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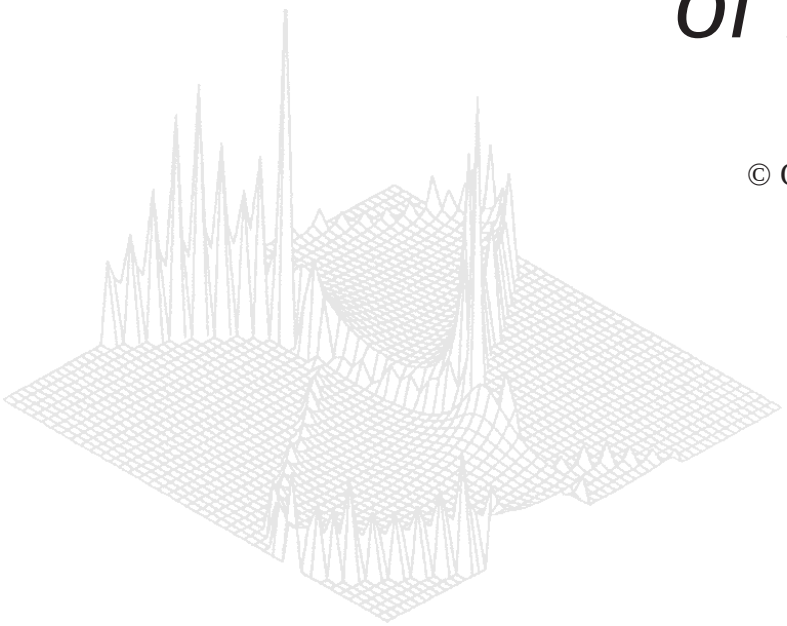

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Glueball Spectrum of SU(3) Lattice Gauge Theory by Plaquette Expansion

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Abstract

By casting the Hamiltonian of pure SU(3) in 3+1 space–time dimensions into approximate tri-diagonal form we study the glueball spectrum of the system. In particular we obtain estimates for the ground state energy density, the string tension σ, and the masses of the lowest lying 0^{++}, 1^{+-} and 2^{++} glueballs. These initial calculations lead to estimates of various mass ratios in general agreement with other studies of the spectrum.

1. Introduction

The particle spectrum of quantum chromodynamics (QCD) is inherently nonperturbative, making its calculation difficult to perform. To study this low energy aspect of the theory a nonperturbative gauge invariant cut-off is required. One such cut-off, as first set out by Wilson (1974), is the introduction of a lattice, where space–time coordinates are discretised and strong coupling calculations can be performed. Physical predictions are extracted from this lattice gauge theory by taking the spacing of the lattice to zero, which corresponds to the coupling g approaching zero as required by asymptotic freedom. Physical quantities, such as particle masses, are expected to match onto renormalisation group forms in this continuum limit so that their ratios become independent of the coupling.

As an important step on the way to calculating the full spectrum of QCD an investigation of the pure SU(3) gauge field theory has been undertaken by many authors. The low lying colourless particle excitations predicted by the theory, called glueballs, have been fairly well established by various studies, either based upon Wilson's Lagrangian formulation or upon the Hamiltonian formulation developed by Kogut and Susskind (1975). Monte Carlo treatments in the Lagrangian formalism require great amounts of computer time, and whilst they provide good estimates for the hadron spectrum of QCD the glueball spectrum is not as well determined, the more recent works being Bali *et al.* (1993), Gupta *et al.* (1991), Vohwinkel and Berg (1989) and Michael and Teper (1989). In another work, a variational ansatz combined with Hamiltonian Monte Carlo methods has been used by Chin *et al.* (1988) to estimate the mass ratio of the two lightest glueballs, the 0^{++} and the 2^{++} glueballs.

Analytic techniques have also been applied to the problem. Hamiltonian strong coupling expansions for various glueball masses were intially carried out by Kogut

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et al. (1976), and later extended by Hamer (1989). There is however a need to extrapolate the perturbative results obtained in the strong coupling region into the weak coupling region by various approximants. The t expansion method (Horn and Lana 1991) provides nonperturbative expressions for the glueball masses in terms of analytically calculated Hamiltonian moments with respect to a well chosen trial state. The computation is in the infinite lattice limit, but still requires an extrapolation of results, this time in the $t \to \infty$ limit.

The plaquette expansion method (Hollenberg 1993), upon which this work is based, is a non-perturbative technique which has previously been applied to various lattice spin systems (Tomlinson and Hollenberg 1994) and lattice gauge theories (Hollenberg 1994*a*, 1994*b*; Hollenberg and Witte 1994), including the first order (analytic) application to the vacuum properties of SU(3) and the calculation of the particle spectrum of full QCD. In this paper the expansion in the vacuum sector is taken beyond the first order calculation performed previously. Also calculations for the string tension and the lowest lying glueballs in the scalar, axial and tensor sectors of the pure SU(3) Hilbert space are carried out beyond first order.

2. The Plaquette Expansion

The plaquette expansion is based upon the Lanczos method, which involves casting the Hamiltonian into tri-diagonal form by choosing a trial state $|v_1\rangle$, and constructing an orthogonal basis using the Lanczos recursion relation,

$$|v_n\rangle = \frac{1}{\beta_{n-1}} \left[(H - \alpha_{n-1})|v_{n-1}\rangle - \beta_{n-2}|v_{n-2}\rangle \right] . \tag{1}$$

The non-zero terms of the Hamiltonian matrix defined by this basis are the diagonal and the first off-diagonal terms, $\alpha_n = \langle v_n|H|v_n\rangle$ and $\beta_n = \langle v_{n+1}|H|v_n\rangle$, which depend upon the moments of the Hamiltonian with respect to $|v_1\rangle$, $\langle H^m\rangle \equiv \langle v_1|H^m|v_1\rangle$, where m ranges from 1 to $2n-1$ for α_n and 1 to $2n$ for β_n. By choosing an appropriate trial state the low-lying energy states of the system may be studied by stopping the Lanczos recursion after l iterations, putting the Hamiltonian in the approximate form,

$$H \to T_l = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{l-1} \\ & & \beta_{l-1} & \alpha_l \end{pmatrix} . \tag{2}$$

As l is increased the eigenvalues of T_l are found to converge to the eigenvalues of the original Hamiltonian. However, the analytic evaluation of the α and β is usually severely restricted due to the difficulty in calculating higher moments.

It is now known (Hollenberg 1993) that progress can be made for lattice systems, if the terms of the matrix are initially rewritten in terms of connected Hamiltonian moments (Horn and Weinstein 1984), which depend simply upon the number of plaquettes of the lattice, N_p. In particular, for the Kogut–Susskind

(1975) lattice Hamiltonian of the SU(3) pure gauge theory in 3+1 dimensions, given by

$$H = \frac{g^2}{2}\sum_l E_l^2 + \frac{1}{g^2}\sum_p \left(6 - \mathrm{tr}U_p - \mathrm{tr}U_p^\dagger\right), \tag{3}$$

where E_l is the colour electric flux operator on the link l, $\mathrm{tr}U_p$ is the magnetic flux operator of a plaquette p and g is the coupling constant, connected moments have been calculated by Horn and Lana (1991) for use in the t expansion. Using strong coupling trial states in various sectors of the Hilbert space these connected moments all have the form

$$\langle H^n\rangle_c \equiv c_n N_p + b_n, \tag{4}$$

where the c_n and b_n are independent of the number of plaquettes.

The plaquette expansion, anticipating the $N_p \to \infty$ limit, now involves expanding the α and β in $1/N_p$. This was originally performed (Hollenberg 1993) for the case where the connected moments are purely extensive ($b_n = 0$). A pattern is seen to develop for the smaller values of n, from which general expressions for the elements of the matrix can be written down. The same procedure can be performed for the more general moments (4) leading to the general expansions:

$$\alpha_n = [c_1 N_p + b_1] + (n-1)\left[\frac{c_3}{c_2}N_p + \left(\frac{b_3c_2 - c_3b_2}{c_2^2}\right)\right]\frac{1}{N_p} + \cdots,$$

$$\begin{aligned}\beta_n^2 = n\Big[c_2N_p + b_2\Big] \\ + \tfrac{1}{2}n(n-1)\Bigg[\left(\frac{c_2c_4 - c_3^2}{c_2^2}\right)N_p \\ + \left(\frac{c_2^2b_4 - 2c_2c_3b_3 + 2c_3^2b_2 - c_2c_4b_2}{c_2^3}\right)\Bigg]\frac{1}{N_p} + \cdots.\end{aligned} \tag{5}$$

For the case where the connected moments are purely extensive the first two terms in these expansions have been rigorously established (Witte and Hollenberg 1994). At this stage the validity of the further terms is assumed for all n based upon a direct calculation of the first few iterations. The above expansions are quite general in form, the physics of the problem contained in the determination of the c_n and b_n. Also, as can be seen, the first terms in the expansions depend only upon the lower Hamiltonian moments, those up to $\langle H^6\rangle_c$ being used in the above expressions. Further moments are only required as more terms of the expansion are included.

The general procedure now for handling T_l is to expand each of the elements out to order $1/N_p^r$ for some set r and diagonalising the matrix for the lower eigenvalues. If the moments are purely extensive and only the first two terms

in each of the expansions in (5) are kept (the $r = 0$ calculation), an analytic expression for the lowest eigenvalue E_0 in the infinite lattice limit has been obtained (Witte and Hollenberg 1994):

$$\lim_{N_p\to\infty} \frac{E_0}{N_p} = c_1 + \frac{c_2^2}{c_2c_4 - c_3^2}\left[(3c_3^2 - 2c_2c_4)^{\frac{1}{2}} - c_3\right] . \tag{6}$$

We now return to the explicit Hamiltonian moments calculated by Horn and Lana. Their initial trial state is the vacuum in the strong coupling limit, $|0_s\rangle$, from which they have calculated the connected Hamiltonian moments up to $\langle H^9\rangle_c$. The moments are purely extensive and by direct substitution into equations (5), we obtain with $y = 2/g^2$,

$$\begin{aligned}
\alpha_n = {} & 3yN_p + (n-1)\left[\frac{16}{3y} - \frac{y}{2}\right] + \tfrac{1}{2}(n-1)(n-2)\left[\frac{4}{3y} + \frac{y}{4}\right]\frac{1}{N_p} \\
& + (n-1)(n-2)\left[\left(\frac{1}{6y} + \frac{y}{32}\right)\right. \\
& \left. + (n-3)\left(-\frac{2192}{243y^5} + \frac{92}{81y^3} - \frac{106}{729y} - \frac{293y}{2592}\right)\right]\frac{1}{N_p^2} + \cdots , \\
\beta_n^2 = {} & n\left[\frac{y^2}{2}\right]N_p + \tfrac{1}{2}n(n-1)\left[-\frac{y^2}{4}\right] \\
& + \tfrac{1}{6}n(n-1)(n-2)\left[\frac{161y^2}{432} + \tfrac{2}{3} + \frac{40}{3y^2}\right]\frac{1}{N_p} \\
& + n(n-1)(n-2)\left[\left(\frac{53y^2}{10368} - \tfrac{1}{12} + \frac{7}{27y^2}\right)\right. \\
& \left. + (n-3)\left(\frac{931y^2}{20736} + \tfrac{185}{972} + \frac{10030}{6561y^2} - \frac{1000}{243y^4} + \frac{10972}{729y^6}\right)\right]\frac{1}{N_p^2} + \cdots .
\end{aligned} \tag{7}$$

This form of the Hamiltonian may now be used to study the vacuum sector of the Hilbert space, in particular the ground state energy density, $\mathcal{E}_0 \equiv E_0/N_p$, and the lowest lying scalar glueball, which has quantum numbers 0^{++} in the continuum limit (van den Doel and Horn 1987). Its mass is given by the difference between the two smallest eigenvalues, $M_S \equiv E_1 - E_0$.

As the connected moments are extensive, equation (6) is appropriate for providing the zeroth order ($r = 0$) estimate of the ground state energy density as $N_p \to \infty$ (Witte and Hollenberg 1994),

$$\mathcal{E}_0 = \frac{1}{3y}\left(6y^2 + 32 - (1024 - 192y^2 + 27y^4)^{\frac{1}{2}}\right) . \tag{8}$$

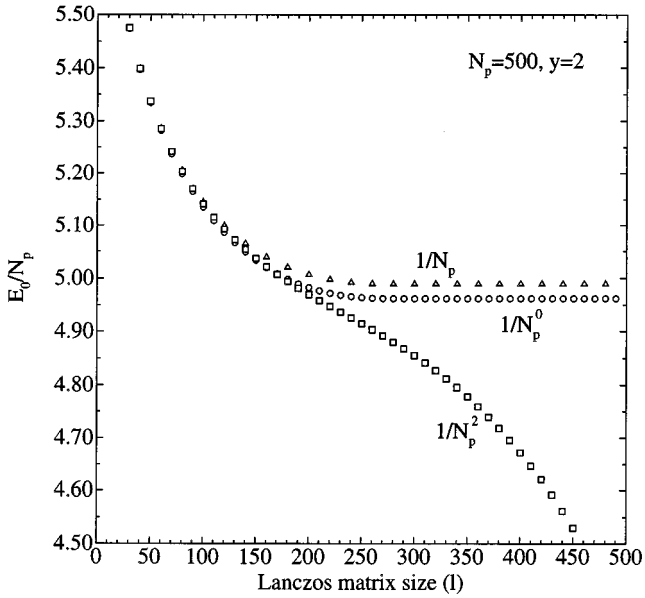

Fig. 1. Ground state energy density, E_0/N_p, as a function of Lanczos matrix size l for the various expansion orders, $r = 0$ to $r = 2$, for values $N_p = 500$ and $y = 2$

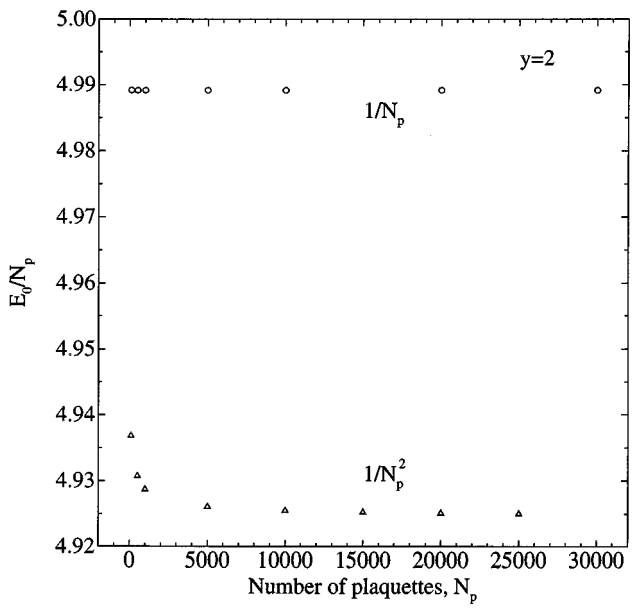

Fig. 2. Convergence of the ground state energy density, E_0/N_p, for increasing number of plaquettes, N_p, for orders $r = 1$ and $r = 2$ at $y = 2$.

Higher order estimates require numerical diagonalisation but these however take less effort than the Monte Carlo simulations. Firstly, T_l is diagonalised for fixed values of y, N_p and l. The size of the matrix is increased and the convergence of the eigenvalues investigated, as shown in Fig. 1. For values of y greater than $y = 1\cdot 7$ using the $r = 2$ expansion the convergence of the eigenvalues is incomplete, as n becomes large enough, so that the last term in the expansions becomes significant in the diagonalisation. In this case the estimate is taken to be the value corresponding to the point of inflection, an appropriate procedure as can be seen, by repeating the process for higher values of N_p at the fixed value of y. As N_p is increased the estimates for the ground state energy density for all values of y for all orders is found to converge (see Fig. 2), providing the final infinite lattice estimates.

The calculation of M_S is numerical for all orders and follows the above procedure. However, the $r = 2$ estimate for the scalar mass is restricted to those values of $y \leq 1\cdot 7$, as it is found that its value does not converge beyond this point if the difference between the eigenvalues is taken at the point at which either the first or the second eigenvalue breaks down.

The connected Hamiltonian moments with respect to trial states in the axial, tensor and string sectors of the Hilbert space were also calculated by Horn and Lana (1991). These trial states are of the form $\Omega|0_s\rangle$, where Ω is an operator such that the trial state has the appropriate quantum numbers associated with the sector of interest. The Hamiltonian moments are read off directly from the t-expansion expressions, and the form of the various trial states leads to non-zero b_n. The c_n in the moments are just the contributions from the vacuum, the same as used previously to study the vacuum sector. By taking the difference between the lowest eigenvalue of the Hamiltonian matrix developed by using these various non-vacuum moments and the ground state energy found previously, estimates of the lowest axial glueball mass (1^{+-}) M_A, the lowest tensor glueball mass (2^{++}) M_T, and the string tension σ are found. Calculations are performed retaining terms in the expansion to a given order, leading to $r = 0$, $r = 1$ and $r = 2$ estimates as previously.

3. The Results

The ground state energy density for the various orders is presented in Fig. 3. At strong coupling (small y) it is seen that the plaquette expansion has already converged at order $r = 0$, as all orders are in agreement below $y \simeq 1\cdot 5$. For values of y beyond this point up to approximately $y = 2\cdot 3$ there are slight discrepancies between the orders but they are still consistent to within a few per cent. However, it is also noted that the sequence of results is not monotonic as the order is increased, with the $r = 0$ closer than the $r = 1$ curve to the $r = 2$ calculation over most of this range. At higher values the first two order calculations, $r = 0$ and $r = 1$, turn downwards after reaching a value of $5\cdot 3$ at approximately $y = 2\cdot 6$ and $y = 2\cdot 4$ respectively. These values represent the extremes at which these results can be considered at all reliable. By including the next order terms, $r = 2$, the situation is improved. The gradient at no stage is negative and the ground state energy density agrees with the results of the t expansion (Horn and Lana 1991) for a greater range of y, with both techniques producing curves that pass through $5\cdot 6$ at $y \simeq 2\cdot 6$. As the continuum limit

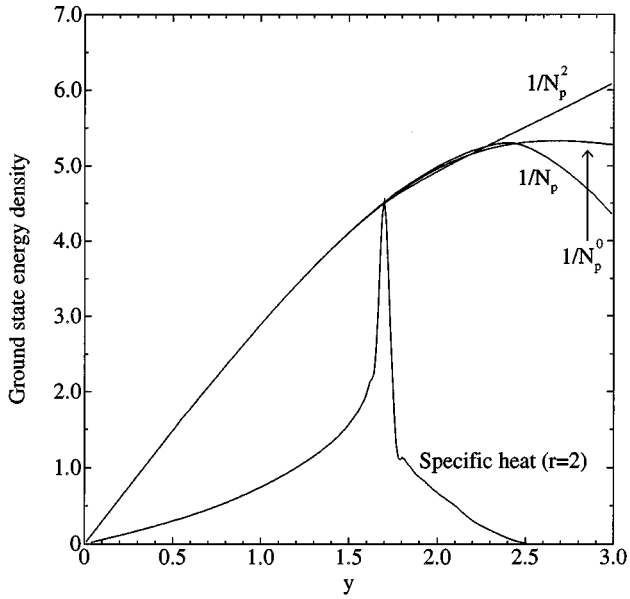

Fig. 3. Ground state energy density as a function of y for the various orders. Also plotted is the $r = 2$ estimate of the specific heat.

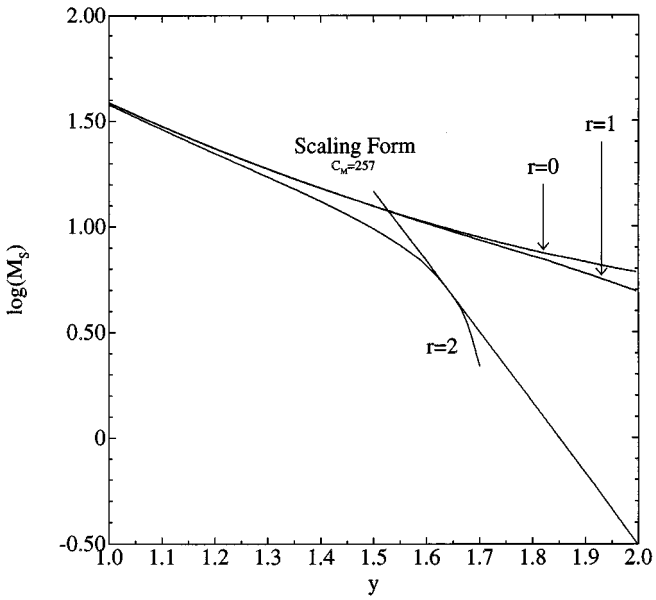

Fig. 4. A plot of $\log(M_S)$ as a function of y for the various orders. Included is the scaling form that matches onto the $r = 2$ ($C_M = 257$) calculation.

is approached this calculation unfortunately also becomes no longer valid as it shows no indication of levelling off to a constant value, as the ground state energy density should do.

Also plotted in Fig. 3 is the specific heat, defined as $-d^2\mathcal{E}_0/dy^2$, for the case $r = 2$. The peak seen is sharper and higher than that obtained using the t expansion, but its position, $y \simeq 1\cdot 7$, is consistent with that of the various t expansion approximants which peak in the region $y = 1\cdot 5$ to $1\cdot 7$. This specific heat peak indicates the position of a transition region between strong coupling and weak coupling, with predictions of physical quantities expected to match onto various scaling forms beyond this point.

The scaling form for any physical quantity which has the dimension of mass is (Politzer 1973; Gross and Wilczek 1973)

$$m(g) \approx C(\beta_0 g^2)^{-\beta_1/2\beta_0^2} \exp\left(-\frac{1}{2\beta_0 g^2}\right), \tag{9}$$

where β_0 and β_1 are the lowest coefficients in the Callan–Symanzik β-function (Caswell 1974; Jones 1974):

$$\beta_0 = \frac{11}{16\pi^2}, \qquad \beta_1 = \tfrac{34}{3}\left(\frac{3}{16\pi^2}\right)^2. \tag{10}$$

We plot M_S versus y in Fig. 4 to determine if the results from the plaquette expansion match onto this weak coupling form beyond the transition region. Again, the plaquette expansion converges quickly in the strong coupling region, before the results start to diverge as the transition region is approached. As the order of the calculation is increased there is a decrease in the value of M_S, with the $r = 0$ and $r = 1$ expansions consistent with one another well into the weak coupling region. They do not however show any sign of turning over onto the scaling form. By including the next terms, the $r = 2$ calculation, it is seen that these curves are already different from the lower orders well before the transition region and continue to diverge beyond this point. Although there is no extended region in which these curves approximate the scaling form (9) they may be matched onto the scaling form at a point within the weak coupling region at $y = 1\cdot 65$ corresponding to a value of $C_{M_S} = 257$.

The quantity $\sigma^{\frac{1}{2}}$ also has the dimensions of mass and thus is also expected to match onto the weak coupling form (9) with a corresponding value for $C_{\sigma^{\frac{1}{2}}}$. A ratio such as $M_S/\sigma^{\frac{1}{2}}$ should then become independent of the coupling constant in the continuum limit. This ratio is plotted in Fig. 5 for the various orders. Again there is no clear monotonic behaviour of the ratio as the order is increased, and the $r = 2$ calculation varies significantly from the lower orders in the weak coupling region. There is also no clear sign of the ratios levelling out to a constant value, but at the start of the transition region, $y = 1\cdot 5$, the $M_S/\sigma^{\frac{1}{2}}$ ratio decreases from $3\cdot 3$ for the $r = 1$ calculation to a value of $3\cdot 0$ for the $r = 2$. Having obtained this value for the ratio it is possible to use the string tension to renormalise the theory as its value, $\sigma = (420\ \mathrm{MeV})^2$, can be determined from slopes of Regge trajectories. This gives an approximant range of 1260 MeV to

1400 MeV for the mass of the lowest scalar glueball. These values may now be compared to other recent analyses. Within the transition region the t expansion estimate for M_S reaches a value of about $3\sigma^{\frac{1}{2}}$ (Horn and Lana 1991), whilst the various curves obtained by extrapolating the strong coupling expansion series by Hamer (1989) give a value of $2\cdot4 \pm 0\cdot5$ for the $M_S/\sigma^{\frac{1}{2}}$ ratio. The most recent Monte Carlo calculation by Bali *et al.* 1993 produces a slightly higher value of 1550 ± 50 MeV for the scalar mass.

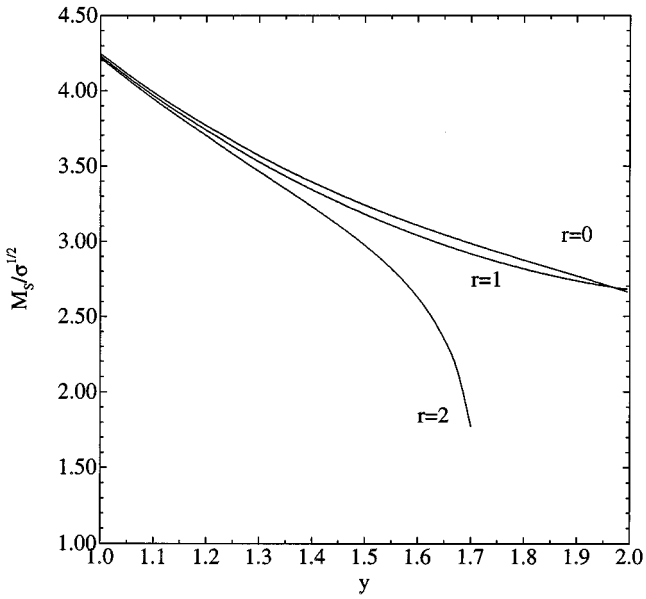

Fig. 5. The ratio $M_S/\sigma^{\frac{1}{2}}$ as a function of y for the various orders, $r = 0$ to $r = 2$.

Finally, in order to obtain estimates for the glueball mass ratios, the ratios $M_T/\sigma^{\frac{1}{2}}$ and $M_A/\sigma^{\frac{1}{2}}$ are plotted in Figs 6 and 7 respectively. For the tensor case the ratio at the transition region increases as the order of the calculation is increased, from a value of $3\cdot2$ for $r = 0$ to $3\cdot9$ for $r = 2$ at $y = 1\cdot5$. The spread in the axial ratio over the various orders, $4\cdot75$ to $4\cdot85$, is less than either of the other two glueball mass ratios, as can be seen in Fig. 7, with all $M_A/\sigma^{\frac{1}{2}}$ ratios reaching a minimum close to the start of the transition region.

With these estimates the various glueball mass ratios can now be estimated. The M_T/M_S and M_A/M_S ratios start off from 1 at $y = 0$ as all trial states are made from single plaquette wave functions. The M_T/M_S ratio remains very close to 1 throughout the strong coupling region, with the tensor and scalar glueballs degenerate over the entire range at order $r = 0$ as the connected moments required at this order are the same for both. The ratio starts to rise as the transition region is approached reaching values in the range $1\cdot05$ to $1\cdot3$ for the various orders, these numbers having being obtained from the corresponding string tension curves, which provide estimates that are consistent with other works. The t expansion (Horn and Lana 1991) provides an estimate of $1\cdot06 < M_T/M_S < 1\cdot3$, whilst the strong coupling expansion curves (Hamer

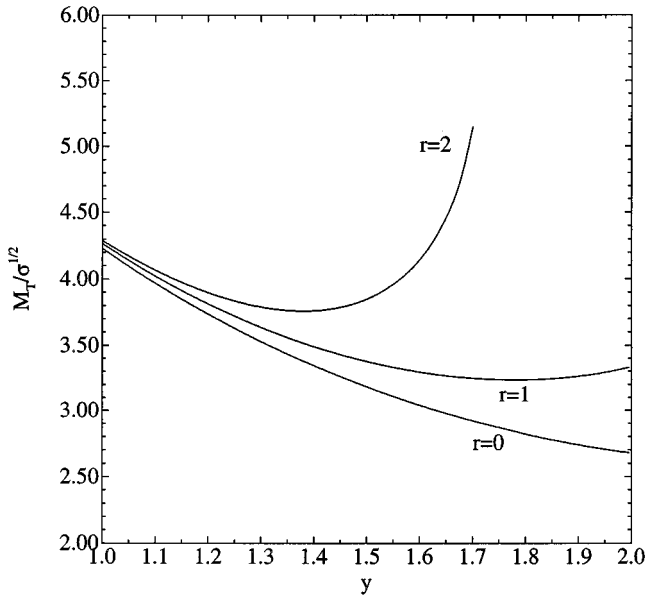

Fig. 6. The ratio $M_T/\sigma^{\frac{1}{2}}$ as a function of y for the various orders, $r = 0$ to $r = 2$.

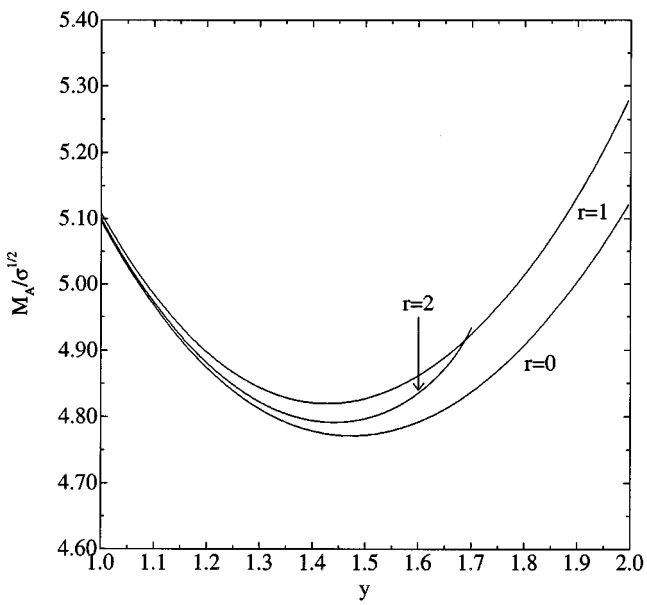

Fig. 7. The ratio $M_A/\sigma^{\frac{1}{2}}$ as a function of y for the various orders, $r = 0$ to $r = 2$.

1989) rise from about $1\cdot 2$ at $y = 1\cdot 5$ to a value of approximately $1\cdot 5$ at $y = 2$. The ratio of Chin *et al.* (1988) rises more slowly over this range reaching a value of $1\cdot 2$ at $y = 2$, whilst Bali *et al.* (1993) give a value of about $1\cdot 46$. The M_A/M_S ratio is greater than 1 even within the strong coupling region and rises to a greater value at the transition region. The various curves give an estimate in the range $1\cdot 4$ to $1\cdot 65$. These may once more be compared to the results for the t expansion which give $1\cdot 4 < M_A/M_S < 1\cdot 8$, the strong coupling expansion results, $1\cdot 5$ at $y = 1\cdot 5$ and 2 at $y = 2$ for M_A/M_S, and the numerical calculations of Bali *et al.* (1993), $M_A/M_S = 1\cdot 9$.

The above numbers for the various mass estimates can only be considered approximate at best, there still being various issues related to the technique that must be resolved. In particular the lack of convergence within the scaling region is disappointing. Although there are no observable scaling ratios observed, when compared to the results obtained by other techniques it appears as though the plaquette expansion results are reaching the transition region without the need for any extrapolation. It may well be that the choice of the strong coupling state as a trial state, although facilitating computation of moments, may be completely inappropriate in the scaling region. Higher order calculations in progress will determine this. However, progress towards an understanding of lattice gauge theory using this technique may be made by considering exponential states as used in variational studies.

References

Bali, G. S., Schilling, K., Hulsebes, A., Irving, I. C., Michael, C., and Stephenson, P. W. (1993). *Phys. Lett.* B **206**, 378.

Caswell, W. E. (1974). *Phys. Rev. Lett.* **33**, 244.

Chin, S. A., Long, C., and Robson, D. (1988). *Phys. Rev. Lett.* **60**, 1467.

Gross, D. J., and Wilczek, F. (1973). *Phys. Rev. Lett.* **30**, 1343.

Gupta, R., Patel, A., Baillie, C. F., Kilcup, G. W., and Sharpe, S. R. (1991). *Phys. Rev.* D **43**, 2301.

Hamer, C. (1989). *Phys. Lett.* B **224**, 339.

Hollenberg, L. C. L. (1993). *Phys. Rev.* D **47**, 1640.

Hollenberg, L. C. L. (1994*a*). *Phys. Rev.* D **50**, 2293.

Hollenberg, L. C. L. (1994*b*). *Phys. Rev.* D **50**, 6917.

Hollenberg, L. C. L., and Witte, N. S. (1994). *Phys. Rev.* D **50**, 3382.

Horn, D., and Lana, G. (1991). *Phys. Rev.* D **44**, 2864.

Horn, D., and Weinstein, M. (1984). *Phys. Rev.* D **30**, 1256.

Jones, D. R. T. (1974). *Nucl. Phys.* B **75**, 531.

Kogut, J., and Susskind, L. (1975). *Phys. Rev.* D **11**, 395.

Kogut, J., Sinclair, D. K., and Susskind, L. (1976). *Nucl. Phys.* B **114**, 199.

Michael, C., and Teper, M. (1989). *Nucl. Phys.* B **314**, 347.

Politzer, H. D. (1973). *Phys. Rev. Lett.* **30**, 1346.

Tomlinson, M. J., and Hollenberg, L. C. L. (1994). *Phys. Rev.* B **50**, 1275.

van den Doel, C. P., and Horn, D. (1987). *Phys. Rev.* D **35**, 2824.

Vohwinkel, C., and Berg, B. A. (1989). *Phys. Rev.* D **40**, 584.

Wilson, K. G. (1974). *Phys. Rev.* D **10**, 2445.

Witte, N. S., and Hollenberg, L. C. L. (1995). *Z. Phys.* B **99**, 101.

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