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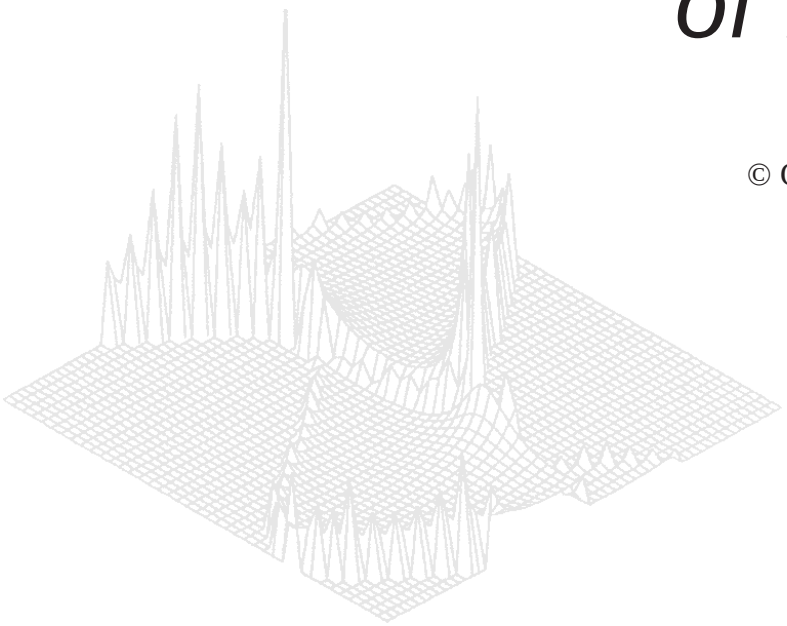

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Diffusion Cooling of Electrons in an A.C. Field*

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Abstract

Boundaries affect the measured values of transport coefficients in all drift tube experiments, to a greater or lesser extent, and nowhere is this more apparent than in the experiment first devised by Cavalleri (1969) and subsequently adapted by Crompton and coworkers in the 1970s. The phenomenon of 'diffusion cooling' is particularly striking and arises essentially from a penetration of the 'boundary layer' (of thickness of the order of the mean free path for energy exchange) throughout a significant portion of the gas chamber. Although this is something of an obstacle to extracting the classical diffusion coefficient from experimental data, it is of great interest in its own right from a theoretical point of view, and the Crompton *et al.* experiments motivated several theoretical treatments which successfully explained diffusion cooling, albeit for zero applied field and on the basis of the 'two-term' spherical harmonic representation of the velocity distribution function. The present paper puts these theories in the context of the modern, generalised eigenvalue theory, which may be used as a basis for describing all swarm experiments. In addition, the earlier zero-field studies are generalised to the extent that an a.c. heating field is included, as was the case for the original Cavalleri experimental set-up. This field is found to enhance diffusion cooling effects for a simple model cross section.

1. Introduction

The influence of boundaries upon charged particle transport properties remains a challenging and interesting theoretical problem (Kumar 1991), which the author embarked upon over two decades ago with the encouragement of Professor R. W. Crompton, who at the time was conducting a series of experiments, using the Cavalleri chamber (Cavalleri 1969; Huxley and Crompton 1974), aimed at determining the thermal value of the diffusion coefficient of electrons in noble gases. In this experiment, electrons produced in a cylindrical chamber by ionising x-rays diffuse freely through the gas to the walls of the container, with no space-charge effects (see however the important comments by Liley 1993) where they are absorbed. The number of electrons remaining in the chamber is monitored as a function of time, with an exponential decay usually being observed at sufficiently long times, from which a diffusion coefficient may be inferred (Huxley and Crompton 1974). The experiment as originally devised by Cavalleri (1969) had an r.f. field applied along the cylindrical axis, but was employed by Crompton and co-workers at the Australian National University

* Dedicated to Professor Robert W. Crompton on the occasion of his seventieth birthday.

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with zero applied field (Gibson *et al.* 1973; Crompton *et al.* 1975; Rhymes and Crompton 1975*a*, 1975*b*; Rhymes 1976; Crompton *et al.* 1980; Hegerberg and Crompton 1980, 1983) . The results sometimes showed significant pressure dependence, which was correctly attributed to the effect of boundaries, and associated with the very interesting phenomenon of 'diffusion cooling'. This pressure dependence, and the diffusion cooling effect itself, was something of an obstacle to the experimentalist in the quest to find the true thermal diffusion coefficient, and could be eliminated in most (but not all) cases by simply raising the gas pressure, thereby causing the 'boundary layer', of thickness of the order of the mean free path for energy transfer, and therefore inversely proportional to pressure, to become negligibly small in comparison with the size of the chamber. What was a something of a nuisance in the experiment, however, proved most attractive to theoreticians, and several papers dealing exclusively with diffusion cooling followed. Parker (1965) had previously developed the basic kinetic theory necessary to describe the free (as distinct from the ambipolar case, studied even earlier by Biondi (1954)) diffusion cooling effect, by formulating an eigenvalue equation from the Boltzmann equation in the case of simple model cross sections and plane-parallel plate geometry. Following the experiments of Crompton *et al.*, Leemon and Kumar (1975) reformulated the problem in a more general way for more realistic geometry and cross sections and developed a reasonably accurate numerical method of solution of the eigenvalue equation. Robson (1976*a*, 1976*b*) then extended the eigenvalue problem to include inelastic collisions and developed a variational principle to furnish approximate solutions quickly. Ness (1977) went one step further and accounted for simultaneous diffusion and attachment cooling effects. Brandt and Arista (1979) later analysed positron diffusion in finite geometry, independently of these results. Robson (1980) gave a brief outline of the influence of an applied r.f. heating field, but the details were never published. More recently Robson and Prytz (1993) developed a highly accurate numerical procedure for calculating the eigenvalue spectrum.

The qualitative explanation in terms of the energy-dependence of cross sections and thermal contact between the swarm and the gas has been well-documented (Rhymes and Crompton 1975*b*; Robson 1976) and we merely repeat a few of the most essential ingredients here: At low pressures, the mean free path for energy exchange between electrons and the gas is comparable with the dimensions of the cell, the thermal contact between electrons and gas is weak, and electrons will therefore be lost to the walls before they have had a chance to thermalise. (One could even say that the electrons are in better contact with the walls than with the gas at low pressures; the situation is not unlike the Knudsen regime in rarified neutral gases.) Furthermore, if the collision cross section varies less rapidly with energy than $\sqrt{\epsilon}$, then even simple mean free path arguments indicate that the gas is more transparent to diffusion of higher energy electrons, which are therefore lost preferentially to the walls, resulting in a lowering of the overall mean energy (and therefore temperature) of the remaining electrons. The *effective* diffusion coefficient is also lowered. On the other hand, if the cross section rises more rapidly with energy than $\sqrt{\epsilon}$, lower energy electrons are lost at a relatively greater rate, and there is actually a 'diffusion heating' effect. However, the effective diffusion coefficient is in all cases *lower* than the thermal value, which it approaches asymptotically as the pressure increases, and

the boundary layer recedes from the bulk of the swarm. Note that thermal contact between the electrons and noble gases may also be enhanced (and the diffusion cooling effect correspondingly reduced) in an experiment through the addition of trace amounts of a molecular gas (Rhymes and Crompton 1975*b*). It is interesting to compare this boundary-associated phenomenon with the boundary-free cooling/heating effects induced by preferential loss of electrons in the bulk of the gas by an energy-dependent attachment cross section, observed in later experiments by Crompton *et al.* (1980) and Hegerberg and Crompton (1983). The combined effects of boundary and bulk phenomena, as represented by diffusion and attachment cooling respectively, pose a most interesting theoretical problem (Ness 1977).

The author's interest in boundary effects was rekindled in recent years, again largely through the influence of Professor Crompton, at the Second Japan–Australia Workshop on Gaseous Electronics and Its Applications (Crompton and Hayashi 1990). In what follows, the earlier analyses of the 1970s are put in the context of general eigenvalue theory given at this workshop (Robson 1991*a*; Kondo and Tagashira 1993) and the effect of an applied a.c. field is discussed, with calculations for two simple models.

2. Transport Coefficients as Eigenvalues

(2a) Some General Theoretical Observations

In the last two decades or so, there have been tremendous advances in the theory and calculation of drift velocities and diffusion coefficients for idealised, boundary–free models of swarm experiments, but as far as inclusion of electrode and wall effects and analysis of real drift chamber experiments is concerned, some progress has been made (England and Skullerud 1993) but much remains to be done (Kumar 1991); in particular, whenever boundary or initial condition-induced large-gradient circumstances are encountered in experiment, the corresponding theoretical picture remains somewhat clouded. Measured transport properties may then be significantly influenced by the size and shape of the gas chamber and will almost assuredly not be comparable with classical transport coefficients, which are usually defined and calculated theoretically in connection with the idealised, infinite-medium, weak-gradient situation. The theory of the Cavalleri experiment is something of an exception to this observation, insofar as large gradients can be readily accommodated, although it has to be emphasised that the boundary condition incorporated in all the solutions of the Boltzmann equation mentioned in the Introduction, and persevered with in this paper, is something of an approximation to reality, corresponding to zero 'extrapolation length'. Furthermore, in the Cavalleri experiment, as well as in many other cases where bulk properties of the swarm are measured, the large-gradient effect can often be eliminated in practice by the simple expedient of increasing the gas pressure, as explained in the Introduction. The boundary layer recedes from the bulk of the swarm, where the measurements are made, the experiment then approximates idealised circumstances, and it is justifiable to label the quantity thus determined as a classically-defined transport coefficient. On the other hand, if the principal quantity measured is either at the bounding electrode or is in any case always within the boundary layer, no matter how small, there seems no

way of escaping the large-gradient effects, and the quantity measured may not be classifiable in terms of standard transport coefficients. The Townsend–Huxley experiment appears to fall into this category, but the author is happy to leave further discussion of this difficult problem to the companion paper by England and Skullerud (1997, present issue p. 553), and proceed with the relatively simple analysis of the Cavalleri cell.

(2b) Eigenvalue Theory

Over thirty years ago Parker (1965) identified the thermal diffusion coefficient in terms of the limit of an eigenvalue of a certain differential equation, for the particular case of electron transport in a finite medium, but it is only fairly recently that the eigenvalue nature of swarm transport properties in general has been recognised (see the reviews by Kumar 1991 and Robson 1991*a*) and calculations made accordingly (Date *et al.* 1992, 1993; Kondo and Tagashira 1993). The generalised eigenvalue theory enables us to:

(i) Present a unified theoretical picture of swarm experiments, illustrating how they may be all characterised by the one, common feature, namely the fundamental 'dispersion' relation, which in particular contains information about all the classically defined 'hydrodynamic' transport properties;
(ii) Clarify and standardise key definitions, in order to eliminate the confusion often surrounding the labelling of transport coefficients, as highlighted by Brennan and Ness (1992)—see also Robson (1991*b*);
(iii) Provide a framework for theoretical calculation of properties actually measured in real experiments, be they in large or weak density gradient situations.

The Cavalleri arrangement has already been discussed in this context for one-dimensional, infinite parallel-plate geometry (Robson 1991*a*), and this can be readily extended to the cylindrical geometry actually used in experiment. The generalised eigenvalue theory subsumes the special eigenvalue problem formulated originally by Parker (1965) and in more sophisticated forms by others (Leemon and Kumar 1975; Robson 1976*a*, 1976*b*). Note that all the latter theories were based on the 'two-term' spherical harmonic representation of the electron velocity distribution function, whereas the generalised approach avoids any such assumption, and provides a medium for correspondingly more accurate and satisfying computation.

The fundamental eigenvalue problem for *zero* field can be written as

$$(\omega + i\boldsymbol{k}\cdot\boldsymbol{c} + J)\Psi = 0\,, \tag{1}$$

where $\boldsymbol{c}$ is the velocity of a swarm particle (electron, positron, muon or ion) of mass m, and J is the Boltzmann collision operator, accounting for elastic, inelastic, superelastic, attaching, ionising, etc. interactions between swarm particles and neutral gas molecules. In what follows, properties of the latter are designated by subscripts '0'. The number density of charged particles n is assumed to be so small in comparison with n_0, the neutral gas number density, that space-charge effects and mutual interaction between charged particles are negligible, an assumption that Liley (1993) has critically examined for the Cavalleri experiment. Eventually

we shall specialise to electron swarms and give the specific form of J, but for the present this is not necessary. The quantities measured in any zero-field experiment can be characterised in terms of the eigenvalues of (1), which can generally be shown to depend upon k^2 only:

$$\omega = \Omega^{(\mu)}(k^2)\,, \tag{2}$$

and the corresponding eigenfunctions $\Psi^{(\mu)}(\boldsymbol{c}|\boldsymbol{k})$ of equation (1), where $\mu = 0, 1, 2, \ldots$ is an index ordering the allowed 'velocity modes'. Values of the 'wave number' $\boldsymbol{k}$ are determined by the boundary conditions associated with the particular experimental arrangement. For a finite medium, the spatial modes are characterised by a discrete set of wave numbers, while for an unbounded medium $\boldsymbol{k}$ is continuous. In any case, the classical *hydrodynamic* diffusion coefficient is defined in terms of the fundamental 'dispersion relation',

$$\omega = \Omega^{(0)}(k^2)\,, \tag{3}$$

as

$$D = -\tfrac{1}{2}(\partial^2\Omega^{(0)}/\partial k^2)_{k=0}\,, \tag{4}$$

while the net reactive production rate is

$$\nu_I = \Omega^{(0)}(0)\,. \tag{5}$$

If $\boldsymbol{k}$ is sufficiently small in magnitude (gradients are weak), then the rhs of (3) can be expanded in a Taylor series:

$$\omega = \Omega^{(0)} = \nu_I - D_{\text{eff}}k^2\,, \tag{6}$$

where

$$D_{\text{eff}} = D + Q^{(4)}k^2 + Q^{(6)}k^6 + \ldots \tag{7}$$

can be thought of as an 'effective' diffusion coefficient, differing from the classical quantity (4) unless k is vanishingly small. The quantities $Q^{(2n)}$, with $n = 2, 3, \ldots$, are higher order derivatives of $\Omega^{(0)}$. Since D_{eff} is k-dependent, it must therefore depend upon the size and shape of the container, i.e. upon the particular experimental arrangement. (Note that D does *not* depend upon either geometry or the type of experiment.) The velocity distribution function in the asymptotic long-time limit is given by

$$f(\boldsymbol{r}, \boldsymbol{c}, t) = \exp(-t/\tau^{(0)}) \int d\boldsymbol{k} P(\boldsymbol{k}) \Psi^{(0)}(\boldsymbol{c}|\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \tag{8}$$

where

$$\tau^{(0)} = -1/\Omega^{(0)} \tag{9}$$

is the fundamental time constant, $\Omega^{(0)}$ is evaluated at the lowest value of $\boldsymbol{k}$ satisfying (12) below, and $P(\boldsymbol{k})$ is a function which selects the allowed values of $\boldsymbol{k}$ according to the boundary conditions imposed on f.

The total number of swarm particles remaining in the container of volume V at any time t is given by

$$n_{\text{total}}(t) = \int d\boldsymbol{c} \int d\boldsymbol{r} f(\boldsymbol{r}, \boldsymbol{c}, t) \sim e^{-t/\tau^{(0)}} \tag{10}$$

and measurement of this quantity, as in the Cavalleri experiment, thus determines $\tau^{(0)}$ and hence the effective, geometry-dependent diffusion coefficient (6). Of course, for vanishingly small k (large container) and/or small $Q^{(2n)}$ (high pressure),

$$D_{\text{eff}} \rightarrow D\,. \tag{11}$$

Note that a perturbation procedure was employed by Parker (1965), and discussed further by Leemon and Kumar (1975) specifically for the Cavalleri cell. This type of expansion is equivalent to the density gradient expansion procedure (Kumar *et al.* 1980) by now standard in the kinetic theory of swarms.

For the present suppose that on the boundary surface S, the even part of the velocity distribution function vanishes, i.e.

$$f^{(e)}(\boldsymbol{r}, \boldsymbol{c}, t) = \tfrac{1}{2}[f(\boldsymbol{r}, \boldsymbol{c}, t) + f(\boldsymbol{r}, -\boldsymbol{c}, t)] = 0 \tag{12}$$

for $\boldsymbol{r} \in S$, since then the number density

$$n(\boldsymbol{r}, t) = \int d\boldsymbol{c} f^{(e)}(\boldsymbol{r}, \boldsymbol{c}, t) \tag{13}$$

also vanishes on S. However, current to the walls, related to the odd part of f, is non-zero. Equation (12) is only an approximation to reality, with some drawbacks which are unphysical in principle, but which in practice can be ignored. The allowed values of $\boldsymbol{k}$ are then found by substituting (8) into (12) and using the symmetry condition

$$\Psi(-\boldsymbol{c}|\boldsymbol{k}) = \Psi(\boldsymbol{c}|-\boldsymbol{k}) \tag{14}$$

which follows from (1). If the container is a cylinder of radius a and height h, it can thus be shown that the allowed values of the components of $\boldsymbol{k}$ perpendicular and parallel to the cylindrical axis are

$$k_{\perp} = \xi_0/h\,, \tag{15}$$

$$k_{\parallel} = \pi/a\,, \tag{16}$$

respectively, for the lowest order spatial mode, where ξ_0 is the smallest zero of the Bessel function $J_0(x)$. The number density and the even part of the velocity distribution function are given by

$$n(\boldsymbol{r},t) \sim e^{-t/\tau^{(0)}} J_0(k_\perp \rho)\sin(k_\parallel z) \tag{17}$$

for cylindrical coordinates (ρ, z, θ) and

$$f^{(e)}(\boldsymbol{r},\boldsymbol{c},t) \sim n(\boldsymbol{r},t)\Psi^{(e)}(\boldsymbol{c}|\boldsymbol{k}) \tag{18}$$

respectively, where the even part of the lowest order eigenfunction is defined by

$$\Psi^{(e)}(\boldsymbol{c}|\boldsymbol{k}) = \tfrac{1}{2}\{\Psi^{(0)}(\boldsymbol{c}|\boldsymbol{k}) + \Psi^{(0)}(-\boldsymbol{c}|\boldsymbol{k})\}\,. \tag{19}$$

Finally, the quantity

$$\tfrac{3}{2}k_B T_{\text{eff}} = \int d\boldsymbol{c}\tfrac{1}{2}mc^2\Psi^{(e)}(\boldsymbol{c}|\boldsymbol{k}) \Big/ \int d\boldsymbol{c}\Psi^{(e)}(\boldsymbol{c}|\boldsymbol{k}) \tag{20}$$

is a measure of the energy of the swarm particles. This is clearly dependent upon $\boldsymbol{k}$ and therefore also upon geometry in general.

(2c) Electron Diffusion, Two-term Approximate Theory

So far the theory as outlined above is applicable to swarm particles of any mass, but now we specialise to electron swarms, for which $m/m_0 \ll 1$, where m_0 denotes the mass of a neutral gas molecule, and certain simplifying approximations are possible. If collisions between electrons and the gas molecules are predominantly elastic, the distribution of electron velocities is very nearly isotropic in general and the 'two-term' approximation for both distribution function and eigenfunction,

$$\Psi(\boldsymbol{c}|\boldsymbol{k}) \approx \Psi_0(c|\boldsymbol{k}) + \boldsymbol{\Psi}_1(c|\boldsymbol{k})\cdot\boldsymbol{c}/c\,, \tag{21}$$

holds quite well. In this case, the even part of the eigenfunction is just the first term in the expansion, i.e.

$$\Psi^{(e)} = \Psi_0\,. \tag{22}$$

Substitution of (21) into (1), followed by appropriate integration over angles of velocity, together with some algebraic manipulation, leads to the following eqaution for Ψ_0:

$$\left(\omega + k^2\frac{c}{3n_0 Q} - L\right)\Psi_0 = 0\,, \tag{23}$$

where Q is the momentum-transfer cross section and L is a linear operator, whose explicit form in the absence of inelastic processes is given by

$$L\Psi = \frac{1}{c^2}\partial_c\left\{\tfrac{1}{3}n_0 c^3 Q\left[\frac{3k_B T_0}{m_0}\partial_c\Psi + \frac{3m}{m_0}c\Psi\right]\right\}, \tag{24}$$

where T_0 denotes the gas temperature and k_B is Boltzmann's constant. Using energy

$$\epsilon = \tfrac{1}{2}mc^2 \tag{25}$$

as the independent variable instead of speed c, there follows

$$2\frac{m}{m_0}\frac{d}{d\epsilon}\left\{n_0\epsilon^2 Q\left(\Psi_0 + k_B T_0 \frac{d}{d\epsilon}\Psi_0\right)\right\} + \left[\left(\frac{m\epsilon}{2}\right)^{\frac{1}{2}} D_{\text{eff}} - \frac{\epsilon}{3n_0 Q}\right]k^2\Psi_0 = 0\,, \tag{26}$$

where ω has been written as in (6) in terms of D_{eff}, which now assumes the role of eigenvalue. For simplicity, we have assumed no ionisation or attachment and have set $\nu_I = 0$.

Equation (26) forms the starting point for earlier analyses (Parker 1965; Leemon and Kumar 1975; Robson and Prytz 1993); a slightly modified form is required for gas mixtures and inelastic processes (Robson 1976*a*, 1976*b*). These papers should be consulted for details on solution procedures and results in particular cases. Notice that T_{eff} as defined by equation (20) now assumes the role of a geometry-dependent effective temperature of the electrons, differing in general from the free-space, equilibrium value T_0.

Possibly a more satisfactory treatment, which avoids the limitations inherent in the two-term approximation (21), would be to solve equation (1) directly, using a 'multi-term' expansion in spherical harmonics. Such an approach would seem highly desirable, especially if inelastic processes were significant. It is clear that the general eigenvalue equation, therefore offers both conceptual and practical advantages over earlier two-term formulations. However, we persevere with the two-term approximation and move on to the case of an oscillatory applied field.

3. Eigenvalue Problem for an A.C. Field

(3a) Setting up the Eigenvalue Equation

There is no general eigenvalue theory for charged particle transport in alternating fields,

$$E = E_0 \cos(2\pi\nu t)\,, \tag{27}$$

and therefore it is necessary to start from scratch and develop a theory to suit each case. The following summarises the steps taken for an electron swarm: *Firstly*, we assume that the applied frequency lies in the range

$$\nu_m \ll \nu \ll \nu_\epsilon\,, \tag{28}$$

where ν_m and $\nu_\epsilon = (2m/m_0)\nu_m$ are the collision frequencies for momentum and energy transfer respectively. We also assume that the macroscopic time scale is much larger than the relaxation time for momentum–transfer:

$$\omega \ll \nu_m\,. \tag{29}$$

Secondly, if collisions are predominantly elastic, the electron distribution function may be well represented by the first two terms of a spherical harmonic expansion (cf. equation 21). For conditions under which (28) holds, the energy distribution of the electrons (as represented by the first term in this expansion) is only weakly

modulated during one cycle of the field, and can be taken to be effectively constant for averaging purposes. *Finally*, the Boltzmann equation is decomposed into its scalar and vector parts by appropriate angular integrations and averaged over one period of the oscillating field.

It is thus found that the (cycle-averaged) number density is again given by (17), with relaxation time defined according to (6) and (9), the effective diffusion coefficient being the eigenvalue of

$$\frac{2m}{m_0}\frac{d}{d\epsilon}\left\{n_0\epsilon^2 Q\Psi_0+\left[n_0\epsilon^2 Qk_BT_0+\frac{m_0}{6m}(eE_{\rm rms})^2\frac{\epsilon}{n_0Q}\right]\frac{d\Psi_0}{d\epsilon}\right\}$$
$$+\left[\left(\frac{m\epsilon}{2}\right)^{\frac{1}{2}}D_{\rm eff}-\frac{\epsilon}{3n_0Q}\right]k^2\Psi_0=0\,, \tag{30}$$

where $E_{\rm rms}=E_0/\sqrt{2}$ and k^2 is the sum of the squares of (15) and (16). [Cavalleri (1969) argued that the actual height h of the cylinder should be corrected to account for boundary effects, but any discussion along those lines would take us beyond the scope of this work.] Equation (30) appears not greatly different from the zero-field counterpart (26), but in reality it is much more difficult to solve. Before discussing the solution, we observe that only *one* diffusion coefficient appears in the equation, instead of the expected two distinct quantities, parallel and lateral to the field directions respectively (Huxley and Crompton 1974). This is because *time-averaged* diffusion in an a.c. field is *isotropic* when (28) holds and the coefficient found is in fact the *lateral* diffusion coefficient.

(3b) Calculation of $D_{\rm eff}$

It is convenient to first of all introduce dimensionless quantities. Let Q_0 be a constant characterising the magnitude of the cross section and define:

$$u=\frac{\epsilon}{k_BT_0}\,, \tag{31}$$

$$q=\frac{Q}{Q_0}\,, \tag{32}$$

$$\overline{\lambda}=\frac{1}{n_0Q_0}\,, \tag{33}$$

$$\overline{\lambda}_\epsilon=\overline{\lambda}\left(\frac{m_0}{2m}\right)^{\frac{1}{2}}. \tag{34}$$

Then equation (30) can be written as

$$\frac{d}{du}\left\{qu^2\left[\Psi_0+\left(1+\frac{(E^\star)^2}{q^2u}\right)\frac{d\Psi_0}{du}\right]\right\}+\kappa^2\left(\theta u^{\frac{1}{2}}-\frac{u}{q}\right)\Psi_0=0\,, \tag{35}$$

where

$$E^\star = \frac{eE_{\rm rms}\bar{\lambda}}{\sqrt{3}k_BT_0}\,, \tag{36}$$

$$\kappa = \frac{\bar{\lambda}_\epsilon}{\sqrt{3}k}\,, \tag{37}$$

$$\theta = \frac{3D_{\rm eff}}{\bar{v}\bar{\lambda}}\,, \tag{38}$$

$$\bar{v} = \sqrt{2k_BT_0/m}\,. \tag{39}$$

If we further define

$$\xi(u) = \int \frac{du}{1+(E^\star)^2/uq^2} \tag{40}$$

and write

$$\Psi_0 = e^{-\xi(u)}\phi(u)\,, \tag{41}$$

then (35) takes the Sturm–Liouville form

$$\frac{d}{du}\left(p(u)\frac{d\phi}{du}\right) + [\theta r(u) - s(u)]\phi = 0\,, \tag{42}$$

where

$$p(u) = qu^2e^{-\xi}/\xi'\,, \qquad r(u) = \kappa^2u^{\frac{1}{2}}e^{-\xi}\,, \qquad s(u) = \kappa^2ue^{-\xi}/q\,. \tag{43}$$

A generalisation of the variational principle established earlier (Robson 1976*a*, 1976*b*) for zero field is that the eigenvalues θ are minimum values of the functional

$$\Theta = \int_0^\infty [p(\phi')^2 + s\phi^2]du \Big/ \int_0^\infty r\phi^2 du\,. \tag{44}$$

Only the lowest eigenvalue $\theta^{(0)}$ is required to determine the asymptotic behaviour of the swarm. Further discussion of the variational method can be found in Robson (1980).

We now consider two special models:

(i) *Constant collision frequency* $q = u^{-\frac{1}{2}}$, $\nu_m =$ constant.

Equation (35) can be solved *exactly* for this model with

$$\theta^{(0)} = \frac{3}{4\kappa^2}(\sqrt{1+4\beta\kappa^2} - 1)\,, \tag{45}$$

$$\theta^{(\mu)} = \theta^{(0)} + (2\alpha+1)\mu/\kappa^2\,, \tag{46}$$

$$\Psi^{(\mu)} = e^{-\xi}\phi^{(\mu)} = \exp[-(1+\alpha)u/\beta]L_\mu^{(\frac{1}{2})}[(2\alpha+1)u/\beta]\,, \tag{47}$$

where

$$\alpha = \tfrac{1}{2}(\sqrt{1+4\beta\kappa^2}-1)\,, \tag{48}$$

$$\beta = 1+(E^\star)^2\,. \tag{49}$$

From (38) and (45) it then follows that the effective diffusion coefficient for the lowest mode ($\mu = 0$) is

$$D_{\text{eff}} = \frac{\overline{v}\overline{\lambda}}{4\kappa^2}(\sqrt{1+4\beta\kappa^2}-1)\,, \tag{50}$$

while the lowest order eigenfunction can be written as (reverting now to dimensional quantities):

$$\Psi^{(0)}(\epsilon) = e^{-\epsilon/k_B T_{\text{eff}}}\,. \tag{51}$$

The effective electron temeprature is given by

$$T_{\text{eff}} = T_e/(1+\alpha)\,, \tag{52}$$

where

$$T_e = \beta T_0 = k_B T_0 + \tfrac{1}{3}m_0(eE_{\text{rms}}/m\nu_m)^2 \tag{53}$$

is the free-space electron temperature. That is

$$T_{\text{eff}} < T_e \tag{54}$$

and the electrons are 'cooled' by the walls. Note that for a large container and/or high gas pressure,

$$\kappa \to 0\,, \tag{55}$$

$$\alpha \to 0\,, \tag{56}$$

and the effective diffusion coefficient (50) becomes

$$D_{\text{eff}} \to \beta\overline{v}\overline{\lambda}/2 = k_B T_e/m\nu_m = D\,, \tag{57}$$

where D is the free-space diffusion coefficient and $\nu_m = (\overline{v}/\overline{\lambda})$ is the collision frequency. It follows from (50) and (57) that

$$D_{\text{eff}} < D\,. \tag{58}$$

The diffusion cooling phenomenon is clearly illustrated by (54) and (58).

The zero-field limit is regained by setting $\beta = 1$ in the above formulas. When the field is strong $\beta \gg 1$ and even though the large container/high pressure

condition $\kappa \ll 1$ may pertain, the *product* $\beta\kappa^2$ may not be negligible, i.e. α and the cooling effect may still be appreciable: The field 'pumps' the high-energy tail of the distribution function, the region which suffers the greatest loss to the walls. This is why diffusion cooling is actually *enhanced* by application of a field.

(ii) *Cross section proportional to speed* $q = u^{\frac{1}{2}}$.

Direct substitution in (42) shows that for this model the lowest eigenvalue and eigenfunction are respectively

$$\theta^{(0)} = 1\,, \tag{59}$$

$$\phi^{(0)} = 1\,, \tag{60}$$

which indicates that the diffusion coefficient and temperature retain their free-space values regardless of the presence of boundaries. As explained elsewhere (Rhymes and Crompton 1975*b*; Robson 1976*a*, 1976*b*) for this model electrons diffuse at the same rate to the walls, independently of their energy: There is no preferential loss from any energy range to promote any cooling effect, even in the presence of a field.

Apart from these two simple models, it would seem that any analytic discussion, even approximate, would be difficult, and therefore further elucidation must await a full numerical treatment.

4. Concluding Remarks

After exploring the connection between the eigenvalue problem estalished over thirty years ago by Parker (1965) for diffusion in finite geometry, and the modern eigenvalue theory covering all swarm experiments, this paper focussed on the peculiar nature of diffusion of electrons in the Cavalleri cell, both with and without an applied a.c. field for model elastic collisional cross sections, employing short arguments and simple mathematics. It was shown for the constant collision frequency model how a field actually enhances the diffusion cooling phenomenon, by pumping electrons into the energy regime preferred for loss to the walls. Clearly much work remains to be done in the a.c. field case, not only for further model cross sections, but especially for actual gases, including development of a workable theory to account for inelastic processes. Extension of the Discrete Ordinate method described elsewhere (Robson and Prytz 1993) is one obvious avenue to be explored.

Acknowledgment

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References

Biondi, M. (1954). *Phys. Rev.* **93**, 1136.
Brandt, W., and Arista, R. (1979). *Phys. Rev.* A **19**, 2317.
Brennan, M. J., and Ness, K. F. (1992). *Nuovo Cimento* D **14**, 933.

Cavalleri, G. (1969). *Phys. Rev.* **179**, 186.
Crompton, R. W., and Hayashi. M. (Eds) (1990). Proc. 2nd Japan–Australia Workshop on Gaseous Electronics and its Applications, Gotemba, Japan.
Crompton, R. W., Hegerberg, R., and Skullerud, H. R. (1980). *J. Phys.* B **13**, L 455.
Crompton, R. W., Kumar, K., Leemon, H. I., and Rhymes, T. (1975). Proc. 12th Int. Conf. on Phenomena in Ionised Gases (Eds J. G. R. Holscher and D. C. Schram).
Date, H., Kondo, K., Yachi, S., and Tagashira, H. (1992). *J. Phys.* D **25**, 1330.
Date, H., Kondo, K., and Tagashira, H. (1993). *J. Phys.* D **26**, 1211.
England, J. P., and Skullerud, H. R. (1993). Proc. 8th Int. Swarm Seminar, Norwegian Institute of Technology, Trondheim (Eds T. H. Lovaas and H. R. Skullerud).
England, J. P., and Skullerud, H. R. (1997). *Aust. J. Phys.* **50**, 553 (present issue).
Gibson, D. K., Crompton, R. W., and Cavalleri, G. (1973). *J. Phys.* B **6**, 1118.
Hegerberg, R., and Crompton, R. W. (1980). *Aust. J. Phys.* **33**, 989.
Hegerberg, R., and Crompton, R. W. (1983). *Aust. J. Phys.* **36**, 831.
Huxley, L. G. H., and Crompton, R. W. (1974). 'The Diffusion and Drift of Electrons in Gases' (Wiley: New York).
Kondo, K., and Tagashira, H. (1993). *J. Phys.* D **26**, 1948.
Kumar, K. (1991). Proc. 20th Int. Conf. on Phenomena in Ionised Gases (Eds V. Palleschi, D. P. Singh and M. Vaselli).
Kumar, K., Skullerud, H. R., and Robson, R. E. (1980). *Aust. J. Phys.* **33**, 343.
Leemon, H. I., and Kumar, K. (1975). *Aust. J. Phys.* **28**, 25.
Liley, B. S. (1993). *Aust. J. Phys.* **46**, 773.
Ness, K. F. (1977). Honours thesis, James Cook University.
Parker, J. H. (1965). *Phys. Rev.* **139**, A1792.
Rhymes, T. (1976). Ph.D. thesis, Australian National University.
Rhymes, T., and Crompton, R. W. (1975*a*). *Phys. Rev.* A **12**, 776.
Rhymes, T., and Crompton, R. W. (1975*b*). *Aust. J. Phys.* **28**, 675.
Robson, R. E. (1976*a*). *Phys. Rev.* A **13**, 1536.
Robson, R. E. (1976*b*). *Aust. J. Phys.* **29**, 171.
Robson, R. E. (1980). Proc. Conf. on Boundary and Interior Layers (Ed. J. J. H. Miller), p. 407 (Boole Press: Dublin).
Robson, R. E. (1991*a*). *In* 'Gaseous Electronics and its Applications' (Ed. R. W. Crompton *et al.*), p. 89 (Kluwer: Dordrecht).
Robson, R. E. (1991*b*). *Aust. J. Phys.* **44**, 685.
Robson, R. E., and Prytz, A. (1993). *Aust. J. Phys.* **46**, 465.

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