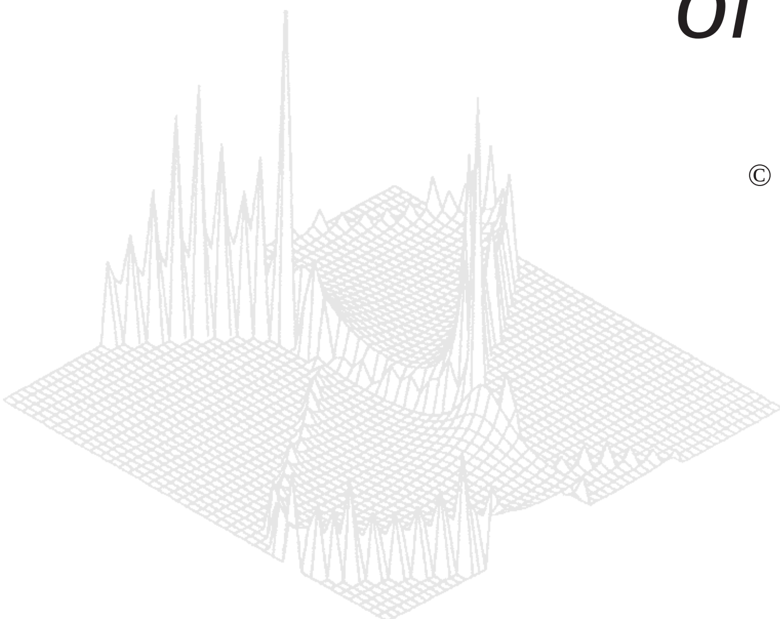
CSIRO PUBLISHING

Australian Journal of Physics

Volume 50, 1997



A journal for the publication of
original research in all branches of physics

www.publish.csiro.au/journals/ajp

All enquiries and manuscripts should be directed to
Australian Journal of Physics
CSIRO PUBLISHING
PO Box 1139 (150 Oxford St)
Collingwood
Vic. 3066
Australia

Telephone: 61 3 9662 7626
Facsimile: 61 3 9662 7611
Email: peter.robertson@publish.csiro.au

Published by **CSIRO** PUBLISHING
for CSIRO Australia and
the Australian Academy of Science

Aust. J. Phys., 1997, **50**, 263–79

Control of Chaos in the Rössler System

Stuart Corney

Department of Physics, University of Tasmania,
GPO Box 252C, Hobart, Tas. 7018, Australia.

Abstract

The control method of Ott, Grebogi and Yorke (1990) as applied to the Rössler system, a set of three-dimensional non-linear differential equations, is examined. Using numerical time series data for a single dynamical variable the method was successfully employed to control several of the unstable periodic orbits in a three-dimensional embedding of the data. The method also failed for a number of unstable periodic orbits due to difficulties in linearising about the orbit or the tangential coincidence of the stable manifold and the motion of the orbit with external parameter.

1. Introduction

The proposal by Ott, Grebogi and Yorke (OGY) (1990) for a method of controlling chaos in experimental systems has stimulated much work in the area, resulting in the control of chaos in many situations, a number of new methods for controlling chaos based on the work of OGY, and a way of perturbing a chaotic system so that it quickly enters a small, desired region of phase space (targetting). For a review of this work see, for example, Shinbrot *et al*. (1993). Most of this work has been of a theoretical nature, with some applications to real experimental systems and other test applications using model dynamical systems, but there has to date been no systematic study of the different methods, their advantages and disadvantages. This paper seeks to do just that for the OGY method. It describes a concrete implementation of the procedure and discusses the successes and difficulties that arise.

In this paper the method of OGY is applied to the Rössler (1976) system, a set of three first order differential equations. Continuous systems have been controlled using a number of hybrid methods (Pyragas 1992; Kittel *et al*. 1992; Azevedo and Rezendo 1991; Roy *et al*. 1992), but no attempt has been made to control chaos in such a system using the original method of OGY.

This method is described in the original paper (Ott *et al*. 1990; see also Corney 1995) and shall not be repeated in full here. It assumes a continuous experimental time series of a scalar variable. A continuous delay coordinate vector is formed

$$\mathbf{u}(t) = \{x(t), x(t-T), \ldots, x(t-(d-1)T)\}, \tag{1}$$

10.1071/PH96015 0004-9506/97/020263$10.00

where for the Rössler system $d = 3$ and the delay T is determined from topological considerations (Liebert *et al.* 1991). It is also assumed there is a parameter which can be varied in a small range about some nominal value.

The system is then discretised by way of a Poincaré section. The result is a set of points $\xi_1, \xi_2, \ldots, \xi_k$ where ξ_n denotes the coordinates on the section of the n^{th} piercing by the orbit $\mathbf{u}(t)$. The most common choice for the Poincaré section is $x(t-(d-1)T) = \text{const}$, and therefore

$$\xi_n = \{x(t_n), x(t_n - T), \ldots, x(t_n - (d-2)T)\}, \tag{2}$$

where t_n is the time of the n^{th} piercing. From this set of points the location and stability properties of the orbit can be determined (Lathrop and Kostelich 1989).

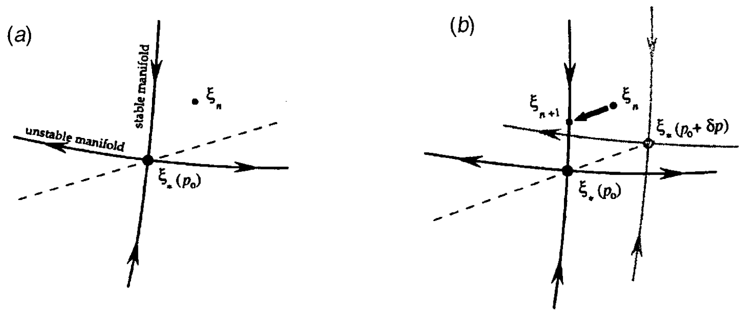


Fig. 1. (*a*) Period one point $\xi_F(p_0)$, its stable and unstable manifold, and the line (dashed) along which ξ_F moves if a small perturbation is applied. (*b*) Result of perturbing p to $p+\delta p$; the stable and unstable manifolds of $\xi_F(p_0+\delta p)$ are shown as grey lines through $\xi_F(p_0+\delta p)$.

For illustrative purposes, consider a period one orbit on a two-dimensional Poincaré section; the extension to higher periods or dimensions is straightforward. In the vicinity of such a fixed point the future path of iterates is directed by the positions of the stable and unstable manifolds. For a typical initial condition a trajectory will wander chaotically until it enters the region near the fixed point where its behaviour is dominated by the stable and unstable manifolds. The idea, as seen in Fig. 1, is to to alter a system parameter, and thus the location of the two manifolds, so that the next iterate is directed onto the stable manifold of the unperturbed fixed point. The system parameter is then returned to its original value and the subsequent iterates now lie on the stable manifold of the fixed point and so approach that point. Thus control is achieved.

The size of the perturbation, as given by OGY is

$$p_n = \frac{\lambda_u}{\lambda_u - 1} \frac{\xi_n \cdot f_u}{g \cdot f_u}, \tag{3}$$

where λ_u is the unstable eigenvalue, f_u is the contravariant basis vector corresponding to the unstable eigenvector e_u and g is a vector which approximates the movement of the fixed point with small perturbations.

For a noisy system the method is still valid, except that once the trajectory enters the region where perturbations may be applied repeated perturbations are required as the noise will necessarily knock a trajectory off the stable manifold of the fixed point.

This is the method devised by OGY to control chaos. Unfortunately there are two practical difficulties encountered when implementing the method; firstly in theory all fixed points can be approximated by a linear system in some vicinty of the point, although in practice this linearisation may be very difficult to obtain. Secondly the time taken for the trajectory to enter the region where perturbations can be applied could be rather long. The first of these flaws is without a solution so far and shall be discussed further in Section 3. The second formed the genesis of the targetting method developed subsequently (Shinbrot *et al*. 1990).

2. Linear Estimation

The control method of OGY can be applied to an experimental system in which the equations governing the motion are unknown, indeed even the dimension of the system is not initially required. The Rössler system is a mathematical system, but throughout this paper it has been treated as an experimental system where the equations of motion are not known. After consideration of the system the control parameter was chosen to be b, whilst the dynamical variable used was the x coordinate. For an explanation of these choices see Corney (1995)

In keeping with the original method proposed by OGY the Rössler system is written

$$\begin{aligned} \frac{dx}{dt} &= -y - z, \qquad (4)\\ \frac{dy}{dt} &= x + ay, \\ \frac{dz}{dt} &= c + z(x - b - p). \end{aligned}$$

The control parameter is, therefore, p with a nominal value $p = p_0 = 0$.

The fundamental period of the system can be determined by measuring the time between intersections of the trajectory and a plane and then averaging this time for a great many such intersections. The averaging process was carried out by allowing one hundred randomly-generated initial conditions to evolve over a time interval of 1000 s. This process showed that there is little variation in the time between intersections and yielded an overall average period of $\tau = 5{\cdot}84$ s.

The stability properties of each periodic orbit must now be determined. This involves finding a linear fit to the dynamics around each point near an unstable periodic orbit. Following the method of OGY, the scalar time series $x(t)$ is embedded in a three-dimensional phase space

$$\mathbf{u}(t) = \{x(t), x(t-T), x(t-2T)\}, \qquad (5)$$

where $T = 0{\cdot}83$, as calculated by Liebert *et al*. (1991). This is then discretised by way of a Poincaré section. For this experiment the section took the form of the plane $x(t-2T) = 0$, so the series becomes two-dimensional with coordinates

$$\xi_n = \{x(t_n), x(t_n - T)\}. \qquad (6)$$

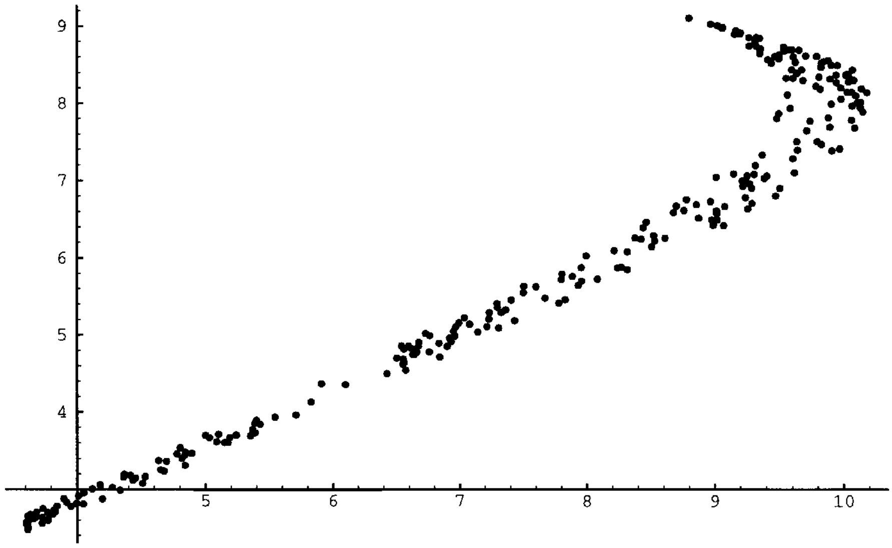


Fig. 2. Poincaré return map for the delay coordinates using the time delay of Liebert *et al.* (1991).

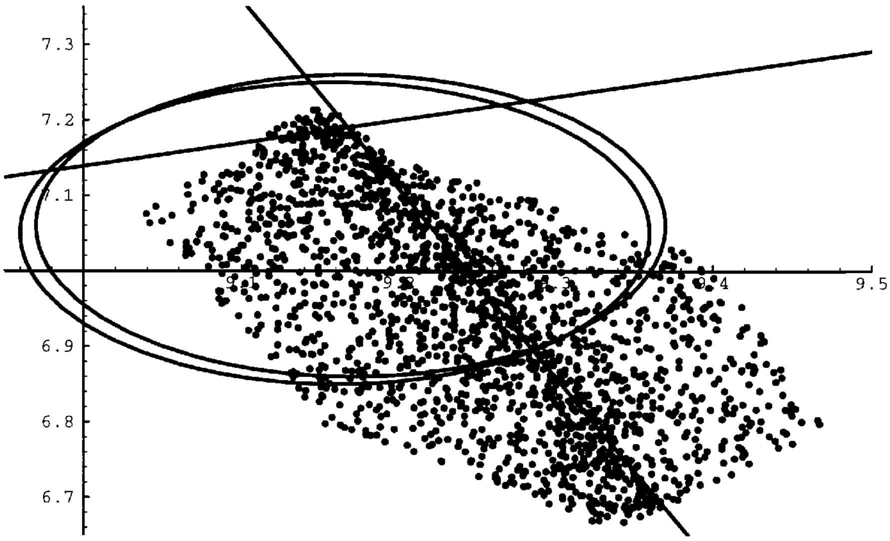


Fig. 3. Set of points on the return map attractor which have a first return number of $m = 1$ together with the stable and unstable manifolds superimposed on the position of the unstable fixed point.

The result of this process is a set of points which lies in a region which is almost linear, save for a hook at one end. The Poincaré section for a run of 1500 s can be seen in Fig. 2. Although the time delay as suggested by Liebert *et al.* (1991) was used, the shape of the attractor agrees well with that suggested by Fraser and Swinney (1986). From examining Fig. 2 it can be seen that the highest density of points occurs on the hook of the attractor. In fact for a time series of 34000 points about 10000 points (29%) are in the region of the hook. It is probable that much of the stretching and folding occurs in this high density region and different unstable periodic orbits may be difficult to distinguish. This will be confirmed later in this section. For this reason it is unlikely that on this hook it will be possible to accurately estimate the dynamics of the system. If perturbations need to be applied on the hook then more sophisticated methods than those considered here will have to be employed.

For a time series of more than 34000 points, the first return number m of each point was calculated. Table 1 shows the number of points of each period up to period 15 for the unperturbed system ($b = 5{\cdot}7$). Of the 34000 points in the time series, in excess of 28000, or 84%, have a period ≤ 15.

Table 1. Number of points with each m value up to 15 for the unperturbed ($p = 0$) and perturbed system ($p = 0{\cdot}005$)

The time series was 34100 points long. 84% of points have an m value less than 15

Period	No. of points		Period	No. of points	
	$p = 0$	$p = 0{\cdot}005$		$p = 0$	$p = 0{\cdot}005$
1	2015	1797	8	2457	2553
2	1301	1168	9	1597	1689
3	8447	8532	10	1466	1222
4	882	1114	11	1644	1688
5	1339	1251	12	883	845
6	2908	3323	13	1091	878
7	849	735	14	1004	1223
			15	673	591

For the control algorithm used it is also necessary to calculate the position of the periodic orbits for a system which is slightly perturbed from the original. The first return number m of a time series with $b = 5{\cdot}705$ (i.e. $p = 0{\cdot}005$) is also shown in Table 1.

Lathrop and Kostelich (1989) proposed in their paper that the stability properties could be estimated from an average of the linear fits of neighbourhoods around each point with a given m value. Fig. 3 shows the set of points with $m = 1$; it can be clearly seen that they are all clustered together and that the unstable fixed point must be somewhere in the cluster. This figure also indicates that it is unnecessary to perform a linear fit around each point in the set, as the neighbourhoods around different points will contain almost entirely the same points. Therefore, instead of performing a large number of linear fits (one for each point with $m = 1$), it was thought that one linear fit using almost all points (as not quite all points are within the cluster shown) with $m = 1$ would suffice. This method has an advantage over that proposed by Lathrop and Kostelich (1989); their method included points with a different m value in the linear fit around each reference point being considered. These points are not directly under the influence of the period one orbit (as $m \neq 1$) and are probably more influenced by another periodic orbit. It was therefore thought that the fitting technique would be corrupted by these points. The proposed method uses only points with the correct m value, which are expected to be directly under the influence of the period m orbit.

Table 2. Stability properties of the period one orbit

Basin 1	$n = 2005$
$\xi_F = \{9\cdot16762, 7\cdot19014\}$	$\xi_{F(p)} = \{9\cdot10635, 7\cdot36531\}$
$A = \begin{pmatrix} -2\cdot00384 & -1\cdot14477 \\ -0\cdot866581 & 0\cdot502326 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 35\cdot7691 \\ 11\cdot522 \end{pmatrix}$
$\lambda_u = -2\cdot35146$	$e_u = \{0\cdot372303, -0\cdot928111\}$
$\lambda_s = 0\cdot849947$	$e_s = \{-0\cdot956857, -0\cdot290559\}$

(2a) Period One

For the period one orbit a least squares fit was done to find the best fit of

$$\xi_{n+1} = A\xi_n + \mathbf{b}\,, \tag{7}$$

where ξ_n is a period one point. Ten of the period one points were not used in this fitting as these points did not occur in the central cluster (which from now on will be called the basin of the period one orbit) and so are probably anomalies in the attractor. The result of this fitting can be seen in Table 2. The unstable fixed point can be determined by solving for the fixed point of the linear approximation, i.e.

$$\xi_F = A\xi_F + \mathbf{b}\,, \tag{8}$$

where A and $\mathbf{b}$ are the matrix and vector determined by the linear fit. The perturbed system has almost identical stability properties, but the slightly different fixed point $\xi_F(p)$ is crucial to the control algorithm and so both are given in Table 2.

Fig. 3 shows a superposition of the eigenvectors, the unstable fixed point (by the linear fit method) and the period one basin. The fixed point as determined by the linear fit is on the edge of the cluster of points making up the period one basin. This is not what would be expected and suggests that the numerical linearisation may not be a good approximation to the true linearisation in the vicinity of the unstable fixed point.

(2b) Period Two

For the $m = 2$ points one would expect two basins, corresponding to the two points of a period two orbit. As can be seen in Fig. 4, there are four basins, containing 233, 73, 270 and 176 points respectively. This is most likely to

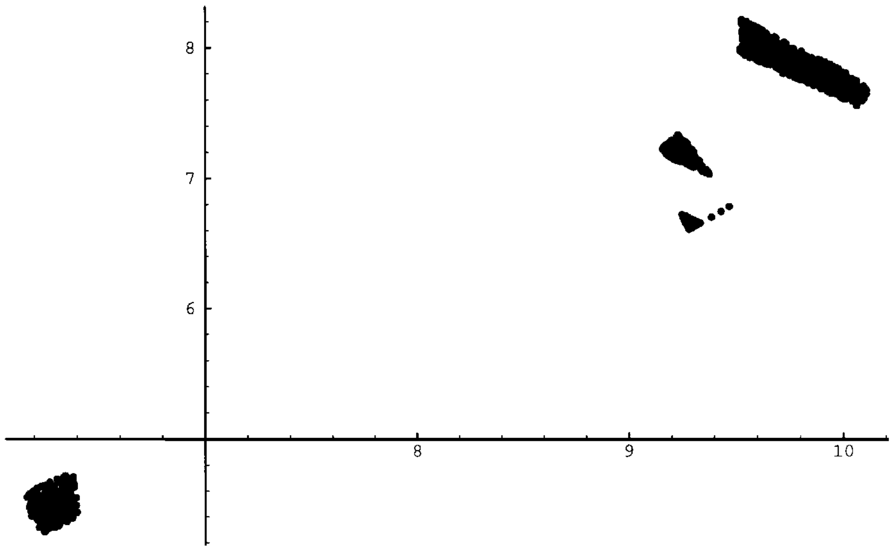


Fig. 4. Four basins of the $m = 2$ set. The two outer basins are from the one orbit, as are the two inner basins.

correspond to two distinct period two orbits on the attractor. A linear fitting to each of these basins, using the initial point and its image two strikes of the section later, confirms this. The stability data of the four separate basins can be seen in Table 3. Two pairs exist with similar eigenvalues and eigenvectors, but are noticeably different from the remaining two basins. The eigenvalues and eigenvectors of an unstable periodic orbit are constant on that orbit, thus implying that there exist two period-two orbits in the Rössler system. These orbits shall be called the $p21$ and $p22$ orbits.

(2c) Period Three

There are five distinct basins of period three points. The linear fitting technique applied to each of these yields the results shown in Table 4. As can be seen, there exist two pairs of basins (corresponding to two orbits) and a fifth basin, which lies on the hook region of the attractor. As was stated, it is unlikely that an accurate linear fit can be employed in this basin, a situation reflected in the results of the linear fit. In all probability this basin is actually a mixture of the remaining two basins of the two period-three orbits identified from the first four basins.

(2d) Higher Periods

All the higher periods (sets of points with an m value greater than three) have points on the hook region of the attractor; the set of period four points consists of four basins, three of which appear to be accurately represented by the linearisation (see Table 5) and a fourth on the hook. For some m values over half of the points are on the hook, whilst for others the percentage is much lower. Any point on the hook is lost to the linear analysis used, and therefore represents useless data. If a serious attempt were to be made to control higher periods of the Rössler system then a method would have to be devised which

Table 3. Stability properties of the period two orbit
The total number of period two points was 1301

$p21$	$n = 233$
$\xi_F = \{6\cdot 28867, 4\cdot 47693\}$	$\xi_{F(p)} = \{6\cdot 28485,\ 4\cdot 47739\}$
$A = \begin{pmatrix} -3\cdot 3485 & -0\cdot 248862 \\ -2\cdot 18873 & -0\cdot 109648 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 28\cdot 4439 \\ 18\cdot 7242 \end{pmatrix}$
$\lambda_u = -3\cdot 50874$	$e_u = \{-0\cdot 840773,\ -0\cdot 541387\}$
$\lambda_s = 0\cdot 0505987$	$e_s = \{0\cdot 0730188,\ -0\cdot 997331\}$
$p21$	$n = 176$
$\xi_F = \{9\cdot 69457, 8\cdot 03744\}$	$\xi_{F(p)} = \{9\cdot 64709,\ 8\cdot 10376\}$
$A = \begin{pmatrix} -1\cdot 63348 & -2\cdot 02754 \\ -1\cdot 41098 & -0\cdot 805985 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 41\cdot 8267 \\ 28\cdot 1941 \end{pmatrix}$
$\lambda_u = -2\cdot 96098$	$e_u = \{-0\cdot 836628,\ -0\cdot 547771\}$
$\lambda_s = 0\cdot 521518$	$e_s = \{0\cdot 68524,\ -0\cdot 728318\}$
$p22$	$n = 73$
$\xi_F = \{9\cdot 19793, 7\cdot 48998\}$	$\xi_{F(p)} = \{9\cdot 13803,\ 7\cdot 52794\}$
$A = \begin{pmatrix} 5\cdot 03717 & 0\cdot 912868 \\ 0\cdot 969308 & 1\cdot 10125 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} -43\cdot 9668 \\ -9\cdot 67314 \end{pmatrix}$
$\lambda_u = 5\cdot 25043$	$e_u = \{0\cdot 973781,\ 0\cdot 227489\}$
$\lambda_s = 0\cdot 887995$	$e_s = \{-0\cdot 214873,\ 0\cdot 976642\}$
$p22$	$n = 270$
$\xi_F = \{9\cdot 43883, 6\cdot 60771\}$	$\xi_{F(p)} = \{9\cdot 488,\ 6\cdot 52707\}$
$A = \begin{pmatrix} 3\cdot 88803 & 1\cdot 46086 \\ 2\cdot 77427 & 2\cdot 20076 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} -36\cdot 9126 \\ -34\cdot 1203 \end{pmatrix}$
$\lambda_u = 5\cdot 22719$	$e_u = \{0\cdot 737147,\ 0\cdot 675733\}$
$\lambda_s = 0\cdot 861607$	$e_s = \{-0\cdot 434707,\ 0\cdot 900572\}$

could accurately determine the linearisation of these unstable periodic points. This supports the claim that the hook is a region of much folding and stretching as a number of different periodic points are mixed together, making effective linearisation very difficult.

Another phenomenon which becomes more pronounced with higher period orbits is the presence of isolated points of a given m value, or of very small basins (<30 points). These are also useless data points, as the dynamics around a small set cannot be accurately estimated by any fitting procedure. Nevertheless, for all m values up to period ten ($m = 10$), which was the highest period examined, there existed basins large enough so that the dynamics in that region could be estimated, although for many of these basins the linearised dynamics did not appear to accurately represent the actual behaviour of a trajectory in the neighbourhood.

It was hoped that through estimating the Lyapunov exponents of the first fifteen unstable periodic orbits an estimate of the Lyapunov exponents of the system as a whole could be achieved (Lathrop and Kostelich 1989). However, due to the significant proportion of points for which the numerical linearisation method failed this was not possible.

Table 4. Stability properties of the period three orbit
The total number of points in the period three orbit was 8447

$p31$ good	$n = 1265$
$\xi_F = \{3\cdot 99033, 2\cdot 83285\}$	$\xi_{F(p)} = \{3\cdot 99018,\ 2\cdot 83534\}$
$A = \begin{pmatrix} -2\cdot 10545 & -0\cdot 0742427 \\ -1\cdot 4763 & -0\cdot 102041 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 12\cdot 6021 \\ 9\cdot 01285 \end{pmatrix}$
$\lambda_u = -2\cdot 15874$	$e_u = \{-0\cdot 812381,\ -0\cdot 583128\}$
$\lambda_s = -0\cdot 05$	$e_s = \{0\cdot 0360745,\ -0\cdot 999349\}$
$p32$ good	$n = 688$
$\xi_F = \{2\cdot 8352, 3\cdot 43558\}$	$\xi_{F(p)} = \{4\cdot 84045,\ 3\cdot 43558\}$
$A = \begin{pmatrix} 4\cdot 86774 & 0\cdot 234634 \\ 3\cdot 28956 & 0\cdot 286732 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} -19\cdot 5025 \\ -13\cdot 4552 \end{pmatrix}$
$\lambda_u = 5\cdot 02948$	$e_u = \{0\cdot 821696,\ 0\cdot 569927\}$
$\lambda_s = 0\cdot 12399$	$e_s = \{-0\cdot 0494117,\ 0\cdot 998778\}$
$p31$ good	$n = 987$
$\xi_F = \{7\cdot 18809,\ 5\cdot 16694\}$	$\xi_{F(p)} = \{7\cdot 1899,\ 5\cdot 16851\}$
$A = \begin{pmatrix} -2\cdot 10629 & -0\cdot 21629 \\ -1\cdot 68049 & -0\cdot 137017 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 23\cdot 4459 \\ 17\cdot 9548 \end{pmatrix}$
$\lambda_u = -2\cdot 27621$	$e_u = \{-0\cdot 786361,\ -0\cdot 617767\}$
$\lambda_s = 0\cdot 0328102$	$e_s = \{0\cdot 100599,\ -0\cdot 994927\}$
$p32$ fair	$n = 614$
$\xi_F = \{8\cdot 51994,\ 6\cdot 28903\}$	$\xi_{F(p)} = \{8\cdot 52524,\ 6\cdot 30563\}$
$A = \begin{pmatrix} 4\cdot 48625 & 0\cdot 949183 \\ 3\cdot 51294 & 0\cdot 943364 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} -35\cdot 6397 \\ -29\cdot 5721 \end{pmatrix}$
$\lambda_u = 5\cdot 25585$	$e_u = \{0\cdot 775318,\ 0\cdot 631571\}$
$\lambda_s = 0\cdot 170162$	$e_s = \{-0\cdot 214956,\ 0\cdot 976624\}$
basin 5 very bad	$n = 4892$
$\xi_F = \{9\cdot 27156,\ 8\cdot 06643\}$	$\xi_F(p) =$ undetermined
$A = \begin{pmatrix} 0\cdot 787045 & 0\cdot 199019 \\ -0\cdot 070614 & 0\cdot 802149 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 0\cdot 369047 \\ 2\cdot 25066 \end{pmatrix}$
$\lambda_u = 0\cdot 794597 + 0\cdot 118\mathrm{i}$	$e_u = \{0\cdot 859133,\ 0\cdot 0325997 + 0\cdot 510\mathrm{i}\}$
$\lambda_s = 0\cdot 794597 - 0\cdot 118\mathrm{i}$	$e_s = \{0\cdot 859133,\ 0\cdot 0325997 - 0\cdot 510\mathrm{i}\}$

3. Control

Once the stability properties and movement of the periodic points have been determined for a given orbit it should be a simple matter of applying small perturbations to the system in accordance with the control algorithm of OGY to control chaos in the Rössler system. In this section control will be attempted for the six orbits with period four or less.

However, as may be expected, the situation is not so straightforward. For the procedure to work, a number of assumptions are made about the dynamics of the system, for example that the fixed point does not move in a direction parallel to the stable manifold. The most important assumption is that the motion of a trajectory near an unstable fixed point can be accurately modelled by the

Table 5. Stability properties of the period four orbit

The total number of period four points was 882

$p41$ good	$n = 49$
$\xi_F = \{5{\cdot}61418,\ 4{\cdot}00583\}$	$\xi_{F(p)} = \{5{\cdot}61782,\ 4{\cdot}0039\}$
$A = \begin{pmatrix} -15{\cdot}9867 & -1{\cdot}02419 \\ -11{\cdot}7349 & -0{\cdot}530427 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 99{\cdot}4692 \\ 72{\cdot}0126 \end{pmatrix}$
$\lambda_u = -16{\cdot}7287$	$e_u = \{-0{\cdot}80982,\ -0{\cdot}586679\}$
$\lambda_s = 0{\cdot}211551$	$e_s = \{0{\cdot}0631022,\ -0{\cdot}998007\}$
$p42$ good	$n = 197$
$\xi_F = \{8{\cdot}69834,\ 6{\cdot}42493\}$	$\xi_{F(p)} = \{8{\cdot}70641,\ 6{\cdot}4287\}$
$A = \begin{pmatrix} -9{\cdot}907434 & -2{\cdot}2266 \\ -9{\cdot}949169 & -2{\cdot}13532 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 101{\cdot}936 \\ 102{\cdot}706 \end{pmatrix}$
$\lambda_u = -11{\cdot}364$	$e_u = \{-0{\cdot}697116,\ -0{\cdot}716958\}$
$\lambda_s = 0{\cdot}15466$	$e_s = \{0{\cdot}234532,\ -0{\cdot}972108\}$
$p43$ good	$n = 242$
$\xi_F = \{9{\cdot}43431,\ 7{\cdot}20418\}$	$\xi_{F(p)} = \{9{\cdot}43912,\ 7{\cdot}21197\}$
$A = \begin{pmatrix} -9{\cdot}3005 & -4{\cdot}66418 \\ -11{\cdot}0697 & -5{\cdot}05809 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 130{\cdot}775 \\ 148{\cdot}079 \end{pmatrix}$
$\lambda_u = -14{\cdot}671$	$e_u = \{-0{\cdot}655678,\ -0{\cdot}75504\}$
$\lambda_s = 0{\cdot}312896$	$e_s = \{0{\cdot}436528,\ -0{\cdot}899691\}$
basin 4 bad	$n = 381$
$\xi_F = \{9{\cdot}63213,\ 8{\cdot}27891\}$	$\xi_{F(p)} = \{9{\cdot}67418,\ 8{\cdot}27685\}$
$A = \begin{pmatrix} 0{\cdot}467671 & -0{\cdot}195715 \\ -1{\cdot}58287 & -1{\cdot}27174 \end{pmatrix}$	$\mathbf{b} = \begin{pmatrix} 6{\cdot}74777 \\ 34{\cdot}0539 \end{pmatrix}$
$\lambda_u = -1{\cdot}43457$	$e_u = \{0{\cdot}120345,\ 0{\cdot}994749\}$
$\lambda_s = 0{\cdot}630525$	$e_s = \{0{\cdot}768688,\ -0{\cdot}639624\}$

linear least squares technique employed. This assumption, although integral to the procedure, is not always valid, and when it is not the method of OGY has little or no controlling influence on the system.

This section reports on attempts made to control the low period unstable orbits of the Rössler system and the difficulties encountered in doing this. It will be shown that of the first six unstable periodic orbits, only two can be fully controlled using the linear approximation of OGY, while a third orbit can be controlled for a limited time.

(3a) Period One

The stability properties of the unstable period one orbit (fixed point) are given in Table 2. Because of the ever-present noise in the system (caused by switching when changing parameters and the limitations of numerical solutions) it was necessary to apply repeated perturbations, i.e. once the trajectory has entered the region where the calculated perturbations are suitably small, a recalculated perturbation is applied on every piercing of the Poincaré section.

A control algorithm was devised as follows. The scalar variable $x(t)$ is embedded and a Poincaré section is taken; the system is then transformed so that the fixed point, as calculated by the linear fit, is at the origin. A trajectory is allowed to evolve and, as each new point on the section is formed, its magnitude is determined. When a point is found which is sufficiently close to the origin, the perturbation required to nudge it onto the stable manifold is calculated. The external parameter is then changed by this amount and the trajectory of this new system is allowed to evolve until it restrikes the section. This process of calculating the perturbations and allowing the system to evolve is repeated for as long as control is desired, or until the required perturbation becomes too large, i.e. the trajectory trajectory is knocked out of the controllable region.

For the period one orbit the maximum perturbation was set at $p_{max} = 0\cdot 3$. This corresponds to approximately 5% of the nominal value of b. A change of this magnitude has no obvious effect on the system (Corney 1995). If the control is effective, then the perturbations should quickly decrease in size. The neighbourhood inside which the initial point must lie for the perturbations to start was chosen to be a circle around the origin of radius $0\cdot 1$. This is smaller than is necessary using equation (2) given by OGY.

The OGY method failed to control the period one orbit of the Rössler system. The controlling perturbations appeared to have no stabilising effect whatsoever, and in fact only three to five perturbations were applied before a randomly chosen chaotic trajectory wandered out of the controllable region. This is not an unexpected result; recall that the fixed point for the linear fit was on the edge of the basin. This would seem to indicate that the linear fitting was not accurate and therefore may pose a problem when control methods are applied. In short, the period one orbit may not be controllable.

The issue of controllability has long been an area of interest in engineering, but almost all this work has dealt with unstable *linear* systems. An unstable linear system is completely controllable provided it satisfies an established criterion, known as controllability (Ogata 1990). A comprehensive treatment of linear control can be found in a range of books, for example, Ogata (1990) or Van de Vegte (1990).

The problem of controlling the period one orbit of the Rössler system is, however, not one of linear controllability; the unstable linear system corresponding to the period one orbit can be easily controlled using the same perturbations as those calculated for the Rössler system. The control method of OGY is not effective for the unstable period one orbit of the Rössler system and this is primarily because of the failure of the linearisation method used, resulting in a poor representation of the true linearisation of the dynamics. This can be seen by defining the set of images of the points in the basin under the Ròssler system as the 'real images', and the set of images of points in the period one basin under the corrsponding linear system, as the 'linear images'. If the Rössler system is close to linear about the fixed point then these sets should be almost identical. Fig. 5 is a plot of the real and linear images, with a line drawn between the corresponding points in the two sets. The distance between almost all pairs of points is quite small, but for points which exist where the real stable manifold would be expected to lie (down the centre of the basin) the real and linear images are on opposite sides of the cluster. This is rather disturbing, for it is near the stable manifold that the perturbations are applied and to which the system tends upon their application. Consequently it is especially important for

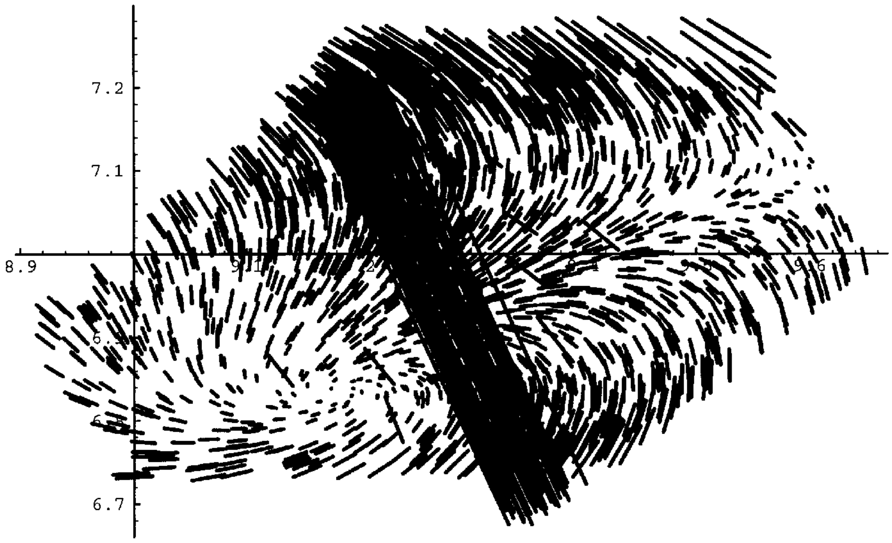


Fig. 5. Comparison of the real and linear images. A line is drawn between the corresponding points in each set. All pairs are reasonably close except those near the centre of the set (near the stable manifold).

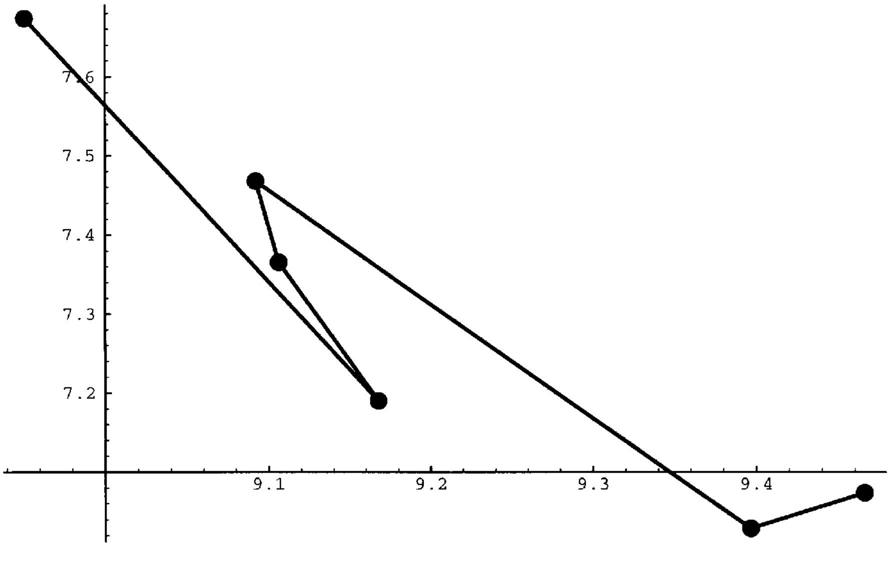


Fig. 6. OGY assume that the fixed point $\xi_F(p)$ moves along a line with small changes in p. This figure shows the location of $\xi_F(p)$ for $p = -0 \cdot 01$ (top left), $0, 0 \cdot 005, 0 \cdot 02, 0 \cdot 1, 0 \cdot 2$, and as can be seen they are far from co-linear.

the linear approximation to hold in this region. As it does not, the control is unlikely to be successful.

Another indicator of the non-linear nature of the fixed point can be seen in the motion of the calculated fixed point with external parameter. This corresponds to the vector **g** in OGY's paper. The assumption of linearity implies that the fixed point $\xi_F(p)$ moves in a straight line (for small variations of p). Fig. 6 gives the location of $\xi_F(p)$ for six values of p; the motion of $\xi_F(p)$ is far from linear. As can be seen the vector **g** changes radically with slight perturbations, and so the equation given by OGY, which relies on **g**, is invalid. In fact, the calculated fixed point is *outside* its corresponding basin for any small perturbation.

In conclusion, the attempt to control the unstable period one orbit of the Rössler system was doomed to failure. The reason for this was the non-linearity of the dynamics of the system in the neighbourhood of the said fixed point. If control of this fixed point is essential then a method which uses polynomial fitting, or some other higher order technique, is necessary so that the stable and unstable manifolds can be successfully modelled.

(3b) Period Two

The set of $m = 2$ points consisted of two unstable period two orbits. As these orbits are distinct they will be controlled separately, starting with the $p21$ orbit which consisted of basins 1 and 4.

For a period two or higher cycle the control algorithm is almost identical to the period one case; for simplicity the system is transformed so that one of the points is at the origin and the system is allowed to evolve. When a point suitably close to the origin occurs, the control perturbation which will nudge the double iterate of this point onto the stable manifold of the origin is determined and applied to the system. The next strike of the Poincaré section is near the second point of the period two orbit, and so the perturbation required to bring the double iterate of this point onto the second point's stable manifold is applied. This two step process is repeated for as long as control is desired, or until the trajectory is forced away from the orbit by non-linearities in the system.

The neighbourhood of the origin was chosen to be a circle of radius $0\cdot 1$ and the maximum perturbation $p_{max} = 0\cdot 3$. Once again, if control is effective the perturbations should quickly decrease in size.

The attempt to control the $p21$ orbit was highly successful. Fig. 7 shows the evolution of the scalar x coordinate with time. As can be seen, the system is controlled almost instantly and kept there for as long as is desired. The controlled orbit is somewhat noisy; the second point of the orbit has variations around the fixed point of amplitude $\approx 0\cdot 3$. This is reasonably large and may be a problem if the controlled period two orbit is necessary in an experimental situation. The first point of the orbit has very little noise.

An interesting feature of the $p21$ orbit is that the control is just as effective with either one or two perturbations applied each period. Even the noise level does not change significantly. The most likely cause of this is a slightly inaccurate linear fit of the second point (Corney 1995).

The initial perturbation is quite large ($\approx 0\cdot 3$), but by the third iterate the applied perturbation has fallen to $\approx 0\cdot 02$. By the seventh iteration the calculated perturbations have settled down into an almost periodic variation around $\delta p = 0$

with an amplitude of approximately $0 \cdot 02$. Again this suggests a slightly inaccurate linear fit.

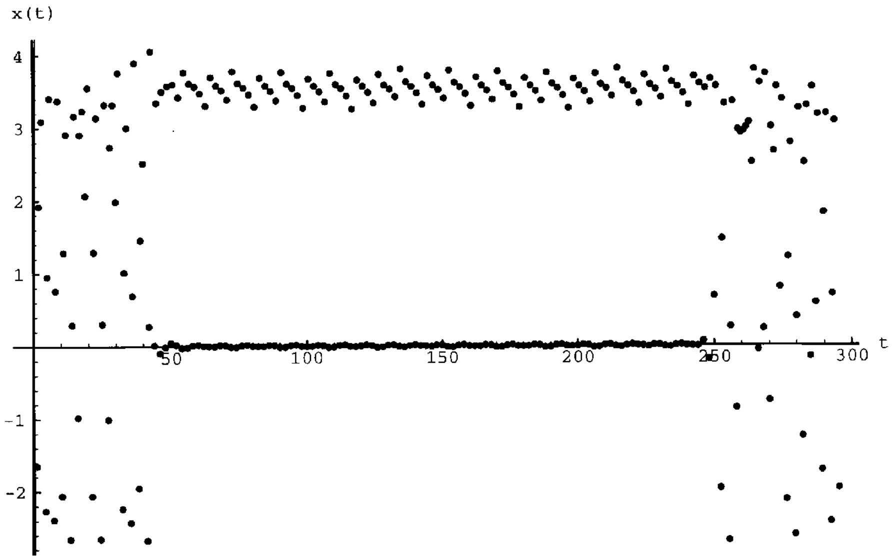


Fig. 7. Controlled $p21$ orbit. The first perturbation is applied at $t = 48$ s and the perturbations are stopped at $t = 248$ s.

The second period two orbit $p22$ is unlikely to be controllable as the linearised fixed points are both well out of their respective basins. This indicates non-linear behaviour near the orbit, perhaps even on a greater scale than the period one orbit. Consequently a trajectory near the linearised fixed point will not be period two and the dynamics near the real fixed point cannot be linear. The attempt to control the $p22$ orbit was unsuccessful. Due to the inaccurate location of the fixed point the system was actually displaying period one behaviour when the perturbations were applied.

(3c) Period Three

The control of a period three orbit is a straightforward extension of the technique for a period two orbit, except that the order of visitation of the points must be known. This can easily be determined by studying the behaviour of a point which has an $m = 3$ value. This shows that the third point of each orbit is indeed on the hook. The fact that the stability properties of only two points of each period three orbit are known should not be a problem as it was shown for the period two orbit that control can be effectively achieved when one of the points is not known.

The attempt at controlling the $p32$ orbit was successful, as can be seen in Fig. 8. For the third point of the orbit the system was left in the perturbed state as calculated using the previous point. The controlled period three orbit was far less noisy than the period two orbit, although the second and third points do have an almost periodic variation in their location. This variation is thought to be indicative of slightly inaccurate linear fitting as well as noise in the system.

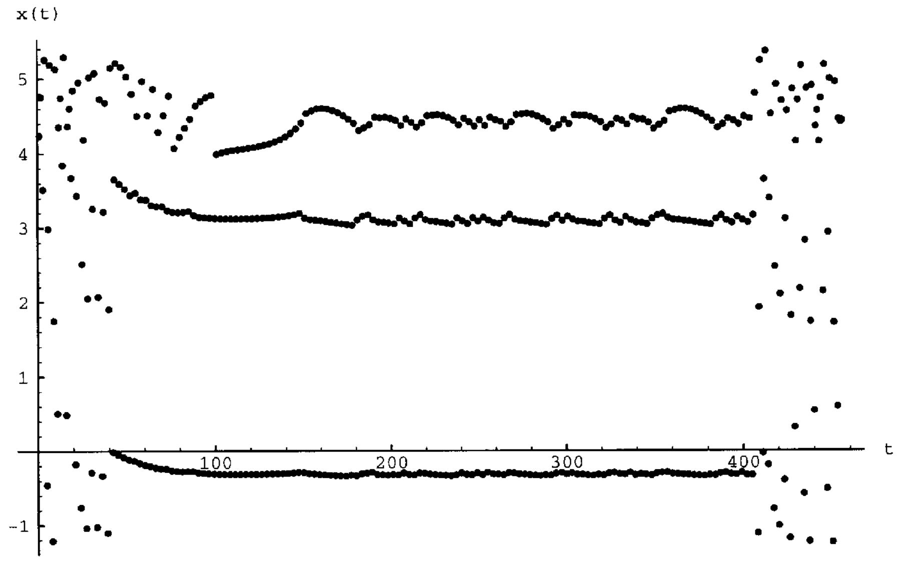


Fig. 8. Controlled $p32$ orbit. Perturbations are first applied at $t = 61$ s and stopped at $t = 405$ s.

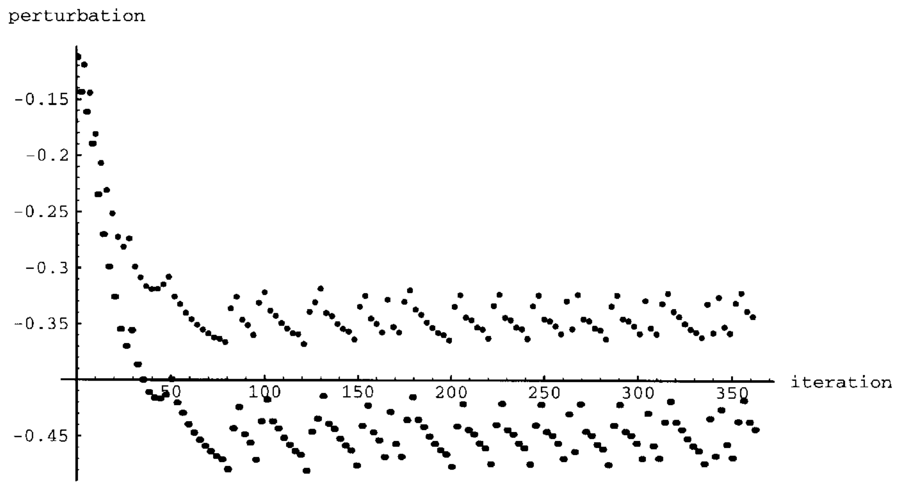


Fig. 9. Perturbations that are applied to control the $p32$ orbit. Note the oscillation of the two perturbations about a non-zero value, indicative of a slight error in the linear fit.

The movement of the controlled orbit away from the location of the $p32$ orbit of the unperturbed system is due, once again, to a slightly inaccurate linear fit; the perturbations actually nudge the system towards an unstable period three orbit of the Rössler system which corresponds to a slightly different parameter value. This can be seen in Fig. 9 by the convergence of the two perturbations towards a non-zero value.

The $p31$ orbit could not be successfully controlled. The reason for this is the failure of the system to satisfy one of the assumptions of the OGY method; this is the assumption that the movement of the fixed point with external control parameter is not parallel to the stable manifold. This results in any required perturbation being large (as the distance between the iterate and the stable manifold remains almost constant with changes in parameter), which is invalid as it changes the system in a fundamental way.

(3d) Period Four

The period four orbit consists of the three linearisable basins and the fourth basin which lies on the hook. The control method used was almost identical to the period three method. The maximum perturbation was set at $p_{max} = 0 \cdot 5$, and due to the small size (in phase space) of each of the basins the control neighbourhood was set with a radius of $0 \cdot 05$.

The OGY method was partially successful at controlling the period four orbit; a trajectory could be kept on the unstable orbit for an interval of the order of twenty periods, but in this time would slowly move further away from the real period four orbit and so larger perturbations were necessary to keep it behaving as a period four orbit. Eventually the required perturbations became too large and the system became uncontrolled.

4. Conclusion

All unstable periodic orbits up to and including period 10 were examined and the stability properties for all basins of these periods were approximated using a linear fitting technique (Corney 1995). For some orbits this process seemed to accurately reflect the dynamics, but for others the linearisations were at best suspect. As the period increased less points on a given orbit could be accurately approximated and so control was not attempted past period four. The control procedure has been shown to be effective in certain cases, as specified by OGY in their original paper (Ott *et al.* 1990). The algorithm is not effective when a good linear fit to the dynamics around a periodic orbit cannot be found, or when the fixed point $\xi_F(p)$ moves parallel to the stable manifold. Neither OGY, Lathrop and Kostelich (1989), nor any other author (see Shinbrot *et al.* 1993 for a review) give any hint that these problems may be a common occurrence; however two of the first six periodic orbits of the Rössler system exhibited such poor linearisations as to render the control method ineffective and a third had the problem of the motion of its fixed point being parallel to the stable manifold. Only two orbits ($p21$ and $p32$) could be controlled for any length of time, whilst the remaining orbit ($p4$) was controllable for only a finite time.

In conclusion, the method of OGY to control unstable periodic orbits in a strange attractor is sound, but only when the dynamics of the system around such an orbit satisfy the assumptions made in their original paper (Ott *et al.* 1990). Unfortunately these assumptions (especially that of being able to successfully linearise around the orbit) are *not* satisfied in many cases.

In order to improve the situation it is necessary to be able to determine the manifolds about unstable orbits that are not easily linearisable. This may involve developing a generalised least squares technique involving polynomial fitting for two, or higher, dimensional data and then being able to obtain the manifolds and

Lyapunov exponents from this fit. This has, as far as the author is aware, not yet been achieved and indeed would be no easy task. Without such a technique it is difficult to see how the percentage of orbits controllable by the OGY method can be increased.

Acknowledgments

The author would like to gratefully thank Prof. Robert Delbourgo for all his guidance and encouragement throughout the preparation of this paper and the thesis that preceeded it.

References

Azevedo, A., and Rezendo, S. (1991). *Phys. Rev. Lett.* **66**, 1342.
Corney, S. (1995). Control of chaos in the Rössler system. Honours thesis, University of Tasmania.
Fraser, A., and Swinney, H. (1986). *Phys. Rev.* A **33**, 1134.
Kittel, A., Parisi, J., and Pyragas, K. (1992). *Phys. Lett.* A **198**, 433.
Lathrop, D., and Kostelich, E. (1989). *Phys. Rev. Lett.* **40**, 4028.
Liebert, W., Pawelzik, K., and Schuster, H. G. (1991). *Europhys. Lett.* **14**, 521.
Ogata, K. (1990). 'Modern Control Engineering', 2nd edn (Prentice–Hall: Englewood Cliffs, NJ).
Ott, E., Grebogi, C., and Yorke, J. (1990). *Phys. Rev. Lett.* **64**, 1196.
Pyragas, K. (1992). *Phys. Lett.* A **170**, 421.
Rössler, O. E. (1976). *Phys. Lett.* A **57**, 397.
Roy, R., Murphy, T., Maier, T., and Gills, Z. (1992). *Phys. Rev. Lett.* **68**, 1259.
Shinbrot, T., Grebogi, C., Ott, E., and Yorke, J. (1993). *Nature* **363**, 411.
Shinbrot, T., Ott, E., Grebogi, C., and Yorke, J. (1990). *Phys. Rev. Lett.* **65**, 3215.
Van De Vegte, J. (1990). 'Feedback Control Systems' (Prentice–Hall: Englewood Cliffs, NJ).

Manuscript received 26 February, accepted 8 August 1996