BUOYANT MOTION IN A TURBULENT ENVIRONMENT

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Summary

Solutions are given of the simultaneous equations for the vertical velocity and temperature of an element of fluid moving under buoyancy and subject to continuous mixing of heat and momentum with its environment. Three distinct modes of behaviour result: (A) ascent followed by damped oscillations, (B) asymptotic ascent to an equilibrium level, (C) *absolute buoyancy* in which the ascent rate increases indefinitely. For an environment in which the lapse rate is subadiabatic the motion is of type A for sufficiently large elements but may become B for the smaller elements; in super-adiabatic lapse rates the mode is C for sufficiently large elements, and B for the smaller elements, which are in no way unstable. The *mode* of motion is independent of the initial conditions but the scale of the motion is not.

The same formulation applies within known limits to the ascent of saturated air, and applications made to atmospheric convection include the verification of a formula for the period of large cloud tops oscillating in a stable layer.

I. Introduction and Scope

The motions which might ensue, and the stability of these motions, when a fluid is heated from below was first discussed in a formal manner by Rayleigh (1916). This classical paper, and those that followed it, require no summary here; one has recently been provided by Sutton (1950), who goes on to offer an explanation for another mode of motion observed under laboratory conditions and described as *columnar*, in contrast to the regular cellular pattern which had been the subject of the earlier work. It has been demonstrated that motion will not necessarily ensue in either case when the layer of fluid is thin, even though its stratification is favourable, owing to the prohibiting effects of viscosity and molecular conduction. The thinner the layer, the more "unstable" the stratification can be without motion occurring.

Valuable as these studies are, they are not entirely suited to the treatment of the phenomenon as it occurs in nature. Their shortcomings in this context are the assumptions that the heating at the bottom is uniform and that the only buoyant forces which operate within the medium are those which arise directly as a result of this heating or by advection from the motion which the heating sets up. Natural surfaces are subject to both small- and large-scale irregularities in their thermal properties; there are similar irregularities in their radiative properties so that in the atmosphere, bounded by a natural surface which emits and receives radiation, the scope for variety is great. The irregularities are passed on in the form of temperature contrasts, and in other ways, to the layers

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of the medium most closely in contact with the surface. Moreover, when the medium is already in turbulent motion there will in general be a continuous source of creation of irregular buoyancy within the medium itself; when an element has been displaced vertically it will in general find itself at a different density from its environment and will henceforward be subject to buoyant forces as well as those inherent in the turbulence until equilibrium is re-established.

Whereas Rayleigh and his followers, by assuming a regular patterned structure, were able to formulate equations which took account of the condition of every element of the medium, this is no longer possible in the natural phenomenon where the buoyant elements are arranged in space, and to some extent also in time, in a disorderly manner. For the latter we require in the first place a treatment of the changes in motion and temperature experienced by an individual buoyant element, preferably in terms which permit of statistical recombination at a later stage. In the normal phraseology of meteorology the approach is along the lines of the *parcel* as opposed to the *slice* method (Normand 1946; Petterssen *et al.* 1946). But the parcel method does not appear to have been extended in any satisfactory manner to allow for the simultaneous mixing, both of heat and momentum, which must take place between the element and its environment according to the laws of turbulent exchange.

An allowance for these processes, and for the consequent need to solve two simultaneous equations instead of one, forms the essential contribution of this paper. This leads to a formulation of single-cell convection which embraces some aspects of the process of *entrainment* of air, though in a manner different from that adopted by other workers (see Austin (1951) for a summary). The present treatment is not confined to the "unstable" stratification since the origins of elemental buoyancy are present in nature even when the medium is cooled from below or a "stable" stratification is otherwise brought about.

For mathematical ease and resultant physical clarity, the assumption has been made that the element is small, in the sense that the mixing of its excess heat and vertical momentum does not affect the average condition of the environment. It is not thought that this assumption is fundamentally restrictive—in other studies it has been found that the slice method modifies the results obtained from the parcel method but does not change their character.

II. The General Equations

The problem to be solved is that of describing the motion and temperature behaviour of an element of fluid moving under its own buoyancy and subject to the turbulent transfer of heat and momentum into an environment which is at rest and remains in a steady state. The treatment will first be restricted to a medium of uniform constitution although, as indicated in Section IV, the ascent of saturated air may be brought within the framework of the same equations and solutions.

It will be assumed that the relative variations in density in space and time are slight as compared with those in velocity, so that density becomes important only in so far as it affects the buoyancy (Rayleigh loc. cit.). If then T and T_e

represent the temperature of the element and environment respectively, w the upwards velocity, and Γ the lapse rate of temperature with height experienced by an element ascending adiabatically, the equations are

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{g}{T_e}(T-T_e),$$
$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=-w\Gamma,$$

the effects of molecular viscosity and conductivity being neglected from the start since they are much smaller than others which will appear when the equations are transformed.

It is required to apply these equations to an aggregate of elements which have a common measure of velocity, $\bar{u}$, $\bar{v}$, $\bar{w}$, and of temperature $\bar{T}$. Writing in the usual fashion $u=\bar{u}+u'$ etc., and using the equation of continuity, the equations become

$$\dot{\bar{w}}=\frac{g}{T_e}(\bar{T}-T_e)-\overline{\frac{\partial}{\partial x}(u'w')+\frac{\partial}{\partial y}(v'w')+\frac{\partial}{\partial z}(w'w')},$$
$$\dot{\bar{T}}=-\bar{w}\Gamma-\overline{\frac{\partial}{\partial x}(u'T')+\frac{\partial}{\partial y}(v'T')+\frac{\partial}{\partial z}(w'T')},$$

where the dot denotes the time derivative following the mean motion. Henceforward we shall reserve the term *elements* for the aggregates so defined and *sub-elemental* when reference is made to finer structure. The motion of the element is defined as the mean of that of the composing particles, so that the element is regarded as composed of a changing set of particles and some aspects of entrainment are thereby formally taken into account. It will be assumed that the buoyancy occurs in a medium already in turbulent motion; that turbulent transfer coefficients for momentum and heat, K_1 and K_2, exist and are independent of position on the sub-elemental scale, that they are determined by processes other than the buoyant motion and so are independent of $\bar{w}$. The equations may then be written

$$\dot{\bar{w}}=\frac{g}{T_e}(\bar{T}-T_e)+K_1\overline{\nabla^2 w}, \qquad \text{(1)}$$
$$\dot{\bar{T}}=-\bar{w}\Gamma+K_2\overline{\nabla^2 T}. \qquad \text{(2)}$$

These are to be applied in what follows to elements of constant finite size within and around which there is some overall pattern in w and T apart from the random variations of the turbulence. While it is part of the complete problem to determine just what this pattern will be, this aspect will not be investigated here where the concern will be with broader features of the behaviour, and we may write

$$K_1\overline{\nabla^2 w}=-\frac{c_1K_1}{R^2}\bar{w},$$

$$K_2\overline{\nabla^2 T}=-\frac{c_2K_2}{R^2}(\bar{T}-T_e),$$

where R is taken to characterize the size (radius) of the element and c_1 and c_2 are numerical factors which depend on the exact form of the patterns. Defining then

$$k_1=\frac{c_1K_1}{R^2}, \quad k_2=\frac{c_2K_2}{R^2}, \quad \dots\dots\dots\dots\dots\dots \quad (3)$$

the equations of the element are written as

$$\dot{w}=\frac{g}{T_e}(T-T_e)-k_1w, \quad \dots\dots\dots\dots\dots\dots \quad (4)$$

$$\dot{T}=-w\Gamma-k_2(T-T_e), \quad \dots\dots\dots\dots\dots \quad (5)$$

the bars now being superfluous.

It might have been acceptable to write down equations (4) and (5) without explanation, as affording a satisfactory starting point for treating the problem in hand. This has indeed been done with equation (4) (e.g. Scorer and Ludlam 1953), though apparently not with the two equations together. The derivation indicated above has the advantages of showing that their application is to elements of constant size and of providing a proper background for the physical interpretation of k_1 and k_2. As in most applications of turbulence theory, intuitive or dimensional arguments have replaced rigorous mechanistic ones at some points; in particular the distinction between turbulent components w' and those deriving from the buoyancy is artificial and there must remain some possibility, which is excluded from what follows, that k_1 and k_2 are not strictly independent of w.

k_1 and k_2 will be referred to as *mixing rates* for momentum and heat respectively. The mixing rate will depend on the ratio of surface to volume of the element, and one might intuitively expect that it will be greatest for the smallest elements and vice versa. The issue is not quite so straightforward, since an underlying principle of modern ideas on turbulence is that K is itself a function of R, increasing as R increases: but the strong indication, both from theory at the smaller scales (Weizsäcker 1948) and from empiricism up to the largest scales (Richardson 1926), is that the rate of increase is approximately as $R^{4/3}$, whence from (3) the intuitive expectation is confirmed subject to the constancy of c_1 and c_2. k_1 and k_2, then, will depend on the general level of turbulence in the environment; in particular this will prevent them from becoming discordant in magnitude or, in other words, they may tend to very large or very small values together, but never separately. Apart from this broad control, it is the dependence of k on R which leads to the more interesting novel features in the interpretations of the solutions which follow, and will accordingly be stressed. The general level of turbulence will be treated as given, and k_1 and k_2 regarded as constants during the life history of a given element; distinguishing features of the solutions at different values of k will be interpreted primarily as distinctions between modes of behaviour of elements of different size.

The form factors c_1 and c_2 depend only on the *shape* of the distribution of w and T within the element, not on the intensity. Whether they remain constant, to an order of magnitude, over the lifetime of a given element and over a wide range of sizes cannot be stated until more is known of the characteristics of turbulence at sub-elemental scales : the constancy is here assumed as a reasonable working hypothesis, of a type related to the similarity assumptions common in modern turbulence theory. Some values are given in Appendix I.

It remains to solve (4) and (5) with k_1 and k_2 constant, under the condition for a steady environment

$$\dot{T}_e = w\frac{\partial T_e}{\partial z}. \quad \text{(6)}$$

Writing $T' = T - T_e$, we obtain from (4) and from (5) and (6)

$$\dot{w} = \frac{g}{T_e}T' - k_1 w, \quad \text{(7)}$$

$$\dot{T}' = -w\left(\frac{\partial T_e}{\partial z} + \Gamma\right) - k_2 T', \quad \text{(8)}$$

from which T' can be finally eliminated, yielding

$$\ddot{w} + (k_1 + k_2)\dot{w} + \left[\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z} + \Gamma\right) + k_1 k_2\right] w + \frac{1}{T_e}\frac{\partial T_e}{\partial z} w(\dot{w} + k_1 w) = 0. \quad \text{(9)}$$

This is to be regarded primarily as the general equation for the buoyant motion of a single element, with assigned values for k_1 and k_2 and known environmental conditions. Having solved for w, the corresponding solution for T'/T_e is obtained from (7). An alternative viewpoint would be that, if the behaviour of elements in a known environment can be observed in detail, equation (9) becomes an equation providing evidence about the turbulence factors k_1 and k_2 and their dependence on scale.

For ease in what follows, auxiliary "rates" $\varkappa$, λ, μ are defined by

$$\varkappa^2 = \frac{g}{T_e}\left(\frac{\partial T_e}{\partial z} + \Gamma\right) + k_1 k_2,$$

$$\lambda^2 = \left|\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z} + \Gamma\right)\right|,$$

$$\mu^2 = \left|\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z} + \Gamma\right) - \frac{(k_1 - k_2)^2}{4}\right| = \left|\varkappa^2 - \frac{(k_1 + k_2)^2}{4}\right|.$$

III. Properties of the Basic Solutions

While equation (9) can be solved numerically under any known conditions, it is more enlightening to derive analytical solutions under particular environmental conditions which make this possible.

In nature it is observed that the temperature variations at a given level are at all times small as compared with the absolute temperature. Close above a factory chimney or open fire this will not be so, but excluding these cases it may be seen from (7) that the last term in (9) is

$$\frac{g}{T_e}\frac{\partial T_e}{\partial z} w \times \frac{T'}{T_e},$$

in which w is multiplied by a quantity which remains small. So long then as $\varkappa^2$, the other multiplier of w, is not small the last term in (**9**) may be neglected. It must be stressed that this simplification is made without putting any restriction on the value of w itself, so that large velocities are not excluded.

It then becomes evident that the type of motion will depend on the multipliers of the remaining terms, that is, on $\varkappa^2$ and (k_1+k_2). Although in practice $\varkappa$ may vary with height, the chief characteristics of the possible modes of behaviour may be most clearly brought out by solving the equation for $\varkappa$ constant ; simple analytical solutions are then obtained which can be recombined when practical application so requires. Natural convection in a layer of constant $\varkappa$, which for practical purposes can be identified as one of constant lapse rate, is therefore governed by the second-order equation with constant coefficients,

$$\ddot{w}+(k_1+k_2)\dot{w}+\varkappa^2 w=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{10})$$

the form of whose solutions is familiar. The discussion of them will be eased by reference to Figures 1 and 2 ; the first represents the special case $k_1=k_2$ but the second is typical of the more general conditions where $k_1 \neq k_2$.

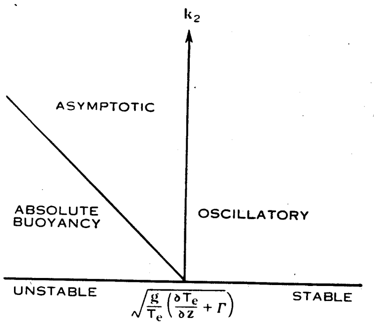

Fig. 1

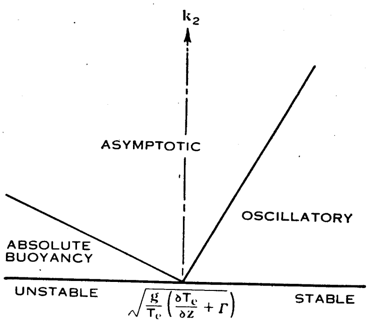

Fig. 2

Fig. 1.—Modes of motion as a function of lapse rate and mixing rates for $k_1=k_2$.
Fig. 2.—Modes of motion as a function of lapse rate and temperature mixing rate k_2 for $k_1=2k_2$.

(i) When

$$\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)>\left(\frac{k_1-k_2}{2}\right)^2,$$

the solutions of (**10**) and (**7**) are given by

$$w=A\mathrm{e}^{-\frac{k_1+k_2}{2}t}\sin(\mu t+\varepsilon_1), \quad \ldots\ldots\ldots\ldots \quad (\mathbf{11})$$

$$\frac{T'}{T_e}=\frac{A\lambda}{g}\mathrm{e}^{-\frac{k_1+k_2}{2}t}\sin(\mu t+\varepsilon_1+\varepsilon_2), \quad \ldots\ldots\ldots \quad (\mathbf{12})$$

where A and ε_1 are determined by the initial conditions and $\tan\varepsilon_2=2\mu/(k_1-k_2)$. This mode of motion will be called *oscillatory*, the element transcending an equilibrium level about which it subsequently executes damped harmonic oscillations. This solution can obtain under inversion, isothermal, or sub-adiabatic lapse rates ; if $k_1=k_2$, it holds for all elements under these conditions, but if $k_1 \neq k_2$, only for sufficiently large ones (small k). For the very largest elements ($k \to 0$) the motion becomes strictly periodic with period $2\pi/\lambda$. This formula for the period in the extreme case has been given by Brunt (1927), but it does not appear to have been applied except as a possible explanation of microbarograph oscillations.

(ii) When

$$\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)=\left(\frac{k_1-k_2}{2}\right)^2,$$

there results a motion which approaches its equilibrium level asymptotically, either from below or from above after a single overshooting according to the initial conditions. This is a special case of restricted interest and will not be discussed further.

(iii) When

$$\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)<\left(\frac{k_1-k_2}{2}\right)^2,$$

the solutions are

$$w=Ae^{-\frac{k_1+k_2}{2}t}\sinh(\mu t+\varepsilon_1), \qquad \text{(13)}$$

$$\frac{T'}{T_e}=\frac{A}{g}e^{-\frac{k_1+k_2}{2}t}\left\{\frac{k_1-k_2}{2}\sinh(\mu t+\varepsilon_1)+\mu\cosh(\mu t+\varepsilon_1)\right\}. \qquad \text{(14)}$$

These represent two quite different modes of behaviour according as the exponential or the hyperbolic terms ultimately become dominant.

When $\varkappa^2$ is positive, the resulting mode will be called *asymptotic*; the exponential term is finally dominant and the element approaches its equilibrium level asymptotically without overshooting. For $k_1=k_2$ this mode is confined to the smaller elements in superadiabatic lapse rates, but for $k_1 \neq k_2$ it applies to sufficiently small elements under any conditions of lapse rate. It is of interest to note that in both asymptotic and oscillatory motion T'/T_e remains small if small initially, so that these solutions are solutions of the complete equation (**9**) as well as of (**10**).

When $\varkappa^2$ is negative, that is, for sufficiently large elements in superadiabatic lapse rates, (**13**) and (**14**) give expressions for the velocity and buoyancy which ultimately increase exponentially with time. This condition will be referred to as one of *absolute buoyancy*. It is clear that (**13**) will ultimately fail to satisfy equation (**9**) but it may readily be seen that the effect of the last term in (**9**), which has been neglected, will be to enhance rather than suppress the acceleration. The condition of absolute buoyancy is therefore real, the criterion for its realization being

$$-\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right) \geqslant k_1k_2. \qquad \text{(15)}$$

Significance of Element Size

Apart from the recognition of the three characteristic types of motion, the important result of the foregoing analysis is that the mode of behaviour is determined entirely by the values of λ, k_1, and k_2. Physically this means that the type of motion depends on the environmental conditions (lapse rate and general level of turbulence) *and on the size of the element*, and is independent of the initial conditions of velocity and buoyancy.

Buoyancy Length and Static Instability

As opposed to the *type* of motion, the *scale* of the motion does depend on the initial conditions. It is most conveniently characterized by the distance of the final equilibrium above the starting level which, following Priestley and Swinbank (1947), is called the *buoyancy length*; this concept applies in all instances of oscillatory and asymptotic motion and, since then w and $\dot{w}$ tend to zero as t tends to infinity, the length is obtained by direct integration of (**10**) as

$$L=\int_0^\infty w\mathrm{d}t=\frac{1}{\varkappa^2}[\dot{w}_0+(k_1+k_2)w_0],$$

whence from (**7**)

$$L=\frac{1}{\varkappa^2}\left[\frac{g}{T_e}T'_0+k_2w_0\right], \qquad (\mathbf{16})$$

the suffix zero denoting initial values. For motion starting from rest

$$L=T'_0\bigg/\left(\frac{\partial T_e}{\partial z}+\Gamma+\frac{k_1k_2T_e}{g}\right). \qquad (\mathbf{17})$$

It is seen from (**16**) that an element given an infinitesimal impulse or temperature disturbance will come to rest after an infinitesimal distance, unless it is absolutely buoyant. The idea that elements are statically unstable when in a superadiabatic lapse rate is therefore true only in a limited sense. In order to create a finite motion the disturbance must be of finite intensity, or alternatively an infinitesimal disturbance must be applied to an element of sufficiently large size. This result is the counterpart of the classical one of Rayleigh which was referred to in the opening paragraph.

Critical Size for Absolute Buoyancy

The theory provides a criterion for the realization of absolute buoyancy in the form of a condition relating the mixing and lapse rates. At the present stage only rather crude empiricism can transform this into a condition for the critical size of element, but this is worth while even though it can hope to indicate no more than the order of magnitude involved. Since k_1 and k_2 are of the same order of magnitude we shall equate them in the present context, whence from $K=aR^{4/3}$ and equations (**3**) and (**15**) is obtained

$$R=(ca)^{3/2}\left\{-\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)\right\}^{-3/4}. \qquad (\mathbf{18})$$

Richardson (1926) gives $a=0{\cdot}2$ c.g.s. units and taking $c=8$ (see Appendix I) Table 1 results.

TABLE 1

CRITICAL ELEMENT SIZE

$-(\partial T_e/\partial z+\Gamma)$ (°C/cm) ..	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
R (m)	1500	250	40	8	$1\frac{1}{2}$	$\frac{1}{4}$

In the free atmosphere, if the lapse rate exceeds the adiabatic by only a small fraction of the latter's value (10^{-4} °C/cm), the critical size is of the order of hundreds of metres, whereas in strong lapse rates very close to the ground it is of order of a metre. The sizes thus obtained support the application of the different modes of solution both to atmospheric convection and to the motion of, and transport of heat by, small elements close to the ground. The first application will be discussed in Section IV and the last, which involves a statistical recombination of the solutions, will be given in a later paper.

IV. Application to Atmospheric Convection

The scope for application of the foregoing to the problem of single-cell convection in the atmosphere is clear enough, but there remain a few points which merit some further discussion.

Dry and Saturated Ascent

The basic equations were derived for a fluid of uniform constitution and it is necessary to state to what extent they remain valid for the mixture of air, water vapour, and water drops which occurs in the atmosphere. So long as the air is not saturated the effect of the presence of water vapour on the specific heat and density is very slight and (**1**) and (**2**) are valid. Γ may then be identified as the dry adiabatic lapse rate Γ_d.

When the air becomes saturated (**2**) and (**5**) no longer hold because of the heat involved in changes in the water phase. The consequences of this may be represented by the same set of equations and solutions provided Γ is taken as some value between Γ_d and Γ_s, the saturated adiabatic rate, and equations (**3**) are modified to

$$k_1=\frac{c_1K_1}{R^2}, \qquad k_2=\frac{\Gamma}{\Gamma_d}\cdot\frac{c_2K_2}{R^2}. \qquad \ldots\ldots\ldots (\mathbf{19})$$

Γ is dependent, within its known limits, on the rate of mixing between element and environment, but it cannot strictly be regarded as constant during the lifetime of an individual element. Nor, therefore, can k_2. But for the largest elements the mixing becomes unimportant, k_1 and k_2 tend to zero, and Γ becomes equal to Γ_s, and the behaviour is strictly determinate.

In general the formulation is not complete because Γ has not been exactly specified. A complete treatment would involve appropriate modification of (**5**) and addition of a third simultaneous equation to express the mixing of water. More complicated motions will then ensue. For example, a cloud element in asymptotic motion, when near its equilibrium level as at present defined, is likely if anything to exceed the environment in moisture content and so will evaporate some of its water and fall back towards a lower level.

Generation of Convective Motion

The vertical velocities which occur in association with clouds and dry thermals are considerably larger than those which are otherwise present and it follows that these are initiated by absolute buoyancy since this is the only mode of motion capable of magnifying an initial disturbance by an order of magnitude.

Table 1, in indicating critical sizes which are rather smaller than the dimensions of most clouds, would suggest the same conclusion.

The question then arises as to how buoyant elements of sufficient size may be generated if external disturbances on this scale are absent. Attention has recently been drawn by Scorer and Ludlam (1953) to some interesting consequences of wake formation, one of which, when considered against the background of the present solutions, suggests an appropriate mechanism. An element so small that its motion is bounded (asymptotic) will leave a residual wake of greater size, though of smaller intensity, than the element itself. On the present theory it is size rather than intensity which is critical and, given a superådiabatic lapse rate, repetitions of this process should eventually produce an element large enough to be absolutely buoyant, provided only that the wind shear is not so strong as to disrupt the coherence of the successive wake elements.

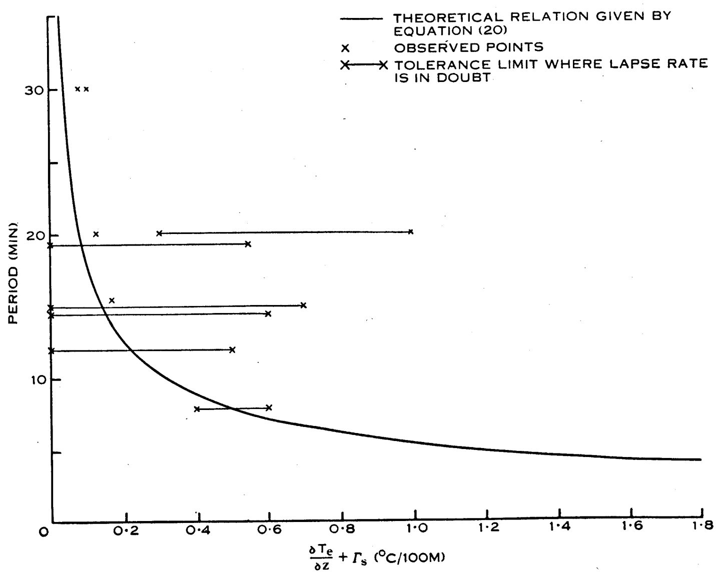

Fig. 3.—Period of cloud oscillation *v.* lapse rate.

Final Stages

Generated by absolute buoyancy, the motion will eventually die away when a more stable layer is reached. Figures 1 and 2 show that, if all the variables are continuous, the motion cannot pass from absolute buoyancy to the oscillatory mode without passing through the asymptotic mode. Depending on the characteristics of the upper layer, the element will be brought to rest asymptotically or may penetrate further and become subject to oscillations; both types of final motion are observed in practice. The oscillatory mode has received little attention, but it occurs by no means rarely (Workman and Reynolds 1949; Smith 1951). For sufficiently large clouds the motion will then approach a simple harmonic oscillation of amplitude L and period

$$2\pi\bigg/\sqrt{\frac{g}{T_e}\left(\frac{\partial T_e}{\partial z}+\Gamma\right)}. \qquad (20)$$

Measurements of the periods of oscillating cloud tops have been made from aircraft operating near Sydney, and may be checked against lapse rates shown by the radiosonde ascent from Rathmines on the same day. The results are shown in Figure 3; in view of the type of data employed the agreement is regarded as good, bearing in mind that (**20**) represents a prediction of the order of magnitude of the period as well as of its variation with lapse rate.

On some occasions the radiosonde showed $\partial T_e/\partial z+\Gamma_s$ changing abruptly with height from a zero or negative to a markedly positive value, the levels of the stable layer being close to those of the reported oscillations. These are the conditions most favourable for large oscillations, as was confirmed by the reports. There was little difficulty on these occasions in identifying the appropriate value of $\partial T_e/\partial z+\Gamma_s$ but in others, where the change was more gradual, this could be estimated only within rather wide tolerance limits. Differences in time and place (not exceeding a few hours and 100 miles) add to the uncertainty.

A final point of quantitative appeal is provided by the observed amplitude of the oscillations. Taking average values from the 10 occasions, the amplitude of 800 ft with $\partial T_e/\partial z+\Gamma_s=+0\cdot25\times10^{-4}$ °C/cm yields $T_0'=0\cdot6$ °C in (**17**); such a value is in harmony with other evidence (Byers and Braham 1949, Table 6).

V. Acknowledgments

The observations of cloud oscillations were made by Dr. E. G. Bowen and his colleagues in the Division of Radiophysics, C.S.I.R.O., and the Rathmines radiosonde data made available by the Director of the Meteorological Branch. The writer is also indebted to Mr. W. C. Swinbank for helpful criticism in the preparation of the paper.

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Appendix I

The Form Factor c

Equations (**3**) involve a numerical factor c which is characteristic of the form of the distribution of w or T within the element. In the absence of information about this distribution, the likely magnitude of c can only be assessed by assuming reasonable analytic forms for the distribution and evaluating. The quantity considered is supposed to be symmetrical about a maximum central value, and using r for radial distance we shall evaluate two functions, the normal form

$$\text{(i)} \qquad w = w_0 e^{-r^2/R^2},$$

which has a finite value (w_0/e) and finite slope at the limits ($r=R$), and

$$\text{(ii)} \qquad w = w_0\left(1 - \frac{r^2}{R^2}\right),$$

which falls sharply to zero at the limits. Since there is interest also in the condition of cylindrical symmetry (columnar convection) we include under (ii) the case where r and R represent distances from a central axis. The results of averaging yield :

	(i) Spherical	(ii) Spherical	(iii) Cylindrical
$\overline{w}$	$0 \cdot 57w_0$	$0 \cdot 4w_0$	$0 \cdot 5w_0$
$\overline{\nabla^2 w}$	$-\frac{2 \cdot 2w_0}{R^2}$	$-\frac{6w_0}{R^2}$	$-\frac{4w_0}{R^2}$

whence

$\overline{\nabla^2 w}$	$-\frac{4\overline{w}}{R^2}$	$-\frac{15\overline{w}}{R^2}$	$-\frac{8\overline{w}}{R^2}$

These values have led to the adoption of $c=8$ in the one place in the paper where such appeal is necessary, but a large measure of uncertainty remains at present in this or other instances where conversion from values of k to values of R is required.