, $e = 0.3$ and $e = 0$ respectively. An examination of the left-hand panel of this figure shows that, in terms of coordinate time, the act of firing the rocket delays the collision with the central singularity. However, the time as measured by each observer displays a quite different behaviour; firing the rocket in this circumstance increases the proper time between the horizon and the singularity. However, it is clear that the observer who settles on the path with $e = 0$ experiences the greatest proper time, with

⁶ An examination of a uniformly accelerating observer in special relativity displays the same relationship.

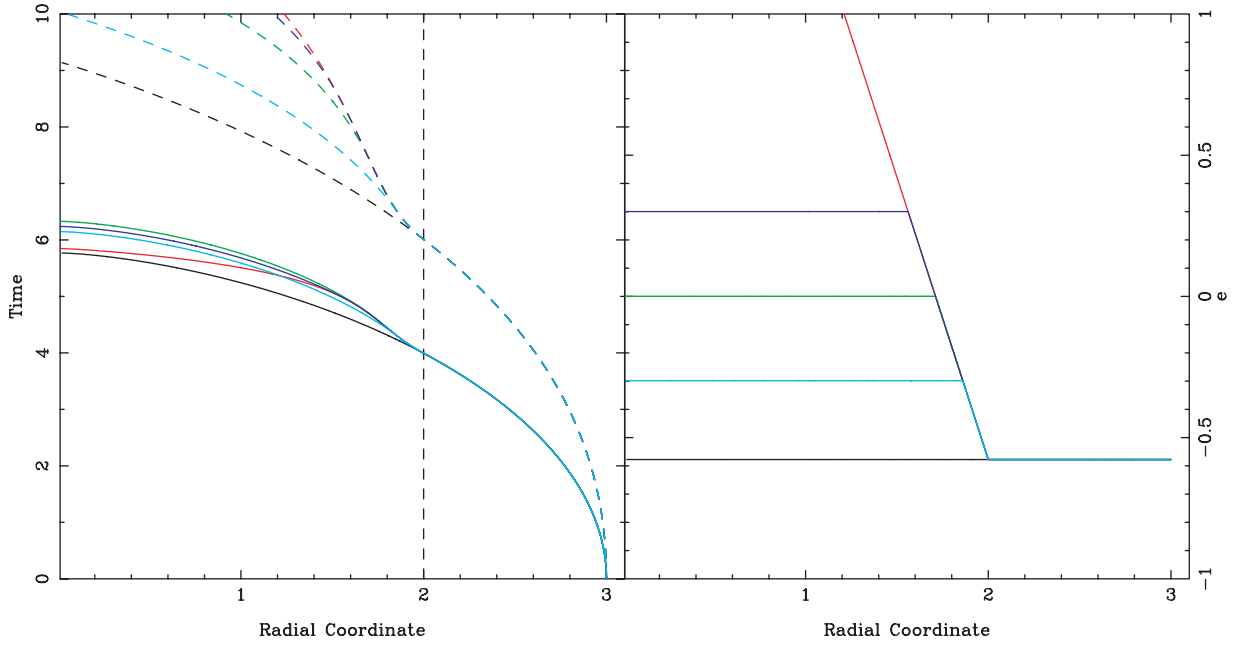


Figure 4 As in Figure 1, with the black line representing a free faller from $r = 3m$. The other lines correspond to an observer who free falls to the event horizon and then fires their rocket with $a = 2$. For the red line, the faller fires their rocket all the way to the singularity, while the dark blue, light blue, and green turn off their rocket when $e = 0.3$, $e = -0.3$ and $e = 0$ respectively. An examination of the proper time in the left-hand panel reveals that it is the path that settles on $e = 0$ that possesses the longest proper time.

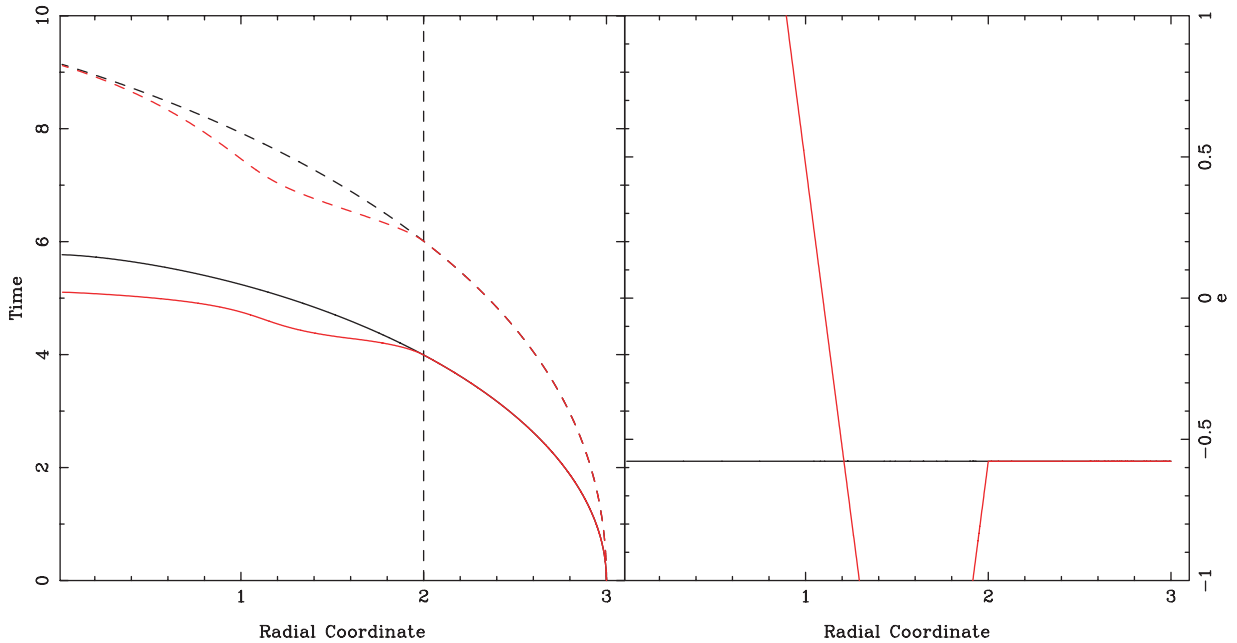


Figure 5 As in Figure 1, with the black line representing a free faller, while the red line represents a rocketeer who, once across the event horizon, accelerates inwards for a short while and then accelerates outwards. The amount of acceleration is tuned so that both the free faller and the rocketeer arrive at the central singularity at the same coordinate time (dotted paths). As revealed by the solid paths, the free faller experiences the greater proper time in the journey below the event horizon.

those that burn their rocket for shorter or longer periods experiencing shorter proper times.

4.3 Clearing Up a ‘Mythconception’

As noted previously, black holes have fired the imagination of the general public and many websites can be found that are dedicated to discussing their strange properties.

However, some authoritative websites carry statements like the following⁷

A consequence of this is that a pilot in a powerful rocket ship that had just crossed the event horizon who tried to accelerate away from the singularity would reach

⁷ <http://cosmology.berkeley.edu/Education/BHfaq.html>

it sooner in his frame, since geodesics (unaccelerated paths) are paths that maximize proper time

The results of this study show that this clearly is not the case; anyone who falls through the event horizon should fire their rockets to maximize the time they have left before impacting the central singularity. In dropping from rest at the event horizon, the firing of a rocket does not extend the time left, it only diminishes it.

While the quote is ambiguous about the initial conditions for the faller, it appears that the error lies in the assumption that the impact onto the central singularity is the same event for the free faller and the rocketeer; if they were then the above statement would be correct and the free faller would experience the maximal proper time. As an example of this, consider Figure 5. Again, the two fallers start from rest and drop towards the event horizon. After crossing the horizon, one continues the free fall path towards the central singularity while the second accelerates inwards for a short while and then swings their rocket round to accelerate outwards such that both fallers arrive at the central singularity at the same coordinate time (the dotted path). In considering the two paths connecting the two identical events in spacetime, the free faller, travelling along the geodesic, experiences the maximum proper time, and nothing the rocketeer does can exceed this.

5 Conclusions

Black holes remain amongst the most studied theoretical consequences of general relativity, although standard texts say little about the use of rockets once you are below the event horizon. This paper has considered this very scenario, showing that a rocketeer can enhance their survival time by firing a rocket once across the event horizon. However, the rocketeer is still doomed to impact on the central singularity in less than the maximal free fall time between the event horizon and the centre.

Additionally, this paper has considered an apparent confusion on the use of rockets below the event horizon which suggest they hasten a fallers demise. This is at odds with this study which shows that rockets can increase survival time for virtually all fallers.

Finally, it should be remembered that ingoing light rays in Eddington–Finkelstein coordinates travel at 45° . A simple examination of Figure 4 reveals something quite interesting; while the constantly accelerating observing within the event horizon (red line) experiences less proper time in their fall to the singularity than the path that settles on $e=0$, an examination of the paths in coordinate time shows that the constantly accelerating observer sees a longer period of time pass in the outside universe than the path on $e=0$. A more detailed study of this effect will be the subject of a future contribution.

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