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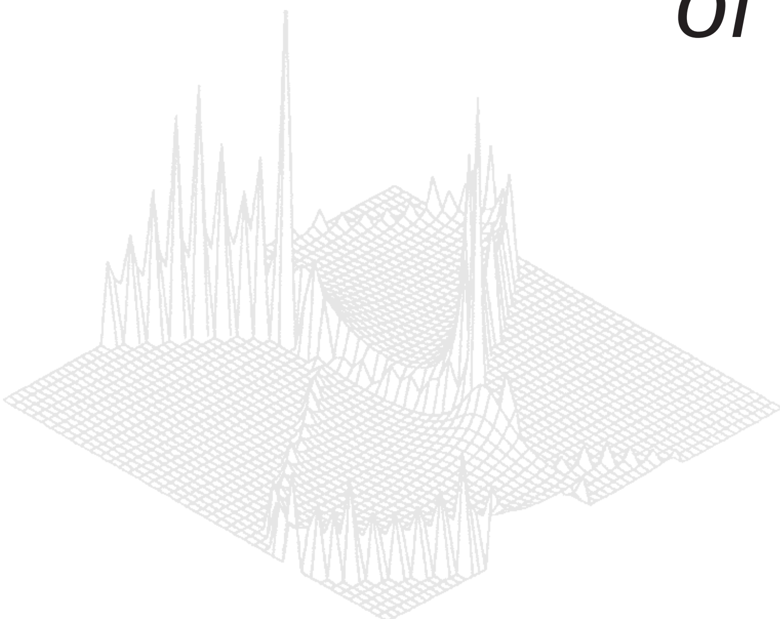

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Quantum Theory of Gravitation

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Abstract

It is possible to construct the non-euclidean geometry of space–time from the information carried by neutral particles. Points are identified with the quantal events in which photons or neutrinos are created and annihilated, and represented by the relativistic density matrices of particles immediately after creation or before annihilation. From these, matrices representing subspaces in any number of dimensions are constructed, and the metric and curvature tensors are derived by an elementary algebraic method; these are similar in all respects to those of Riemannian geometry. The algebraic method is extended to obtain solutions of Einstein's gravitational field equations for empty space, with a cosmological term. General relativity and quantum theory are unified by the quantal embedding of non-euclidean space–time, and the derivation of a generalisation, consistent with Einstein's equations, of the special relativistic wave equations of particles of any spin within representations of $SO(3) \otimes SO(4,2)$. There are some novel results concerning the dependence of the scale of space–time on properties of the particles by means of which it is observed, and the gauge groups associated with gravitation.

1. Relativistic Wave Mechanics of Neutral Particles

Much of modern physics is based on two fundamental but disconnected theories: Einstein's theory of space, time and gravitation, and the theory of quantum mechanics, each of which has not greatly changed since its formulation in the first quarter of the present century. There have been many attempts to reconcile the two theories, but it is evident in the recent literature (Ellis and Matravers 1995; Zalaletdinov *et al.* 1996) that some problems remain, centred around the difficulty in identifying local systems of coordinates in general relativity corresponding to those in which quantal calculations are made.

In his formulation of the special theory of relativity, Einstein recognised the importance of the observation of light signals in establishing the geometrical framework. Most of our information concerning physical space and time is in fact derived from the detection of photons, whose momentum determines the apparent direction of their source, and whose energy, in a cosmological context, is used to determine the distance of the source. It is mainly from the accumulation of such information that our knowledge of the geometry of the universe has been derived. On the other hand, it has become usual more recently to assume that the geometrical framework in which general relativity is formulated is fundamental, and that space–time coordinates can then be assigned to quantal events within this framework. It would therefore seem useful to reexamine to what extent a

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geometry of space and time can be constructed from the observation of individual events, especially the quantal events associated with the emission and absorption of neutral particles. In the present paper we shall investigate the possibility of developing a theory, not inconsistent with Einstein's general theory of relativity, but based solely on the quantal events that form the substrate of observation.

The interest of such an investigation is enhanced by the increasing amount of information concerning distant sources derived from neutrinos (Hiraka *et al.* 1987; Bionta *et al.* 1987; Fogli *et al.* 1995). The experimental evidence (Mikheyev and Smirnov 1988; Mureika and Mann 1996) suggests that these particles, unlike photons, have a non-vanishing rest-mass. It is not yet established that a physical geometry constructed from the observation of neutrinos would be the same as that derived from the observation of light, and one of the aims of this paper is to provide some indication of differences which in the future could be detected experimentally. Of course the results depend to some extent on the hypotheses adopted, and we shall assume that neutrinos are Majorana particles (Majorana 1937), with a very small mass if the experimental evidence derived from double β-decay (see Moe 1995) is taken into account, and normally if not always emitted and absorbed in left-handed states.

Since a neutrino with non-vanishing mass cannot be identical with its anti-particle, this evidence suggests that it is a parity doublet (Green and Hurst 1957) and in the following we shall replace the mass m in Dirac's equation for the electron by a matrix $m\tau$, where τ is an helicity conjugation matrix, so that the wave equation for neutral leptons in the interaction representation becomes

$$i\gamma^\lambda \psi_\lambda = m\tau\psi, \quad \psi_\lambda = \frac{\partial \psi}{\partial x^\lambda} \quad (\lambda = 0, 1, 2, 3)\,. \tag{1}$$

As usual in particle physics, units are chosen so that $c = \hbar = 1$, where c is the velocity of light, and $2\pi\hbar$ is Planck's constant; this leaves only the unit of length unspecified. In the Majorana representation, the γ-matrices (including $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$) are all imaginary, so that there are solutions ψ which are purely real or imaginary, provided that the matrix τ is real. Instead of the real and imaginary parts of solutions of Dirac's equation, ψ may be resolved into even and odd components $\psi_{\rm e}$ and $\psi_{\rm o}$ which are unchanged and change sign, respectively, under the transformation $x^\lambda \to -x^\lambda$. We may then write $\tau = \tau_1\gamma_5$, where τ_1 is the imaginary Pauli matrix, here defined, together with τ_2 and τ_3, by

$$\tau_1(\psi_{\rm e}, \psi_{\rm o}) = i(-\psi_{\rm o}, \psi_{\rm e}), \quad \tau_2(\psi_{\rm e}, \psi_{\rm o}) = (\psi_{\rm o}, \psi_{\rm e}), \quad \tau_3(\psi_{\rm e}, \psi_{\rm o}) = (-\psi_{\rm e}, \psi_{\rm o})\,. \tag{2}$$

It is possible for neutrinos of non-vanishing mass to exist in eigenstates of the helicity, if this is represented by the matrix $\tau_2\gamma_5$, which anticommutes with both $\gamma^\lambda$and τ.

It is well known (see Gilmore 1974) that Pauli matrices define a spinor representation of $SU(2)$ locally isomorphic with $SO(3)$. In the context of a theory of gravitation, the use of imaginaries is not appropriate, and we shall therefore regard the imaginary unit i as a real antisymmetric matrix similar to though distinct from $i\tau_1$. Then γ^0 is the direct product of two antisymmetric matrices, and is therefore symmetric, while the other γ^λ are antisymmetric. In the Majorana representation the matrices $i\gamma^\lambda$ are in an irreducible representation of $SO(4,2)$, but, because of the introduction of the factor τ, the generalised Majorana equation (1) is in a representation of $SO(3) \otimes SO(4,2)$. This equation has symmetries associated with permutations of τ_1, τ_2 and τ_3, but distinct from the charge conjugation symmetries stemming from the use of the imaginary unit i in Dirac's equation. Such symmetries are naturally broken by interactions as well as by the special role of these matrices. The interchange of τ_1 and τ_3 affects the helicity of the neutrino, and even permutations could find a use in the representation of 'flavour'. With this interpretation of the τ-matrices, we shall find that in transmission over sufficiently large distances and also in a gravitational field, the mixing of different states of helicity as well as flavour is possible.

The wave equation for a photon in the interaction representation may be written in a form similar to (1), but with Kemmer matrices replacing the Dirac–Majorana matrices:

$$i\beta^\lambda \psi_\lambda = m\tau\psi, \quad \psi_{\mathrm{e}} = (A^\lambda, F^{\mu\nu}), \quad \psi_{\mathrm{o}} = (A_{\mathrm{d}}^\lambda, F_{\mathrm{d}}^{\mu\nu}), \tag{3}$$

where $\tau = \tau_1\beta_5$, and τ_1 is defined as in (2); A^λ and $F^{\mu\nu}$ are field potentials and intensities in the usual notation, and A_{d}^λ and $F_{\mathrm{d}}^{\mu\nu}$ are their duals, resulting from the interchange of electric and magnetic field variables ($\mathbf{E}_{\mathrm{d}} = \mathbf{B}$, $\mathbf{B}_{\mathrm{d}} = -\mathbf{E}$). The action of the Kemmer matrices on the 10-component vector ψ_{e} is given by

$$i\beta^\lambda \psi_{\mathrm{e}} = (F^{\lambda\mu}, \eta^{\lambda\nu}A^\rho - \eta^{\lambda\rho}A^\mu), \quad i\beta_5\psi_{\mathrm{e}} = (0, F_{\mathrm{d}}^{\mu\nu});$$

where $\eta_{\lambda\mu}$ is the pseudo-euclidean metric tensor with diagonal elements $(1,-1,-1,-1)$. If these substitutions are made in (3), the latter equation reduces to Maxwell's equations in the absence of charge, and the eigenvalue zero of β_5 on the four-vector A^λ ensures that the mass m appears only in the relation between the intensities and the potentials, so that the mass of the photon vanishes. As usual in the interaction representation, photons with a definite spin are created by electromagnetic interactions in eigenstates of the helicity.

The interactions associated with gauge therories may result in permutation of the τ-matrices, and then other solutions of (3) with non-zero rest-mass can be found which could represent the neutral heavy vector boson in electro-weak theories with isospin, but, because of this particle's instability, such solutions are not of interest in the present context.

2. Preliminaries

In this paper, an interpretation will be given of Einstein's law of gravitation in the context of the above formulation of the quantum mechanics of neutral particles, generalised to take account of the curvature of space–time associated with cosmology and the gravitational field. A point of space time will be identified with *an event* in which a neutral particle is emitted or absorbed, and a geodesic with the path of the particle, or what, in a projective geometry, is the *join* of the points of emission and absorption. The emission and absorption of a particle may be treated as separate events, and if the particle propagates over a distance which is large by microscopic standards the energy, momentum and helicity of the particle are selected observables. The state of the particle in transmission will be represented by a relativistic density matrix z, invariant under coordinate transformations and normalised so that its trace is 1; the states of the microscopic systems emitting and absorbing the particles will be represented by density matrices $\rho_{\rm s}$ and ρ, respectively, in Dirac's (1935) sense.

Immediately following emission, the relativistic matrix $z_{\rm s}$ of the particle is strongly correlated with the density matrix $\rho_{\rm s}$ of its microscopic emitter; the latter is normally a component of a more extended source. In a similar way, in the process of absorption, the relativistic matrix z of the particle becomes strongly correlated with the density matrix ρ of its microscopic absorber. Assuming that the particle is observed, the absorber is a component of an extended detector, and with a suitable detector it is possible to measure the energy–momentum, polarisation and type of the particle, so that its density matrix z at the instant of absorption may be inferred to have been in a pure state, which provides information concerning not only the particle itself but the direction and other characteristics of its source. We shall show how a non-euclidean projective geometry of space–time may be constructed from this and similar information.

It deserves to be emphasised that the quantum theory of the present paper is concerned primarily with properties of neutral particles which are either observed or in principle observable; however, the effect of quite general gauge fields on these particles, including those associated with gravitation, will be taken into account in a way that is consistent with the quantisation of those fields. The emission and absorption of a particle are usually in different inertial frames. According to the usual principles of quantum mechanics, the relativistic matrices z and $z_{\rm s}$ are therefore connected by a transformation which is pseudo-unitary or pseudo-orthogonal, depending on the representation:

$$z = u z_{\rm s} \bar{u}, \quad u = u_{\rm c} u_{\rm g} \bar{u}_{\rm c}\,, \tag{4}$$

where the factors $u_{\rm c}$ and $u_{\rm g}$ represent a cosmological and gravitational transformation, respectively. The cosmological factor includes a local Lorentz transformation, responsible for aberration and the Doppler shift in the energy of the observed particle, in addition to the cosmological red shift, whereas the gravitational factor is responsible for a change of gravitational potential and the gravitational shift in frequency.

We shall begin by giving a more precise definition of the relativistic matrix z and establish a representation space for it in accordance with the procedures of quantised field theory. Within a sufficiently small region of space–time, the relativistic wave equations (1) and (3) are assumed to be valid in the interaction representation. These wave equations for a neutral particles can be generalised for any spin s in the form

$$i\alpha^\lambda \psi_\lambda = m\tau\psi\,, \tag{5}$$

where the α^λ-matrices are imaginary $(\frac{1}{2}\gamma^\lambda)$ for spin $\frac{1}{2}$, β^λ for spin 1 and τ is the real pseudoscalar given by

$$\tau = i\alpha_4, \quad \alpha_{c+3} = \tau_c \epsilon_{\lambda\mu\nu\rho}\alpha^\lambda\alpha^\mu\alpha^\nu\alpha^\rho/(24s^3) \quad (c = 1, 2, 3), \tag{6}$$

which also defines three antisymmetric matrices anticommuting with one another, and, in the spinor representation, with the Majorana matrices as well. There is always a real symmetric matrix η (γ_0 for spin $\frac{1}{2}$, $2\beta_0^2-1$ for spin 1), satisfying $\eta^2=1$ and commuting with α^0 but anticommuting with the other α^λ and τ, so that

$$i\bar{\psi}_\lambda\alpha^\lambda = m\bar{\psi}\tau, \quad \bar{\psi}=\psi^{\mathrm{t}}\eta\,, \tag{7}$$

where ψ^{t} is the column to row transpose of ψ. As the α^λ anticommute with τ, $\bar{\psi}\alpha^\lambda\psi$ is as usual a conserved current density.

Since the α-matrices in (2) and (5) are imaginary and τ is real, the solutions of these equations may be purely real or imaginary. They are satisfied by the field variables of quantised field theory in the interaction representation, where ψ and $\bar{\psi}$ are normally expanded in terms of a complete set of orthonormal solutions ζ_p and $\bar{\zeta}_p$, which reduce to Fourier series within a rectangular region of volume V. Thus

$$\psi=\sum_p c_p\zeta_p/\mid p^0V\mid^{\frac{1}{2}}, \quad \bar{\psi}=\sum_p \bar{c}_p\bar{\zeta}_p/\mid p^0V\mid^{\frac{1}{2}}\,, \tag{8}$$

where $\pm p^0$ is the (positive) energy of a created particle and c_p and $\bar{c}_p$ are creation or annihilation operators, depending on the sign of p^0. The relativistic density matrix of a neutral particle, normalised to 1, is then defined as an outer product of the type $z_p=\zeta_p\bar{\zeta}_p$, and is always real. In a cosmological context a similar expansion is possible but the rectangular region must be deformed and extended to the horizon, and the volume is then the (finite) volume of the observable universe.

But in cosmology and general relativity the equations of Dirac and Kemmer also require generalisation, for charged as well as neutral particles. This is usually done (see, e.g., Zhelnorovich 1987, 1996) by the substitution of coordinate-dependent matrices for the Dirac and Kemmer matrices. We shall follow this approach to obtain a generalisation of (5) in Sections 7 and 8 of this paper, but for the present we simply accept the matrices α^λ and τ_c as providing the algebraic substructure of a generalised theory.

3. Representations for Arbitrary Spin

When expressed in terms of the α-matrices, the commutation relations satisfied by the elements of both the Dirac–Majorana and Kemmer algebras are

$$[\alpha_{jk},\alpha_l]=i(\eta_{kl}\alpha_j-\eta_{jl}\alpha_k), \quad [\alpha_j,\alpha_k]=i\alpha_{jk}\,. \tag{9}$$

These relations are also applicable for any spin. Where the subscripts are restricted to values $(0,1,2,3)$, they are replaced by greek characters, so that the $\alpha_{\lambda\mu}$ are generators of a representation of the Lorentz group. But here the interpretation of the subscripts of α_{jk} and η_{jk} may be extended include the values 4, 5 and 6 with α_4, α_5 and α_6 defined as in (6) and $\eta_{44}=\eta_{55}=\eta_{66}=-1$ in (9). With this extended range of subscripts, the α_{jk} are generators of representations of $SO(6,1)$ and the α_j and α_{kl} together are generators of irreducible representations of $SO(6,2)$, within the reducible group $SO(3)\otimes SO(4,2)$ resulting from the inclusion of the τ_c. The matrices $\alpha_{\lambda 4}$ can be interpreted as generators of translations in a de Sitter space of radius R and, together with the $\alpha_{\lambda\mu}$, can be used to construct the factor $u_{\rm c}$ in (1). In a local region, the de Sitter space approximates very closely to the Minkowski space of special relativity. The scalar matrices α_{45}, α_{56} and α_{64} are generators of gauge transformations. The other elements $\alpha_{\lambda 5}$ and $\alpha_{\lambda 6}$ of the Lie algebra may be interpreted as generators of boosts for neutral particles and therefore have a natural role in a theory of gravitation where they will be used to construct the gauge transformation $u_{\rm g}$ in (1). Although these matrices do not commute exactly in general, they have projections onto the chiral states of special relativity which do so.

We have already noticed that the matrices α_λ are imaginary and τ is real in the Majorana representation, and it is quite possible for the solution ψ of (5) to be real. In quantised field theory it is usual to employ complex solutions which are eigenvectors of observables, such as the energy and momentum, that are represented by imaginary differential operators in the coordinate representation. But geometry, and the theory of neutral particles, are traditionally formulated in terms of real quantities, and this has been achieved in the present context by interpreting the imaginary unit as a real asymmetric matrix and the ζ_p in (8) are therefore real even though they are eigenvectors of the energy and momentum.

The representation of the τ_c is independent of the spin, but there are both spinor and tensor representations of the factor $SO(4,2)$ of $SO(3)\otimes SO(4,2)$. The spinor representations of $SO(4,2)$ are real analogues of the complex 4-dimensional spinor representations that are often referred to as unitary and are isomorphic with the group $SU(2,2)$, while the irreducible vector representation is 10-dimensional. As shown in the previous Section, the real spinor representation may be used for neutrinos and the vector representation for photons. In the following, though we are most interested in the applications to neutrinos and photons, it will be found possible to formulate a geometrical basis for a theory of gravitation in a form which is independent of the spin and even of the representation. All of the irreducible finite-dimensional representations of $SO(4,2)$ can be obtained from spinor (Dirac or Majorana) representations by a construction similar to that used earlier in formulating the theory of parafermionic fields (Green 1953). For spin s, we may write

$$\alpha_j = \sum_{r=1}^{2s} \alpha_j^{(r)}, \quad \alpha_{kl} = \sum_{r=1}^{2s} \alpha_{kl}^{(r)}\,, \tag{10}$$

where the $\alpha_j^{(r)}$ are in spinor representations but commute for different values of r. The general formula for the matrix η in (3) is $\prod_r (2\alpha_0^{(r)})$. Any irreducible representation is characterised by its highest weight vector, whose components are the highest eigenvalues l_1, l_2 and l_3 of the commuting real symmetric matrices α_{03}, $i\alpha_{12}$ and $i\alpha_5$ representing the state, the spin and helicity of a neutral particle, respectively, in a particular Lorentz frame at the optical horizon. The quadratic invariant of $SO(4,2)$ is

$$\sum_{j=0}^{5}\left(\alpha^j\alpha_j + \sum_{\kappa=0}^{5}\alpha^{jk}\alpha_{jk}\right) = 2[l_2(l_2+4) + l_3(l_3+2) + l_4^2]\,.$$

To avoid the well known problems arising from the use of more general representations, we shall later adopt representations for particles of spin s of the type used for parafermions of order $2s$ (Bracken and Green 1972), with highest weight vector $(s, s, \pm s)$, noting that the Dirac and Kemmer representations for spin $\frac{1}{2}$ and spin 1, respectively, are of this type. However, the nature of the representation will be not be needed until the final Sections of this paper, where it will appear that the state of highest weights plays a physically important part in the emission of neutral particles, in the interaction representation.

4. Quantum Geometry

We now describe briefly the procedure for constructing a projective geometry of space–time in terms of the normalised density matrix of neutral particles in the coordinate representation. A point is associated with the emission or absorption of an observed particle, and is therefore represented by a relativistic density matrix z which is idempotent and minimal:

$$z^2 = z, \quad \mathrm{tr}(z) = 1\,. \tag{11}$$

These relations are not affected by pseudo-orthogonal transformations, including gauge transformations, of the type $z \to vz\bar{v}$, under all of which z remains real and symmetric. The normalisation of the trace to unity implies that z may be expressed as an outer (tensor) product of vectors ζ and $\bar{\zeta}$ of the type introduced in (8):

$$z = \zeta\bar{\zeta}, \quad \bar{\zeta}\zeta = \mathrm{tr}(z) = 1\,, \tag{12}$$

where $\bar{\zeta}$ is the conjugate $\zeta^{\mathrm{t}}\eta$ of ζ, and $\bar{\zeta}\zeta$ denotes the corresponding inner (scalar) product. Since z is real, the factors ζ and $\bar{\zeta}$ may also be assumed to be real. When z is identified with the relativistic density matrix of an observed particle at that point, the factorisation is unique except in respect of sign. It is important to note that, since the vectors are real and η is symmetric, the inner product satisfies the condition $\bar{\zeta}\zeta' = \bar{\zeta}'\zeta$.

In modern projective geometry (see Murray and von Neumann 1936), a line is the join the of two distinct points. If the points are represented by z and z', where $zz' \neq z$, then the join is represented by an idempotent matrix $z^{(2)} = z \vee z'$, given explicitly in (14) below, such that $zz^{(2)} = z$, $z'z^{(2)} = z'$ and $\mathrm{tr}(z^{(2)}) = 2$.

More generally, if $z^{(d)}$ is a $(d-1)$-dimensional subspace, with $\mathrm{tr}(z^{(d)}) = d$ and $zz^{(d)} \neq z$, then there is d-dimensional subspace represented by $z^{(d+1)} = z \vee z^{(d)}$, where $zz^{(d+1)} = z$, $z^{(d)}z^{(d+1)} = z^{(d)}$ and $\mathrm{tr}(\,z^{(d+1)}) = d+1$. The join of a point z and any subspace $z^{(d)}$ not containing the point can be defined with the help of the recursive formula

$$z^{(d+1)} = z \vee z^{(d)} = z^{(d)} + (1 - z^{(d)})z(1 - z^{(d)})/[1 - \mathrm{tr}(zz^{(d)})]\,. \tag{13}$$

If $zz^{(d)} = z$, however the subspace $z^{(d)}$ contains the point z, and the join is defined as $z^{(d)}$. If the entire space is d-dimensional, $z^{(d+1)} = 1$.

For $d = 1$, $z'zz' = \mathrm{tr}(zz')z'$, and if this does not vanish the definition of the join can also be written as

$$z^{(2)} = z \vee z' = (z' - z)^2/\sigma^2, \quad \sigma^2 \ \tfrac{1}{2}\mathrm{tr}(z' - z)^2\,, \tag{14}$$

where, according as $\sigma^2 > 0$ or $\sigma^2 < 0$, $|\,\sigma\,|$ is interpreted as a space-like separation or as a time-like interval. When $\mathrm{tr}(zz') = 1$, σ vanishes and is light-like. The join of two points is a geodesic, in the sense of Riemannian geometry.

5. Differential Geometry of Space–Time

The projective geometry is perfectly adapted to a Riemannian manifold; it is only necessary to associate each point z of emission or absorption of a particle with some value of a differentiable matrix function $z(x)$ of arbitrarily chosen space–time coordinates (x^0, x^1, x^2, x^3). According to (14), the interval $\delta\sigma$ of the points z and $z + \delta z$ is given, for arbitrary δz, by $\delta\sigma^2 = \frac{1}{2}\mathrm{tr}(\delta z)^2$, and if δz is sufficiently small we write $dz = \delta z$ and $d\tau = \delta\tau$, so that

$$d\sigma^2 = \tfrac{1}{2}\mathrm{tr}(dz^2) = g_{\lambda\mu} dx^\lambda dx^\mu\,, \tag{15}$$

where

$$g_{\lambda\mu} = \tfrac{1}{2}\mathrm{tr}(z_\lambda z_\mu), \quad z_\lambda = \frac{\partial z(x)}{\partial x^\lambda}\,. \tag{16}$$

Then $g_{\lambda\mu}$ is the covariant metric tensor, which, since $z = \zeta\bar{\zeta}$ and $\bar{\zeta}\zeta_\lambda = \bar{\zeta}_\lambda\zeta = 0$, can also be written

$$g_{\lambda\mu} = \bar{\zeta}_\lambda \zeta_\mu\,. \tag{17}$$

The matrix dz^2 represents the geodesic connecting the points z and $z + dz$, and the equation (15) for the line element can be interpreted as a first integral of the geodesic equation of motion of a classical particle. In the quantum theory, a geodesic can of course be constructed connecting the points of emission and absorption of a particle, but the vector ζ representing a particle satisfies a wave equation which determines the probability that it will be detected at a particular point. This is directly related to the classical geodesic equation of motion through (15), (16) and (17). In Section 7 it will be shown how an arbitrary metric tensor of Einstein's general theory of relativity can be expressed in the form (17).

From (12) and (16), we have

$$z_\lambda z_\mu = (\zeta_\lambda\bar{\zeta} + \zeta\bar{\zeta}_\lambda)(\zeta_\mu\bar{\zeta} + \zeta\bar{\zeta}_\mu) = \zeta_\lambda\bar{\zeta}_\mu + g_{\lambda\mu}\zeta\bar{\zeta}\,. \tag{18}$$

We define $g^{\lambda\mu}$ and z^λ by

$$g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda_\nu, \quad z^\lambda = g^{\lambda\mu} z_\mu \tag{19}$$

in the ordinary way, and it follows that

$$z^{(4)} = z^\lambda z_\lambda - 4z = \zeta^\lambda \bar{\zeta}_\lambda \tag{20}$$

is an idempotent. Since $\bar{\zeta}^\lambda \zeta_\lambda = 4$, this satisfies $\mathrm{tr}(z^{(4)}) = 4$ as required, and represents a three-dimensional subspace, but as $zz^{(4)} = 0$, it does not contain the point z. Then

$$z^{\dagger} = 1 - z^{(4)} = 1 - \zeta^{\lambda}\bar{\zeta}_{\lambda} \tag{21}$$

is also an idempotent, and represents a subspace containing z but not $z^{(4)}$.

The Christoffel affinity, derived in the usual way from the metric tensor in (16) and (17), is

$$\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2}\mathrm{tr}(z^{\lambda} z_{\mu,\nu}) = \bar{\zeta}^{\lambda}\zeta_{\mu,\nu} = -\bar{\zeta},^{\lambda}_{\nu}\,\zeta_{\mu}\,, \tag{22}$$

where the subscript ν denotes differentiation with respect to x^{ν}, so that the covariant derivative of v_{μ} is

$$v_{\mu/\nu} = (v_{\lambda}\bar{\zeta}^{\lambda}),_{\nu}\,\zeta_{\mu} = \bar{\zeta}_{\mu}(\zeta^{\lambda} v_{\lambda}),_{\nu}\ . \tag{23}$$

In particular, the covariant derivatives of ζ_{μ} and z_{μ} are

$$\zeta_{\mu/\nu} = (\zeta_{\lambda}\bar{\zeta}^{\lambda}),_{v}\,\zeta_{\mu} = -z^{\dagger}_{\nu}\zeta_{\mu} = z^{\dagger}\zeta_{\mu,\nu}\,, \tag{24}$$

with $z^{\dagger}$ given by (21), and

$$z_{\mu/\nu} = (z_{\lambda}\bar{\zeta}^{\lambda}),_{\nu}\,\zeta_{\mu} = (z_{\lambda}\bar{\zeta}^{\lambda}),_{\mu}\,\zeta_{\nu}\,. \tag{25}$$

Further covariant derivatives may be defined in the usual way so as to conform with the chain rule, and the Riemann–Christoffel tensor is given by

$$\zeta_{\lambda/\mu/\nu} - \zeta_{\lambda/\nu/\mu} = \zeta_{\rho} R^{\rho}_{\lambda\mu\nu}\,. \tag{26}$$

We note that, since $g_{\lambda\mu/\nu} = 0$ and $\bar{\zeta}_{\lambda/\mu}\zeta_{\nu} = \bar{\zeta}_{\nu}\zeta_{\lambda/\mu}$, where $\zeta_{\lambda/\mu}$ is symmetric, the identity $\bar{\zeta}_{\nu}\zeta_{\lambda/\mu} = 0$ holds, and it follows that

$$R^{\rho}_{\lambda\mu\nu} = \bar{\zeta}^{\rho}(\zeta_{\lambda/\mu/\nu} - \zeta_{\lambda/\nu/\mu}) = \bar{\zeta}^{\rho}_{/\mu}\zeta_{\lambda/\nu} - \bar{\zeta}^{\rho}_{/\nu}\zeta_{\lambda/\mu} \tag{27}$$

From (24) it is evident that this tensor can be constructed by ordinary differentiation, or by purely algebraic operations from ζ_{λ} and $z^{\dagger}_{\mu}$.

6. The Gravitational Field Equations

In its passage between its source and a point of observation, a particle traverses empty space. Einstein's field equation for empty space, with the usual cosmological term, is

$$R_{\lambda\mu} = -R^{\nu}_{\lambda\mu\nu} = 3g_{\lambda\mu}/R^{2}\,, \tag{28}$$

where R is of the order of the distance of the cosmic horizon. With the help of (27), this can be expressed entirely in terms of the vector ζ and its covariant derivatives:

$$\bar{\zeta}^{\nu}_{/\nu}\zeta_{\lambda/\mu} - \bar{\zeta}^{\nu}_{/\mu}\zeta_{\lambda/\nu} = 3\bar{\zeta}_{\lambda}\zeta_{\mu}/R^{2}\,. \tag{29}$$

However, it can be seen from (24) that the derivatives on the left side of this equation are projections of the ordinary derivatives onto a subspace of the representation space of the vectors ζ and $\bar{\zeta}$.

Important solutions of (29), including the well known static and wave-like solutions (Kramer *et al.* 1980), can be obtained by assuming $g_{\lambda\mu,\kappa} = 0$ for at least one value of κ. This does not require the vanishing of ζ_κ, but if we denote the x^κ-dependent component of ζ by υ we may write

$$\zeta_\kappa = \upsilon_\kappa = \epsilon\upsilon, \qquad \epsilon\eta = -\eta\epsilon, \quad \epsilon^2 = -\eta_{\kappa\kappa}\,, \tag{30}$$

so that $\bar{\upsilon}_\kappa = -\upsilon\epsilon$ and $g_{\kappa\kappa,\kappa} = 0$. If f is defined by

$$g_{\kappa\kappa} = f^2\eta_{\kappa\kappa}, \quad \bar{\upsilon}\upsilon = f^2\,, \tag{31}$$

it then follows that f is independent of x^κ. When the component υ of ζ has been obtained by the solution of (29) for the function f, other components may be obtained from (26) and (28), as in Section 8 below.

We shall first show that the function f^2 itself satisfies a generalised wave equation, obtained by substituting $\lambda = \mu = \kappa$ in (29). From $g_{\lambda\mu,\kappa} = 0$ it follows that $\bar{\upsilon}_\lambda\epsilon\upsilon_\mu = 0$, so that the resulting equation reduces to

$$\bar{\upsilon}^\nu\epsilon z^\dagger\epsilon\upsilon_\nu - \eta_{\kappa\kappa}\bar{\upsilon}{,}^{\nu}_{\nu}\,\upsilon = 3\eta_{\kappa\kappa}\bar{\upsilon}\upsilon/R^2\,.$$

Since

$$(-g)^{\frac{1}{2}}\bar{\zeta}^\nu_{/\nu} = ((-g)^{\frac{1}{2}}\bar{\zeta}^\nu)_{,\nu}\,, \quad g = \det(g_{\lambda\mu})\,, \tag{32}$$

the equation may also be written

$$(-g)^{-\frac{1}{2}}((-g)^{\frac{1}{2}}\bar{\upsilon}^\nu\upsilon)_{,\nu} + \eta_{\kappa\kappa}\bar{\upsilon}^\nu\epsilon\upsilon_\rho\bar{\upsilon}^\rho\epsilon\upsilon_\nu = -3f^2/R^2\,. \tag{33}$$

Now

$$\begin{aligned}\bar{\upsilon}^\nu\epsilon\upsilon_\rho\bar{\upsilon}^\rho\epsilon\upsilon_\nu &= \bar{\upsilon}^\kappa\epsilon\upsilon_\rho\bar{\upsilon}^\rho\epsilon\upsilon_\kappa + \bar{\upsilon}^\nu\epsilon\upsilon_\kappa\bar{\upsilon}^\kappa\epsilon\upsilon_\nu \\ &= -2\eta_{\kappa\kappa}f^{-2}\bar{\upsilon}\upsilon_\rho\bar{\upsilon}^\rho\upsilon = -2\eta_{\kappa\kappa}f^{-2}ff_\rho(\bar{\upsilon}^\rho\upsilon)_{,\rho}\,,\end{aligned}$$

so we have

$$\tfrac{1}{2}\Box(f^2) = [f^{-2}g^{\nu\rho}(f^2)_{,\rho}]_{/\nu} = -3/R^2\,, \tag{34}$$

where $\Box$ is a generalisation of D'Alembert's differential operator.

If the symmetry is such that f depends on only one coordinate r, which is chosen so that

$$\frac{dr}{d\sigma} = f, \quad \sigma_r = \int_{r'}^{r} dr/f\,; \tag{35}$$

where $d\sigma_r$ is the line element in the r-direction, then the determinant g reduces to pseudo-euclidean form with the value $\det(\eta_{\lambda\mu})$, and we can substitute $f^2\eta^{\nu\rho}$ for $g^{\nu\rho}$ in (34). Then the differential operator in that equation reduces to the ordinary D'Alembertian:

$$\Box = \eta^{\lambda\mu}\partial_\lambda\partial_\mu\,,$$

and the non-static solutions of (34) are obviously wave-like in character. But for static solutions (with $\kappa = 0$) and spherical symmetry, this equation leads to the well known generalisation of Schwarzschild's solution

$$f^2 = 1 - 2m/r - r^2/R^2 \qquad (g = -1)\,,$$

$$d\sigma^2 = f^2 dt^2 - f^{-2}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{36}$$

in spherical polar coordinates.

It is also well known that the zeros of the function f^2 correspond to horizons near the surface of the Schwarzshild sphere ($r = 2m$) and the beginning of time ($r = R$) respectively. But it is worth noticing that both horizons are an artifact of the choice of the radial coordinate and recede as they are approached. For $m = 0$ the line element can be transformed to those of the various cosmological models of de Sitter and Friedman, by simple changes of coordinates tabulated by Rosen (1965). Even for general values of m, the condition $g = \det(\eta_{\lambda\mu})$ serves merely to define a coordinate r, and the geodesic distance between two points on a radius vector is given by the integral σ_r in (35), from which the singularities near $r = 2m$ and $r = R$ can both be removed by a change of variable involving hyperelliptic functions. The distance σ_r is in fact the separation of the two points, derived from the general definition of σ given in (10).

7. Quantal Embedding

In the last two Sections we have developed an interpretation of Einstein's gravitational and cosmological theories based on the definition of the metric tensor in (16) in terms of the relativistic density matrices of the neutral particles that provide the geometrical information. This could be considered equivalent to the embedding of the Riemannian space of the universe in the vector space of the spin matrices of the elementary particles. The possibility of the classical embedding of non-euclidean manifolds in both flat and curved manifolds has been known for a long time and has been exploited by Kasner (1921), Fronsdahl (1959), Rosen (1965) and Gibbons and Wiltshire (1987) among others. However, the present approach is different, and corresponds to what may be called quantal embedding, since the 'coordinates' of the embedding space are not the components of the vector ζ but parameters of the group of transformations connecting different vectors $\zeta_{\rm o}$ and ζ. Here we shall show how to determine these parameters, and obtain some explicit results, including remarkably simple results (in terms of elementary functions) for the Schwarzschild metric.

As already mentioned, the vector $\zeta_{\rm s}$ representing a neutral particle is in a representation of $SO(4,2)$ of highest weights $(s, s, \pm s)$. It follows that each of the spinor components $\alpha_{03}^{(r)}$, $\alpha_{12}^{(r)}$ or $\alpha_5^{(r)}$ appearing in (10) has eigenvalues $\frac{1}{2}$, $\frac{1}{2}$ or $\pm\frac{1}{2}$ respectively on a highest weight vector of the representation. Thus ζ can be expressed in terms of products of spinor factors:

$$\zeta = u\zeta_{\rm s} = \prod_{r=1}^{2s} u^{(r)}\zeta_{\rm s}^{(r)}\,, \tag{37}$$

where u and the $u^{(r)}$ are pseudo-orthogonal ($\bar{u} = \eta u^{\mathrm{t}} \eta = u^{-1}$) and ζ_{s} is the state vector of the particle at its source, which is either a highest weight vector connected with it by a simple pseudo-orthogonal transformation to be given explicitly in (49) below. From (16) and (37) it follows that the metric tensor $g_{\lambda\mu}$ is given by

$$g_{\lambda\mu} = \bar{\zeta}_{\mathrm{s}} \bar{u}_\lambda u_\mu \zeta_{\mathrm{s}} = \sum_{r=1}^{2s} \bar{\zeta}_{\mathrm{s}}^{(r)} \bar{u}_\lambda^{(r)} u_\mu^{(r)} \zeta_{\mathrm{s}}^{(r)} \,. \tag{38}$$

This result ensures that it will be sufficient for the purposes of this Section to consider the irreducible representations for spin $\frac{1}{2}$. However, it represents the metric tensor as the sum of $2s$ identical terms, which is proportional to the spin. We note that, although this spin dependence of the metric tensor affects only the the apparent scale of the universe, it is feasible that by comparison of of the time of transit of light and neutrinos between a source and a detector, this particular feature of the present interpretation of Einstein's theory could be tested experimentally.

In the classical theory a particle moving freely under gravity moves along a path which is the join of the points z_{s} and z of emission and absorption, computable with the help of (14). In the quantum theory essentially the same calculation yields the variation of the relativistic density matrix of the particle between source and detector, but still requires an explicit form of the metric tensor or the corresponding tetrad vectors. The most general expression that can be written down for u in (37) has 28 independent functions of position, clearly sufficient to reproduce any metric tensor, with considerable redundancy attributable to the possibility of gauge transformations $u \to \overline{v} u v$ which leave the metric tensor unchanged. Here our immediate objective is to show in more detail how the observation of neutral particles can provide information on the nature of space–time, by identifying particular elements u of the group $SO(6,2)$ corresponding to the special types of metric tensor for empty space derived in the previous Section, and we shall therefore not consider the most general form.

Accordingly, we adopt a spinor representation for the $SO(4,2)$ subgroup and substitute

$$u = \exp(2i\omega^j \alpha_j) = \widetilde{q} + 2iq^j \alpha_j, \quad \widetilde{q} = \cos\omega, \quad q^j = \omega^j \sin\omega/\omega \,, \tag{39}$$

where $\omega^2 = \omega^j \omega_j$ and the α_j form a set of seven real anticommuting matrices with $(\alpha_5, \alpha_6, \alpha_7)$ defined in (6), so that $\{\alpha_j, \alpha_k\} = \frac{1}{2}\eta_{jk}$, with $\eta_{jk} = 2\delta_{j0}\delta_{k0} - \delta_{jk}$. The exponential function can easily be resolved into factors, as in (4):

$$u = \bar{u}_{\mathrm{c}} u_{\mathrm{g}} u_{\mathrm{c}}, \quad u_{\mathrm{g}} = \exp(\omega^4 \alpha_4^5) \exp(\omega^5 \alpha_5^6) \exp(2i\omega^6 \alpha_6) \exp(\omega^5 \alpha_6^5) \exp(\omega^4 \alpha_5^4) \,,$$
$$u_{\mathrm{c}} = \exp(\omega^3 \alpha_3^4) \exp(\omega^1 \alpha_1^2) \exp(\omega^2 \alpha_2^3) \exp(\omega^0 \alpha_0^4) \,, \tag{40}$$

representing elementary cosmological and gravitational effects, including rotations and boosts. In units with $R = 1$, the parameters ω_0, ω_1 and ω_2 have the same significance as t, φ and θ in (36), and $\omega_3 = \sin^{-1} r$ is the angular distance from the point z_{o}, while ω_4, ω_5 and ω_5 are the parameters which determine the gravitational field. The form of u given in (40) has the special feature that the metric tensor is independent of the vector ζ_s in (37) and can be expressed simply as

$$g_{\lambda\mu} = \tfrac{1}{2}(\bar{u}_\lambda u_\mu + \bar{u}_\mu u_\lambda) = \widetilde{q}_\lambda \widetilde{q}_\mu + q^j_\lambda q_{j\mu} \quad (\widetilde{q}^2 + q^j q_j = 1)\,, \tag{41}$$

where both left and right sides are multiples of the unit matrix.

It seems likely that the metric tensor of any solution of the gravitational field equations can be expressed in the above form. To obtain static solutions of the type considered in the last Section it is sufficient to substitute

$$q^0 = t_c f \sinh(t/t_c), \quad q^1 = r\sin\theta\cos\varphi, \quad q^2 = r\sin\theta\sin\varphi, \quad q^3 = r\cos\theta\,,$$

$$q^4 = t_c f \cosh(t/t_c), \quad q^5 = \chi\cos\omega, \quad q^6 = \chi\sin\omega, \quad \widetilde{q} = (1 - r^2 - t_c^2 f^2 - \chi^2)^{\frac{1}{2}}\,,$$

so that the line element is given by

$$d\sigma^2 = f^2 dt^2 - dr^2 - t_c^2 df^2 - d\chi^2 - \chi^2 d\omega^2 + d\widetilde{q}^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{42}$$

where f, χ and ω may be be independent functions of position in general. The parameters q^4, q^5 and q^6 are coordinates in a three-dimensional subspace of the embedding space and though the metric tensor is unaffected by rotations in this subspace, such rotations do affect the properties of neutral particles propagating in the gravitational field.

For spherical symmetry, f, χ and ω are functions of r, so that the formula (42) reduces to

$$d\sigma^2 = f^2 dt^2 - (1 + {t_c}^2 f'^2 + \chi'^2 + \chi^2\omega'^2 - \widetilde{q}'^2)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{43}$$

where the primes denote differentiation with respect to r. As f is given by (36), $\chi^2 = 2mt_c^2/r$ and the condition $\det(g_{\lambda\mu}) = -\det(\eta_{\lambda\mu})$ yields the angle ω as the quadrature

$$\omega = \int^r [(1 - f^2 - f^2 f'^2)/f^2 - \chi'^2 + \widetilde{q}'^2]^{\frac{1}{2}} dr/\chi\,, \tag{44}$$

wherein a choice $t_c \approx 4m$ removes the singularity on the Schwarzschild sphere. In the Schwarzschild limit ($r \ll 1$), $\widetilde{q}'^2$ is negligible and ω can be evaluated in terms of elementary functions:

$$\omega = \tfrac{1}{4}(2\gamma + \sinh 2\gamma), \quad \gamma = \sinh^{-1}(r/2m)^{\frac{1}{2}}\,,$$

whereas the corresponding classical embeddings (Fronsdahl 1959; Rosen 1965) are in terms of elliptic functions. Unlike classical embeddings, the quantal embeddings have a direct physical interpretation and their parameters are in principle measurable; they also have the important advantage of providing immediate solutions of the generalised Dirac–Majorana and Maxwell equations which will be considered in the next Section. Use will be made there of the fact that the metric tensor in (41) can be expressed in the form

$$g_{\lambda\mu} = h^j_\lambda h_{j\mu}, \quad h^j_\lambda = q^j_\lambda + q^j \widetilde{q}_\lambda/(1+\widetilde{q})\,, \tag{45}$$

where h^j_λ is a heptad of vectors similar to those appearing in earlier work (Green 1958*a*) on the generalisation of Dirac's equation. But from (40) it is already clear that the gravitational field has the effect of inducing rotations in the space of the τ_c-matrices and hence transitions between different helicities and 'flavours', with a mixing angle that increases initially as the change of the square root of the distance in terms of the radius of the Schwarzschild sphere (~ 3 km for the Sun).

When, in the spinor representation, u is of the form shown in (39), the metric tensor, given by (41), is independent of the vector $\zeta_{\rm s}$ representing the particle at its source. By a simple change of origin $q^j \to q^j + q^j_{\rm s}$, the group parameters q^j in (39) can be chosen so that $u = u_{\rm s}$ at the point $z_{\rm s}$, defined as the source of a particle with a given spin and momentum that may be observed subsequently at some other point z of space–time. These parameters are invariant under all coordinate transformations and are computable by a definite procedure from physical measurements; unlike the coordinates $(t, \mathbf{r})$, they may therefore be regarded as 'physical', and the q^λ as 'physical coordinates' with an observable source as the origin.

8. The Generalised Wave Equation

We may now obtain a generalisation of the equation for the vector ζ, which determines the evolution of the relativistic matrix of a neutral particle propagating between an emitter at the point $z_{\rm s} = \zeta_{\rm s}\bar{\zeta}_{\rm s}$ and an absorber at the point $z = \zeta\bar{\zeta}$, identified with the relativistic density matrix of the particle in some representation of $SO(3) \otimes SO(4,2)$. As in special relativistic theory, the vector ζ is regarded as some component of a quantised field variable ψ, which satisfies the same equation.

To obtain a generalisation invariant under all coordinate transformation, the Majorana–Kemmer matrices α_λ must be replaced by

$$e_\lambda = \alpha_j h^j_\lambda\,. \tag{46}$$

For any spin it follows from (38) and (45) that these matrices satisfy

$$[e_{\lambda\mu}, e_\nu] = g_{\mu\nu} e_\lambda - g_{\lambda\nu} e_\mu, \quad e_{\lambda\mu} = [e_\mu, e_\lambda]\,, \tag{47}$$

so that the $e_{\lambda\mu}$ and e_ν are generators of a representation of $SO(3,2)$ and are general relativistic analogues of the $\alpha_{\lambda\mu}$ and α_ν.

The matrices appearing in (46) and (47) are of course not unique: if w is any pseudo-orthogonal matrix, $\bar{w} e_{\lambda\mu} w$ and $\bar{w} e_\nu w$ satisfy the same commutation relations; in particular, with the choice $w = u^{\frac{1}{2}}$, they are satisfied by $\widetilde{e}_{\lambda\mu} = [\widetilde{e}_\mu, \widetilde{e}_\lambda]$ and $\widetilde{e}_\lambda = i u_\lambda \bar{u}$. There are both local and global transformations which reduce the heptad h^j_λ to a pentad (with vanishing components for $j > 4$), as in the theory of Pauli and Solomon (1932) or a tetrad, as in Einstein's (1928, 1930) unitary theory. Apart from this, it is well known that a coordinate transformation can be found such that $g_{\mu\nu}$ reduces to $\eta_{\mu\nu}$ at a particular point, e.g., at the source $z_{\rm s}$ of a particle, and the relations (47) then reduce at that point to the special relativistic form given in (9).

It is important to notice that, as a consequence of (26) and (28), the vector ζ must satisfy

$$\zeta^{\mu}_{/\mu/\nu} - \zeta^{\mu}_{/\nu/\mu} = 3\zeta_\nu\,,$$

which, according to (24), can be expressed in the algebraic form

$$[z^\dagger_\nu, z^\dagger_\mu]\zeta^\mu = 3\zeta_\nu\,, \tag{48}$$

equivalent to Einstein's equations with a cosmological term. The analysis of the last Section has shown that these equations are satisfied by $\zeta = u\zeta_{\rm s}$, with u given by (40): it is thus a consequence of the present approach that for any initial value $\zeta_{\rm s}$ *the solution* ζ *is already determined by Einstein's equations*, apart from a gauge transformation which leaves the metric tensor unchanged. The vector $\zeta_{\rm s}$ is always simply related to the highest weight vector $\zeta_{\rm o}$ in the irreducible representation of $SO(4,2)$ for spin s, defined by

$$\alpha_{03}\zeta_{\rm o} = s\zeta_{\rm o}, \quad \alpha_{12}\zeta_{\rm o} = s\zeta_{\rm o}, \quad \alpha_5\zeta_{\rm o} = \pm s\zeta_{\rm o}\,. \tag{49}$$

For a particle of zero rest mass the relation is no more than a simple Lorentz transformation: since $\alpha_{03}(\alpha_0 - \alpha_3) = (\alpha_0 - \alpha_3)(\alpha_{03} + 1)$ and α_{03} has no eigenvalue $(s+1)$, the first of these equations implies that $(\alpha_0 - \alpha_3)\zeta_{\rm o} = 0$, which can be interpreted as the equation of a particle of zero rest mass propagating in the x_3-direction. For a particle of non-vanishing mass μ propagating in the same direction, a further transformation of the type

$$\zeta_{\rm s} = u_{\rm s}\zeta_{\rm o}, \quad u_{\rm s} = \exp(w\alpha_{34})\exp(\omega\alpha_{04}), \quad \cosh w = \mathrm{cosec}\,\omega = p_0/\mu$$

is required and corresponds to a translation to $z_{\rm s}$ from a point $z_{\rm o}$ on the cosmological or Schwarzschild horizon.

Analogues of the special relativistic equation (5) consistent with Einstein's equations, and therefore satisfied by ζ in (37), are well known. The most general form (see Green 1958*a*, 1958*b*) can be written

$$ie^\lambda(\zeta_\lambda - \Gamma_\lambda\zeta) = 0\,, \tag{50}$$

where Γ_λ depends on the gauge as well as position, and is invariant under pseudo-orthogonal transformations of the type

$$\zeta \to w\zeta, \quad e^\lambda \to we^\lambda\bar{w}, \quad \Gamma_\lambda \to w\Gamma_\lambda\bar{w} + w_\lambda\bar{w}\,. \tag{51}$$

Thus with $\Gamma_\lambda = u_\lambda\bar{u}$ the choice $w = \bar{u}$, with u given by (39), effects the transformation $\zeta \to \zeta_{\rm o}$.

Gauge transformations, which leave the metric tensor unchanged, are also of this type, and it is accepted today (see de Beauregard 1996) that such transformations are physically significant. For photons, gauge transformations are generally considered to be related to a strongly broken symmetry. But for spin $\frac{1}{2}$ transformations affecting the parameters q^4, q^5 and q^6 in (39) could have a simpler interpretation. Though the differences between neutrinos with different 'flavours' (β-, μ- and τ-neutrinos) are not yet fully understood, the concept of a gauge group connecting them is generally accepted. We have shown that there is a gauge group associated with gravitation which could serve this purpose, and does not require different gravitational constants for different neutrinos (and consequent violations of the Principle of Equivalence) as suggested by Gasperini (1988).

To summarise, in the present paper we have attempted a new synthesis of general relativity and quantum mechanics. The most striking outcome of the analysis is that the relativistic statistical matrix of a neutral particle may be viewed as a microcosm of the observable parts of space–time through which the particle may be transmitted. The geometrical properties, including cosmological and gravitational effects, are reflected in the variation with the space–time coordinates of the parameters of the orthogonal group of transformations of the quantal wave equations. These conclusions are to a large extent independent of the representations of $SO(3) \otimes SO(4,2)$ which have been assumed for the quantal embedding of the Riemannian manifold of general relativity. But there are also some interesting features that depend on the representation and on the wave equation assumed for the neutrinos. The geometry of the physical world has been found to depend at least in scale on the spin of the particles by means of which the universe is observed, and there are gauge groups associated with the wave equation which are independent of, but are closely associated with the Riemannian metric of general relativity, and could provide new insights into properties of neutral particles.

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