ASEG-PESA 2013
'THE EUREKA MOMENT'
11-14 AUGUST 2013 • MELBOURNE, AUSTRALIA

Evaluating the utility of gravity gradient tensor components

Mark Pilkington
Geological Survey of Canada
615 Booth St., Ottawa, ON, Canada K1A 0E9
mpilking@nrcan.gc.ca

SUMMARY

Gravity gradiometry offers multiple single components and possible combinations of components to be used in interpretation. Knowledge of the information content of components and their combinations is therefore crucial to their effectiveness and so a quantitative rating of information level is needed to guide the choice. To this end we use linear inverse theory to examine the relationship between the different tensor components and combinations thereof and the model parameters to be determined. The model used is a simple prism, characterized by seven parameters: the prism location, *xc*, *yc*, its width *w* and breadth *b*, the density ρ, the depth to top *z*, and thickness *t*. Varying these values allows a wide variety of body shapes, e.g., blocks, plates, dykes, rods, to be considered. The Jacobian matrix, which relates parameters and their associated gravity response, clarifies the importance and stability of model parameters in the presence of data errors. In general, for single tensor components and combinations, the progression from well- to poorly-determined parameters follows the trend of ρ, *xc*, *yc*, *w*, *b*, *z* to *t*. Ranking the estimated model errors from a range of models shows that data sets consisting of concatenated components produce the smallest parameter errors. For data sets comprising combined tensor components, the invariants *I1* and *I2* produce the smallest model errors. Of the single tensor components, *Tzz* gives the best performance overall, but those single components with strong directional sensitivity can produce some individual parameters with smaller estimated errors (e.g., *w* and *xc* estimated from *Txz*).

Key words: gravity, gradiometer, inversion.

INTRODUCTION

Gravity gradiometry presents multiple possible single component and combinations of components which can be used in interpretation. The purpose of each survey will dictate the choice of component(s), for example, qualitative mapping of dominantly N-S trending structures would focus on the *Txx* component. The information content of the components and their combinations with respect to the purpose is crucial to their effectiveness. Consequently, we need a quantitative rating of these values to guide our choice, especially when many combinations are available and more are likely to be suggested in the future.

Qualitative comparisons of results of inversions for the underdetermined problem (solving for densities in a 3D volume mesh) using different component combinations have been done by looking at solution character and comparing with known geology (Zhdanov et al., 2004; Martinez and Li, 2011; Martinez et al., 2013) An attempt at a quantitative comparison of tensor components and their combinations was made by Pilkington (2012) who used a measure of information content commonly used in optimal survey design to rate different components and some component combinations. *Tzz* was shown to provide the most information out of the single tensor components and that adding more components to the data vector improves the situation. Nonetheless, several limitations are inherent in using the underdetermined model formulation. One is that data errors are not considered. The second is that the model geometry is unrestricted and so it not possible to investigate individual model parameters and their effects on components. Thirdly, the number of parameters and related quantities quickly become unmanageable for large 3D volumes. Any standard measures of solution appraisal used in inversion like the resolution or covariance matrix have dimensions $m \times m$ (*m* is the number of parameters), too large to easily display or analyse, and often too large to calculate.

Keeping the number of parameters small (tens or less) allows for a much easier assessment of solution appraisal measures. This suggests the use of parametric inversion, which involves the inversion of models described by only a small number of parameters (Oldenburg and Li, 2005). Using this approach, solution appraisal, through the covariance matrix for example, reduces to simply calculating a small number of parameter variances and covariances that are easily examined, even for multiple models (Christensen and Lawrie, 2012).

METHOD

The aim of this study is to use the estimated parameter errors resulting from inversions of single and multiple tensor components to quantitatively rate their information content. Table 1 gives a list of 17 different component quantities consisting of single tensor components, combinations of components, and concatenations of components.

Linear inverse theory provides all the tools to examine the relationship between the different data types and the model parameters (Glenn et al., 1973; Inman, 1975). The model used in the following is a simple prism, characterized by just seven parameters: *xc* and *yc*, the *x* and *y* coordinates of the prism center; *w* and *b*, the prism width (in *x*) and breadth (in *y*); the density ρ, the depth to the top surface, *z* and the vertical thickness, *t*. Varying the dimensions of the prism allows a wide variety of body shapes to be considered, e.g., blocks, plates, dykes, rods.

1 - *Txx*
2 - *Txy*
3 - *Txz*
4 - *Tyy*
5 - *Tyz*
6 - *Tzz*
7 - *Tuv* = 0.5*(*Txx*-*Tyy*)
8 - $I1 = TxxTyy+TyyTzz+TxxTzz-Txy^2-Tyz^2-Txz^2$
9 - $I2 = Txx(TyyTzz-Tyz^2)+Txy(TyzTxz-TxyTzz)+Txz(TxyTyz-TxzTyy)$
10 - $H1 = \mathrm{sqrt}(Txz^2+Tyz^2)$
11 - $H2 = \mathrm{sqrt}(Txy^2+0.25*(Tyy-Txx)^2)$
12 - *Tuv* | *Txy*
13 - *Txz* | *Tyz* | *Tzz*
14 - *Txy* | *Tyz* | *Txz*
15 - *Txx* | *Tyy* | *Txy*
16 - *Tzz* | *Tyz* | *Txz* | *Txy* | *Txx*
17 - *Tyy* | *Tyz* | *Txz* | *Txy* | *Txx*

Table 1. List of component and component combinations used in this paper. Notation *Tuv* | *Txy* means that, for example, data vectors *Tuv* and *Txy* both with length *n* are concatenated to form an augmented data vector with length 2*n*.

Consider the nonlinear forward problem relating the model parameter vector **x** (length *m*) to the data vector **d** (length *n*)

$$\mathbf{d} = f(\mathbf{x}) \ . \tag{1}$$

The data vector **d** may contain a single component, a combination of components, or a concatenation of several components (length > *n*). For the prism model, the parameter or model vector **x** contains the seven parameters *xc*, *yc*, *w*, *b*, *ρ*, *z* and *t*. Linearizing equation 1 with respect to the model parameters gives

$$\Delta\mathbf{d} = \mathbf{A}\,\Delta\mathbf{x} \quad , \tag{2}$$

where the elements of the $n \times m$ matrix **A** are given by $\partial d_i/\partial x_j$. $\Delta\mathbf{d}$ is the incremental change in the data values due to the parameter perturbation $\Delta\mathbf{x}$. Matrix **A** can be decomposed into the singular value decomposition:

$$\mathbf{A} = \mathbf{U}\,\boldsymbol{\Lambda}\,\mathbf{V}^T \quad . \tag{3}$$

The columns of the $n \times m$ unitary matrix **U** are the eigenvectors $\mathbf{u}_i$ and **V** is the $m \times m$ unitary matrix with columns $\mathbf{v}_i$. $\boldsymbol{\Lambda}$ is the $m \times m$ diagonal matrix containing the singular values λ_i. If $\mathbf{A}^+$ is the generalized inverse of **A** given by

$$\mathbf{A}^+ = \mathbf{V}\,\boldsymbol{\Lambda}^{-1}\,\mathbf{U}^T \quad , \tag{4}$$

then the solution of equation 2 is given by

$$\Delta\mathbf{x} = \mathbf{A}^+\,\Delta\mathbf{d} \quad . \tag{5}$$

Once an inversion is completed, the errors in the model parameters can be estimated from the parameter covariance matrix **C**:

$$\mathbf{C} = \mathbf{C}_d\,\mathbf{V}\,\boldsymbol{\Lambda}^{-2}\,\mathbf{V}^T \quad , \tag{6}$$

where $\mathbf{C}_d$ is the data covariance matrix. For most gravity and magnetic survey data, $\mathbf{C}_d$ can simply be written as $\sigma^2\mathbf{I}$, implying a constant data error variance of σ^2. For gravity gradiometer data, tensor components may be characterized by different error levels, so this must be an option for $\mathbf{C}_d$. For combinations of components, $\mathbf{C}_d$ is found from using standard rules for the sum and products of independent measurements (Mellor, 1954). From equation 6 it is apparent that the form of **C** is controlled mainly by the smallest singular values which cause large variances and covariances. Matrix **A** (equation 3) on the other hand is primarily controlled by the larger singular values. The individual model parameters are not limited to association with only the large or the small singular values. A parameter may have components in the eigenvector associated with the largest *λ* and in the eigenvector associated with the smallest *λ*, which means that it is an important contributor to the model response, but it is also prone to large errors.

THE PRISM MODEL

Examining the make up of **A** shows which parameters are important by quantifying their contribution to the model response. Parameters making large contributions are better determined when inverted, and vice versa. Figure 1 shows the singular values and eigenvectors for a prismatic model with parameters *z* = 3 km, *t* = 13 km, *w* = 12 km, *b* = 12 km, *ρ* = 0.2 gcm^{-3}. Calculations were carried out on a 64 × 64 grid with the center of the prism located at *xc* = 32 and *yc* = 32 km. Singular values and their associated eigenvectors are ordered from the largest, λ_1, to the smallest, λ_7. Figure 1 indicates that from λ_1 and $\mathbf{v}_1$ the main contributing parameter is the density, *ρ*. Much smaller contributions to $\mathbf{v}_1$ (not visible on plot) are made by *z*, *t*, *w* and *b*. These minor contributions can be significantly larger when *z*, *t*, *w* or *b* is small. The magnitude of λ_1 increases with larger *w*, *b*, *t*, or smaller *z*. Singular values λ_2 and λ_3 are usually associated with contributions from the prism location parameters *xc* and *yc*. For the more directional components *Txx*, *Txz*, *Tyy* and *Tyz* the contribution from *xc* and *yc* is dependent on body shape and orientation. When sources have a dominant strike direction, then the position parameter perpendicular to strike contributes most.

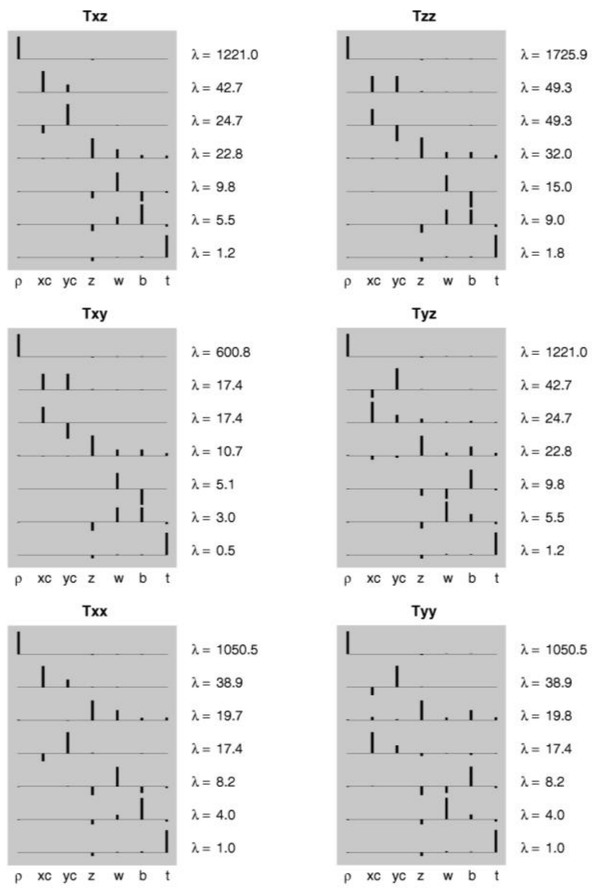

Figure 1. Parameter eigenvectors (columns of matrix V) and eigenvalues for the six basic tensor components. The

model used is a prism with parameters z = 3 km, t = 13 km, w = 12 km, b = 12 km, ρ = 0.2 gcm^{-3}.

Singular value λ_4 is associated mainly with the depth, z, plus lesser contributions from w (greater for *Txx*, *Txz*) and b (greater for *Tyy*, *Tyz*). As the thickness increases, the z contribution in $\mathbf{v}_4$ increases and decreases in $\mathbf{v}_7$, leading to a better determined z. When $z \to 0$, z becomes more important and well-determined and dominates $\mathbf{v}_2$. Increasing z leads to a decrease in the magnitude of all singular values. Larger z increases the z component in λ_5 and λ_6, i.e., the smaller singular values, so the depth becomes less reliably estimated. Singular values λ_5 and λ_6 are most often associated with w and b plus minor contributions from z and t. Again, *Txx* and *Txz* are characterized by larger contributions from w and *Tyy*, *Tyz* equally for b. Again, when w and b are small they contribute more to the larger singular values and eigenvectors accompanying a corresponding increase in ρ contributions to $\mathbf{v}_5$ and $\mathbf{v}_6$. For λ_7, the minimum singular value, $\mathbf{v}_7$ is most often dominated by the thickness, t, with lesser contributions from z, and possibly from w and b. The magnitude of λ_7 decreases as t increases or z increases, or as w or b decrease. Estimating t is therefore difficult, particularly so when it is large (the bottom of the body is not detected) or small (t is not separable from ρ, as only the product ρt is estimable).

In summary, the density ρ can be considered as a well-determined parameter, along with (to a lesser extent) positional parameters *xc* and *yc*. The depth z is only well-determined when small with respect to the prism size. The width and breadth parameters w and b are well-determined mainly when small, while the thickness t is generally a poorly determined quantity.

PARAMETER ERRORS

In determining individual parameter errors resulting from inversion, variable component errors should be addressed because the instrumental set-up can produce different noise levels in the components and also the pre-processing of components often results in changed component noise levels. Of these two sources of variable noise levels, the latter is more important because most pre-processing algorithms combine the original component measurements into a single quantity from which the final (processed) component values are then computed. Since the tensor components provide independent but related measurements of the gravity field, it is appropriate that they are combined by calculating the underlying field (or potential) or equivalent density distribution that models the field (Li, 2001; Barnes and Lumley, 2011). As a result, the two main pre-processing methods currently in use are the Fourier transform approach (to compute the field or potential) and the equivalent source method (to compute a density distribution). I consider just the Fourier method in the following, with the knowledge that for a horizontal observation level and a shallow equivalent source position, the results are comparable. Average values for the ratio of noise levels based on testing of many input noise scenarios are 1: 0.37 : 0.7 : 0.59 for the components *Tzz*, *Txy*, *Txz* (=*Tyz*) and *Txx* (=*Tyy*), respectively. These are the values used to specify the six component noise levels in the following tests. Because components are being compared with each other, only relative noise levels are needed.

To achieve a general picture of how well the different component quantities perform in terms of parameter errors, a range of models was tested. Varying t, w, b and z allows the model to range from shallow to deep prisms, and thin to thick plates and dykes. For each model the standard deviations of the parameter errors were calculated from equation 6.

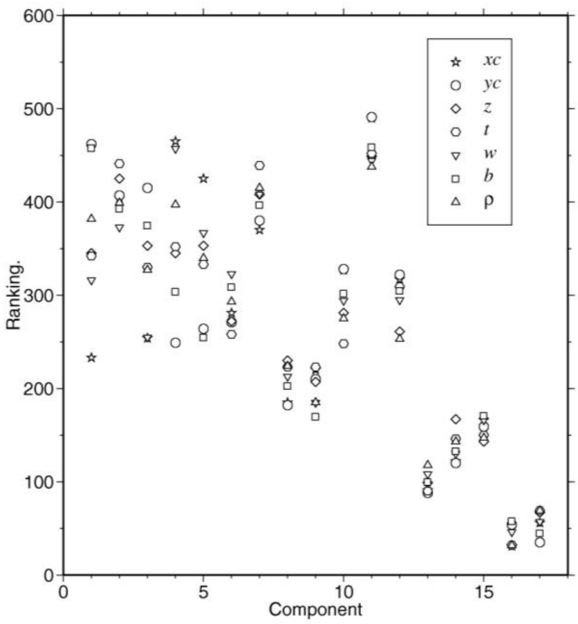

Figure 2. Parameter rankings versus component type based on 29 model inversions. For each inversion the parameter standard deviations were ranked for each parameter by component quantity (Table 1 list) with a value of 1 assigned to the smallest parameter error and a value of 17 to the largest. Plotted values are the sums of the ranked values.

The resulting parameter standard deviations were then ranked for each parameter by component quantity (Table 1 list) with a value of 1 assigned to the smallest parameter error and a value of 17 to the largest. Summing the rank values for the 29 models was used to produce Figure 2, which summarizes the relative performance of the component quantities in terms of individual parameter rankings. For each component quantity, there are seven points plotted, each corresponding to a single model parameter. Even though Figure 2 shows the sum of rankings for widely varying model shapes, the relative rankings are reflective of individual model parameter error behavior. Figure 3 shows parameter errors for a thin plate, demonstrating the general character of the final rankings in Figure 2.

Figure 2 shows that most of the 17 component quantities have parameter rankings clustered fairly close together, implying that all parameters have similar rankings, i.e., none of the parameters are much more reliably estimated than the others. There is, nonetheless, a large spread in the rankings in Figure 2 for the 4 directional single tensor components, *Txx*, *Tyy*, *Txz* and *Tyz*, since these components perform best along one axis, so for example *Txx* and *Txz* recover *xc* and w well, but *yc* and b are poorly estimated.

The optimal design information measure Θ used in Pilkington (2012) varied with source depth and was generally independent of the inverted component quantity at larger source depths. For shallower sources, differences in the component quantities were apparent: the concatenated components (quantities 13-17, Table 1) provided greater information content than the single components, in agreement with Figure 2. *Tzz* is an exception, being associated with high *Θ* but only an average ranking in terms of parameter errors. Nonetheless, *Tzz* shows a higher overall ranking than other single components in Figure 2.

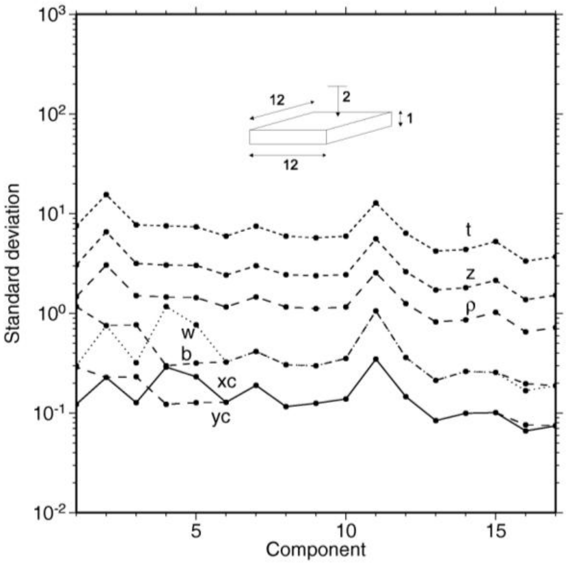

Figure 3. Parameter standard deviations from inversion of component quantities (Table 1 list) for the thin plate model shown.

Some individual parameters are estimated with lower errors than using *Tzz* but these are limited to a few cases of parameters benefitting from directional strengths in some of the single tensor components, e.g., *xc* and *w* are well estimated from *Txx* and *Txz*. The mainly horizontal component combinations *H1* and *Tuv* | *Txy* are ranked at a similar level to *Tzz*. In contrast, the purely horizontal quantities *H2*, *Tuv* and *Txy* rank much lower. These three components comprise just *Txx*, *Tyy* and/or *Txy*. The invariants *I1* and *I2* are ranked in between the higher ranked concatenated combinations and those discussed above. Both *I1* and *I2* have tightly clustered parameter rankings and are the two most highly ranked combined component quantities.

In agreement with the information measure *Θ* (Pilkington, 2012), Figure 2 shows that the concatenated components are the highest ranked, producing the smallest parameter error estimates. This demonstrates that when components are combined into a single quantity through multiplication and addition, it will not perform as well as if the same components were simply concatenated. For example, comparing *H2* and the *Txx* | *Tyy* | *Txy* combination, they both contain the same (purely horizontal) components but provide very different error estimates, the latter being by far the more reliable. Component quantity *Tuv* | *Txy* is a mix of combination and concatenation, and falls between *H2* and *Txx* | *Tyy* | *Txy* in terms of ranking.

In this study, estimated parameter errors for an inverted model are used to measure the quality of a solution. This is not the only measure that could be used to assess the goodness of an inversion. Another useful gauge is how well the inverted model matches geological information, and whether certain known features are present in the inversion results. Adding more components to the data vector was also shown in a detailed study by Martinez et al. (2013) to improve inversion results when compared to known geology.

CONCLUSIONS

Using estimated parameter errors from parametric inversions allows for a quantitative ranking of tensor component quantities comprising single tensor components, combinations of components, and concatenations of components. Furthermore, linear inverse theory allows incorporation of the appropriate relative noise levels of the tensor components after noise reduction processing. Ranking of the estimated model errors from a range of model types shows that data sets consisting of concatenated components produce the smallest parameter standard deviations. For data sets comprising combined tensor components, the invariants *I1* and *I2* produce the smallest model errors. Combinations of the purely horizontal components *Txx* and *Tyy* perform the poorest. Of the single tensor components, *Tzz* gives the best performance overall, but those single components with strong directional sensitivity can produce some individual parameters with smaller estimated errors (e.g., *w* estimated from *Txz*).

ACKNOWLEDGMENTS

James Brewster (Bell Geospace) provided helpful information on component noise levels.

REFERENCES

Barnes, G. and Lumley, J., 2011, Processing gravity gradient data: Geophysics, 72, I33-I47.

Christensen, N. B. and Lawrie, K. C., 2012, Resolution analysis for selecting an appropriate airborne electromagnetic (AEM) system: Exploration Geophysics, 43, 213-227.

Glenn, W. E., Ryu, J., Ward, S. H., Peeples, W. J., and Phillips, R. J., 1973, The inversion of vertical magnetic dipole sounding data: Geophysics, 38, 1109-1129.

Inman, J. R., 1975, Resistivity inversion with ridge regression: Geophysics, 40,798-817.

Li, Y., 2001, Processing gravity gradiometer data using an equivalent source technique: 71st Meeting, SEG, Expanded Abstracts, 1466-1469.

Martinez, C. Y., and Y. Li, 2011, Inversion of regional gravity gradient data over the Vredefort Impact Structure, South Africa: 81st Meeting, SEG, Expanded Abstracts, 841-845.

Martinez, C., Li, Y., Krahenbuhl, R., and Braga, M., 2013, 3D inversion of airborne gravity gradiometry data in mineral exploration: A case study in the Quadrilátero Ferrífero, Brazil: Geophysics, 78, B1–B11.

Mellor, J. W., 1954, Higher mathematics for students of chemistry and physics: Dover Publications.

Oldenburg, D. W. and Li, Y., 2005, Inversion for Applied Geophysics: A Tutorial, in D. K. Butler, ed., Near Surface Geophysics: SEG, 89-150.

Pilkington, M., 2012, Analysis of gravity gradiometer inverse problems using optimal design measures: Geophysics, 77, G25-31.

Zhdanov, M. S., Ellis, R., and Mukherjee, S., 2004, Three-dimensional regularized focusing inversion of gravity gradient tensor component data: Geophysics, 69, 925-93