OPTICAL DIFFRACTION EFFECTS PRODUCED BY AMPLITUDE AND PHASE CHANGES IN THE WAVE FRONT

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[*Manuscript received May* 10, 1954]

Summary

The diffraction which occurs when certain areas of the wave front undergo changes of amplitude or phase or both, relative to the rest of the front, may be readily treated by a method which is virtually a generalization of Babinet's principle. Applications to both Fraunhofer and Fresnel diffraction are given and include phase gratings, striations, haloes, phase contrast techniques, zone plates, and holograms.

I. General Principles

In the general Fraunhofer diffraction problem, a spherical light wave converging on or diverging from a point passes through an aperture. For the present purposes a simple aperture will be defined as one or more transparent areas in an opaque screen ; and the real or virtual diffraction pattern on a plane through the centre of convergence or divergence is assumed known. The transparent and opaque parts of such an aperture will be referred to by their areas σ_1 and σ_2, and the region intercepted by the full cone of rays by its area σ. If, in the general case now to be considered, the area σ is again divided into two parts σ_1 and σ_2, either or both of which may transmit light with changes of amplitude and phase, the arrangement will be called a phase amplitude aperture. Let σ_1 and σ_2 transmit fractions r_1^2 and r_2^2 respectively of the light incident on them, and σ_1 produce a lag in phase φ relative to σ_2. Particular cases are the pure phase aperture, $r_1=r_2=1$, $\varphi\neq 0$, and the pure amplitude aperture, $\varphi=0$. It will be shown that the diffraction pattern due to the phase amplitude aperture can now be expressed in terms of the corresponding simple aperture.

Suppose the wave functions due to the simple aperture, to its complement, and to the aperture of the lens be represented by

$$\Omega_1=A_1e^{-i\Delta_1},\ \Omega_2=A_2e^{-i\Delta_2},\ \Omega=Ae^{-i\Delta}, \quad \ldots\ldots\ldots \quad (1)$$

where A_1, A_2, and A are amplitude factors and Δ_1, Δ_2, and Δ the corresponding phases. Then

$$\Omega=\Omega_1+\Omega_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

With modifications of amplitude and phase the equations corresponding to (1) become

$$\Omega_1'=r_1A_1e^{-i(\Delta_1+\varphi)},\ \Omega_2'=r_2A_2e^{-i\Delta_2},\ \Omega'=A'e^{-i\Delta'}, \quad \ldots\ldots \quad (3)$$

where Ω' is the wave function for the phase aperture. Then

$$\Omega'=\Omega_1'+\Omega_2', \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

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and by elimination of r_2A_2 and Δ_2 with the aid of (**1**), (**2**), (**3**), and (**4**)

$$\Omega'=A'e^{-i\Delta'}=r_1A_1e^{-i(\Delta_1+\varphi)}+r_2(Ae^{-i\Delta}-A_1e^{-i\Delta_1}), \quad \ldots (5)$$

or

$$A'e^{-i(\Delta'-\Delta_1)}=r_2Ae^{-i(\Delta-\Delta_1)}-A_1(r_2-r_1e^{-i\varphi}). \quad \ldots\ldots\ldots (6)$$

From (**6**)

$$\left.\begin{aligned} A'\cos(\Delta'-\Delta_1)&=r_2A\cos(\Delta-\Delta_1)-r_2A_1+r_1A_1\cos\varphi,\\ A'\sin(\Delta'-\Delta_1)&=r_2A\sin(\Delta-\Delta_1)+r_1A_1\sin\varphi,\end{aligned}\right\}\ldots\ldots (7)$$

and

$$(A')^2=r_2^2A^2+(r_1^2+r_2^2-2r_1r_2\cos\varphi)A_1^2-2\{r_2^2\cos(\Delta-\Delta_1)-r_1r_2\cos(\Delta-\Delta_1-\varphi)\}AA_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Let $C=A\cos\Delta$, $S=A\sin\Delta$, $C_1=A_1\cos\Delta_1$, $S_1=A_1\sin\Delta_1$. Then (**8**) becomes

$$(A')^2=r_2^2A^2+(r_1^2+r_2^2-2r_1r_2\cos\varphi)A_1^2-2(r_2^2-r_1r_2\cos\varphi)(CC_1+SS_1) +2r_1r_2(C_1S-S_1C)\sin\varphi. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

C, S, C_1, and S_1 are proportional to the integrals employed in the theory of diffraction, and usually expressed by the same symbols, for calculating the illumination at a given point of a diffraction pattern. They are well known for a rectangle, a circle, a diffraction grating, and other simple cases. In terms of the illuminations E' and E_1, (**9**) may be written

$$E'/E_1=A'^2/A_1^2=(r_1^2+r_2^2-2r_1r_2\cos\varphi)+\{r_2^2A^2-2(r_2^2-r_1r_2\cos\varphi)(CC_1+SS_1) +2r_1r_2\sin\varphi(C_1S-S_1C)\}/A_1^2. \quad \ldots\ldots\ldots\ldots\ldots (10)$$

When $C=S=A=0$, that is, at points where the diffraction pattern due to the lens alone has minimum (zero) values or at points sufficiently far from the geometric image of the source for this assumption to hold, (**10**) gives

$$\frac{E'}{E_1}=\left(\frac{A'}{A_1}\right)^2=r_1^2+r_2^2-2r_1r_2\cos\varphi. \quad \ldots\ldots\ldots (11)$$

Equation (**11**) shows that, in the region to which it applies, the diffraction pattern due to a phase amplitude aperture is everywhere the same as that due to the corresponding simple aperture but with the illumination altered by the factor $(r_1^2+r_2^2-2r_1r_2\cos\varphi)$. Thus for a pure phase aperture with $\varphi=\pi$ there is a fourfold amplification. Babinet's principle in its simple form is a particular case obtained by putting $r_1=0$, $r_2=1$.

The above results also apply to Fresnel diffraction patterns, C and S then being the corresponding integrals expressible in terms of Fresnel integrals. However, equation (**11**) is then restricted in application to the same extent as Babinet's principle when used in connexion with Fresnel diffraction (Wood 1911, p. 239).

It may be noted that, in the general case in which there are n phase amplitude apertures σ_s introducing phase changes φ_s ($s=1, 2, \ldots, n$) relative to a common background or reference aperture and absolute amplitude changes r_s, equation (**5**) is replaced by

$$\Omega'=A'e^{-i\Delta'}=r_cAe^{-i\Delta}+\sum_{s=1}^{n}A_se^{-i\Delta_s}(r_se^{-i\varphi_s}-r_c), \quad \ldots (12)$$

where r_c is the amplitude change introduced by the background. This is equivalent to

$$A'e^{-i\Delta'}=r_c Ae^{-i\Delta}+\sum_{s=1}^{n} q_s A_s e^{-i(\theta_s+\Delta_s)}, \quad \ldots\ldots\ldots \quad (13)$$

where

$$\left.\begin{aligned} q_s \sin\theta_s &= r_s \sin\varphi_s, \\ q_s \cos\theta_s &= r_s \cos\varphi_s - r_c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (14)$$

Thus

$$\left.\begin{aligned} A'\cos\Delta' &= r_c A\cos\Delta + A_\sigma \cos\Delta_\sigma, \\ A'\sin\Delta' &= r_c A\sin\Delta + A_\sigma \sin\Delta_\sigma, \end{aligned}\right\} \quad \ldots\ldots \quad (15)$$

giving

$$(A')^2=r_c^2A^2+A_\sigma^2+2r_cAA_\sigma \cos(\Delta-\Delta_\sigma), \quad \ldots\ldots \quad (16)$$

where

$$\left.\begin{aligned} A'\cos\Delta_\sigma &= \sum_{s=1}^{n} A_s q_s \cos(\theta_s+\Delta_s), \\ A'\sin\Delta_\sigma &= \sum_{s=1}^{n} A_s q_s \sin(\theta_s+\Delta_s), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (17)$$

and

$$(A')^2=\sum_{s=1}^{n} A_s^2q_s^2+\sum_{\substack{s,k=1\\ s\neq k}}^{n} A_sA_kq_sq_k \cos(\theta_s-\theta_k+\Delta_s-\Delta_k). \quad \ldots \quad (18)$$

Two particular cases of interest, which will be referred to later, may be mentioned. When all the apertures are pure phase apertures so that $r_s=r_c=1$ equation (**12**) takes the form

$$A'e^{-i\Delta'}=Ae^{-i\Delta}+2\sum_{s=1}^{s=n} A_s \left|\sin\frac{\varphi_s}{2}\right| e^{-i(\Delta_s+\varphi_s/2\pm\pi/2)}, \quad \ldots \quad (19)$$

the upper or lower sign being taken according as $\sin(\varphi_s/2)$ is positive or negative.

When all the apertures are pure amplitude apertures so that $\varphi_s=0$, we have

$$A'e^{-i\Delta'}=r_cAe^{-i\Delta}+\sum_{s=1}^{s=n} A_s \mid r_s-r_c \mid e^{-i(\Delta_s+\alpha)}, \quad \ldots\ldots \quad (20)$$

where $\alpha=0$ or π according as r_s-r_c is positive or negative. If for one of the phase amplitude apertures σ_p the phase and amplitude factors φ_p and r_p vary continuously over its area it is obvious that the contribution of this aperture to the summation term in (**12**) is

$$\int Pe^{-i\Delta_p}(r_pe^{-i\varphi_p}-r_c)d\sigma_p,$$

where the integration is taken over the area σ_p and P and Δ_p are functions of the position of the element $d\sigma_p$ and of the point at which Ω' is required.

II. Applications

In applying equations (**9**) or (**10**) to any particular problem it is only necessary to know the values of the integrals C, S, C_1, and S_1 (which correspond to the case in which r_1 and r_2 are unity and φ is zero). However, it should be pointed out in connexion with Fraunhofer diffraction that the integrals C_1 and S_1 depend on the position of the area σ_1 as well as its shape and orientation but in such a way that a displacement of σ_1 in its own plane without rotation does not alter $C_1^2+S_1^2$. Thus, as is well known, such a displacement does not alter the diffraction pattern of a simple aperture except as regards the phase Δ_1 associated with it. In the case of a phase amplitude aperture, however, the pattern is modified by such a displacement (except at points at which equation (**11**) applies), as is seen from (**9**). It should be noted further that when an aperture is symmetrically placed with regard to the incident wave the integral S or S_1, as the case may be, is everywhere zero since it represents the integral of an odd function. At the geometric image, even when symmetry is absent, S and S_1 are zero and C and C_1 (or A and A_1) proportional respectively to the areas σ and σ_1 since all rays reach the focus in the same phase.

Examples will now be considered which have been worked out by other methods, generally from first principles, but which may be considered as simple applications of the above formulae.

(*a*) *The Phase Grating*

A phase or laminary grating (Wood 1911, p. 211) consists of a series of transparent strips, alternate strips being of equal width and producing a phase change differing from that of the other strips by a constant. When used to form spectra one set of alternate strips of such a grating represents collectively σ_1 and the other set the complementary area σ_2. Application of (**10**) and (**11**) gives all the effects described qualitatively by Wood (1911, p. 211). In particular when $r_1=r_2=1$ and for a particular wavelength for which $\varphi=\pi$ there is a fourfold intensification of the spectra at points where (**11**) applies. If, in addition, the widths of the strips are all equal the illumination of the central image is zero for this wavelength. These results are deduced, in the first place, for a point source and the principal axis of the pattern but they apply equally to the case of a line source parallel to the laminae.

(*b*) *Effects of Striae in a Lens on the Illumination in the Image Plane*

Striations in optical glass introduce a phase change in the light passing through them. If this phase change is uniform they would act as phase apertures in the sense used here. This assumption of uniformity may seem to introduce an over-simplification but we shall show that it is the mean value of φ over the striation which is important. Suppose a lens has a circular aperture of radius R and a single uniform striation in the form of a narrow transparent strip of length a and breadth b, so that $r_1=r_2=1$, then $S=0$ and $C=A$ at all points of the image plane. For a centrally placed striation,

$$\left.\begin{aligned} &S_1=0, \\ &C_1=ab\left[\frac{\sin\frac{kb}{2}}{\frac{kb}{2}}\right] \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{21})$$

at points on the major axis of the diffraction pattern, k being given by

$$k=\frac{2\pi\theta}{\lambda},$$

where θ is the angle between the optic axis and the line joining the centre of the lens to the point in the image plane at which the illumination is required.

The values of C^2 at the geometric image, first minimum, and second maximum of the diffraction pattern due to the lens alone are proportional to 1, 0, and 0·017 and occur respectively at points for which θ has the values 0, $0{\cdot}61\ \lambda/R$, and $0{\cdot}81\ \lambda/R$. The corresponding values of S are zero. For a large striation such that $ab/R^2=0{\cdot}1$ and $\varphi=\pi$ equations (**9**) and (**21**) give for the illumination at these three points the comparative figures 0·87, 0·004, 0·005. For $\varphi=1$ radian and $ab/R^2=0{\cdot}04$ the corresponding figures are 0·9873, 0·0002, and 0·0160. Françon (1948*a*), using a direct method applicable to his particular problem, considers the effect of a striation which introduces a phase change which varies with the displacement x from its major axis in accordance with the relation $\varphi\propto(1+\cos 2\pi x/b)$. His corrected results (Françon 1948*b*) for a striation of the same dimensions as in the last example and producing the same average phase change are, for the three points mentioned, 0·9852, 0·0001, 0·0195, so that the effects are not very sensitive to the distribution of phase change across the striation.

The effect of a striation, however, depends to some extent upon its position, the diffraction pattern being unsymmetrical for a striation which is off centre except when $\varphi=\pi$. Thus in the first example, if the striation is displaced through a distance $R/2$ from the centre, the illumination at the first and second points is unaffected but at the third it is increased to 0·033.

(*c*) *Haloes etc.*

Consider a large number, n, of diffraction apertures, represented collectively by σ_1, similar as regards shape and orientation, but randomly spaced. At points sufficiently far from the geometric image equation (**11**) is replaced by

$$E'=n(r_1^2+r_2^2-2r_1r_2\cos\varphi)e_1, \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{22})$$

where e_1 is the illumination produced by a single aperture. At the geometric image itself the illumination is given by

$$E'=r_2^2E+(r_1^2+r_2^2-2r_1r_2\cos\varphi)ne_1-2(ne_1E)^{1/2}(r_2^2-r_1r_2\cos\varphi). \qquad \ldots \quad (\mathbf{23})$$

In the special case where $\varphi=\pi$ and each of the constituent areas is $1/2n$ of the area of the lens, the diffraction pattern is intensified at points away from the geometric image by a factor $4n$, as compared to the effect of a simple aperture, and the illumination at the geometric image is reduced to zero. The haloes obtained with transparent disks by R. W. Wood (1911, p. 254) may be treated in this way. A phase grating with random spacing behaves in a similar manner.

(*d*) *Phase Contrast Techniques* (*Zernike* 1942)

The object of these techniques is to render visible, by means of a microscope or similar optical arrangement, details which may be described as pure phase apertures ($r_1=r_2=1$, $\varphi\neq 0$). Suppose that, instead of receiving the diffraction pattern of such an aperture (i.e. the detail to be examined) on the plane conjugate to the original point source, the wave characteristics (amplitude and phase) associated with the diffraction pattern are so modified in passing through this plane as to correspond at all points to those due to a pure amplitude aperture ($r_1\neq r_2$, $\varphi=0$). Then an optical system, placed so as to receive the light passing through this plane and focused on the detail, will image it as such an amplitude aperture. The resulting image will have what is called bright contrast if the ratio $R=r_1/r_2$ is greater than unity and dark contrast if R is less than unity. If φ is small, the wave characteristics for the pure phase aperture in the off-centre part of the diffraction field where $A=0$ are, from (**6**), given by

$$A'e^{-i(\Delta'-\Delta_1)}=-i\varphi A_1=|\varphi|A_1e^{\mp i\pi/2}$$

in which the upper or lower signs are to be taken according as φ is positive or negative. Thus

$$A'=|\varphi|A_1, \quad \Delta'=\Delta_1\pm\pi/2.$$

For the pure amplitude aperture in the same region we have, from (**6**),

$$A'e^{-i(\Delta'-\Delta_1)}=(r_1-r_2)A_1$$

or

$$A'=|r_1-r_2|A_1 \text{ and } \Delta'=\Delta_1 \text{ or } \Delta_1+\pi,$$

according as $r_1>$ or $<r_2$. Thus for a positive value of r_1-r_2 the phase at all points of the off-centre diffraction pattern for the pure phase aperture is in advance of that for the pure amplitude aperture by $\pm\pi/2$ according as φ is positive or negative and (taking $3\pi/2$ as equivalent to $-\pi/2$) vice versa for r_1-r_2 negative. Again the amplitude at all such points due to the pure phase aperture bears the constant ratio $|\varphi|/|r_1-r_2|$ to that due to the pure amplitude aperture. Thus the required correspondence exists for any given value of φ, r_1, and r_2 at all points in this part of the pattern.

For the central part close to the geometric image the phase due to both types of aperture is approximately Δ_1 or Δ and the amplitudes for the two cases respectively are, from first principles or by (**6**), approximately A and $r_1A_1+r_2(A-A_1)$. Thus, to establish for all points of the pattern in the two cases a constant difference of phase and a constant ratio of amplitudes, it is necessary to introduce the small correcting plate used in phase contrast techniques to intercept the rays reaching the vicinity of the geometric image of the point source. This plate should produce a phase change in the transmitted light of $\pm\pi/2$ and have an amplitude transmittance t where

$$\frac{At}{r_1A_1+r_2(A-A_1)}=\frac{|\varphi|}{|r_1-r_2|}, \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{24})$$

or

$$\frac{\sigma t}{\sigma_1R+(\sigma-\sigma_1)}=\frac{|\varphi|}{|R-1|}, \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{25})$$

the contrast being bright or dark according as the phase change introduced is of the same or opposite sign to that of φ. The value of the ratio R determining the contrast is thus related to t by

$$R=\frac{t\sigma\pm\varphi(\sigma-\sigma_1)}{t\sigma\mp\varphi\sigma_1}\simeq1\pm\frac{\varphi}{t}, \quad \ldots\ldots\ldots\ldots\ldots \quad (26)$$

the upper or lower sign being taken according as the plate introduces a phase change of $+\pi/2$ or $-\pi/2$.

Maximum contrast is obtained when $R=\infty$ or 0, i.e. when

$$t=\frac{\sigma_1}{\sigma}\varphi \text{ or } t=\frac{\sigma-\sigma_1}{\sigma}\varphi, \quad \ldots\ldots\ldots\ldots\ldots \quad (27)$$

for bright and dark contrast respectively. For values of t large compared with these the approximate form of (**26**) holds.

When the detail to be examined consists of a number of pure phase apertures introducing small phase changes φ_s relative to a common background it is readily shown by means of equations (**19**) and (**20**) that (**26**) is replaced by

$$R_s=1\pm\frac{\varphi_s\sigma}{t\sigma\mp\Sigma\varphi_s\sigma_s}=1\pm\frac{\varphi_s\sigma}{t\sigma\mp\int\varphi d\sigma}\simeq1\pm\frac{\varphi_s}{t}. \quad \ldots\ldots \quad (28)$$

The integral form for $\Sigma\varphi_s\sigma_s$ is of use when φ changes continuously in any part of the detail.

(*e*) *Zone Plate with Phase Reversal* (*Rayleigh* 1902)

The applications so far considered have been concerned with Fraunhofer diffraction. This and the next are examples involving Fresnel diffraction. For all such applications (except as mentioned in connexion with equation (**11**)) the aperture σ_2 extends effectively from the boundary of σ_1 to infinity. Similarly σ is a region of space of infinite extent.

For a zone plate illuminated by a divergent beam from a point source the light is so concentrated at the focal points of the plate that A is negligible compared to A_1. Thus, if the opaque parts of the plate are rendered transparent but produce a relative phase change of π, substitution of $r_1=r_2=1$, $\varphi=\pi$, $A=0$ in (**8**) gives $(A')^2=4A_1^2$ so that the focal properties of the plate are intensified fourfold.

(*f*) *The Hologram* (*Gabor* 1949, 1951)

If a small object (σ_1, r_1, φ) be placed in a divergent wave from a point source the wave function associated with the diffraction pattern is obtained from (**5**) by setting $r_2=1$, namely,

$$\Omega'=A'e^{-i\Delta'}=Ae^{-i\Delta}-A_1\{e^{-i\Delta_1}-r_1e^{-i(\Delta_1+\varphi)}\}. \quad \ldots\ldots \quad (29)$$

If σ_1 is sufficiently small, A_1^2 is negligible, so from (**8**)

$$A'^2=A^2-2AA_1[\cos(\Delta-\Delta_1)-r_1\cos\{\varphi-(\Delta-\Delta_1)\}]. \quad \ldots \quad (30)$$

Now suppose a photographic plate is exposed to the diffracted wave and then developed by reversal, or its equivalent, with an overall gamma of 2. Then the developed plate or hologram, as it is called, will have an amplitude transmission proportional to $(A')^2$. If the plate is replaced in its original position but with the object removed, so that the direct light from the source falls on it, the wave function Ω'' in the plane immediately behind the emulsion will, except for a constant of proportionality, be represented by

$$\Omega''=A'^2Ae^{-i\Delta}=A^2[Ae^{-i\Delta}-A_1\{e^{-i\Delta_1}-r_1e^{-i(\varphi+\Delta_1)}+e^{-i(2\Delta-\Delta_1)}-r_1e^{i(\varphi-2\Delta+\Delta_1)}\}], \qquad (31)$$

that is, by

$$\Omega''=A^2\Omega'-A^2A_1\{e^{-i(2\Delta-\Delta_1)}-r_1e^{i(\varphi-2\Delta+\Delta_1)}\}, \qquad (32)$$

or

$$\Omega''=A^2e^{-2i\Delta}\Omega'^*-A^2A_1\{e^{-i\Delta_1}-r_1e^{-i(\varphi+\Delta_1)}\}, \qquad (33)$$

where Ω'^* is the conjugate of Ω'.

The interpretation of (**32**) is simple. If it were possible to disregard all but the first term on the right, Ω'' would be proportional to Ω' for a uniform source. Thus from Huyghens' principle the disturbance everywhere behind the plate would be just as though the object were restored to position and the plate removed ; in other words, in the original location of the object there would be a virtual image which could be viewed with a suitable optical system. Actually what is seen is an imperfect image the defects being due to the remaining terms in (**32**).

To interpret the alternative expression given by (**33**) we again consider only the first term on the right of the equation. It should then be noted that in both (**32**) and (**33**) the time or phase origin is the same. Treating first the case in which the object is initially nearer to the source than to the photographic plate, suppose the source to be formed by a wave converging on the point where it is located from the side remote from the plate. Consider now an object in the converging beam at a position which is the mirror image of the original object (σ_1, r_1, φ) in reference to a spherical surface located in the plane of the photographic plate and concentric with the source. Then it may be shown (Appendix I) that this object has the same diffraction pattern (as regards amplitude) in the plane of the plate, and hence the same hologram, as the original object. Thus the hologram reconstruction must contain as an alternative to the virtual image described by (**32**) a second inverted virtual image behind the source in this position. When phase relations are taken into account the complete expression for the wave function associated with the diffraction pattern of the second object (Appendix I, equation (**A4**)) is $A^2e^{-2i\Delta}\Omega'^*$, thus accounting for alternative (**33**). When the original object is closer to the plate than to the source a similar argument applies. However, in this case the mirror image which constitutes the secondary object is on the side of the plate remote from the source and this object would produce the same diffraction pattern in the plane of the plate with a light beam converging on the position of the original point source from the opposite direction. The wave function associated with this pattern (Appendix I, equation (**A7**)) is $A^2e^{2i\Delta}\Omega'$. The conjugate quantity $A^2e^{-2i\Delta}\Omega'^*$ occurring in (**33**) now represents a wave front leaving the hologram and converging on the location of the secondary object to form a real image of the original object. This image may be viewed by means of a short focus optical system such as a microscope with the aid, if necessary, of phase contrast (or similar techniques). Again the neglected terms in (**33**) represent defects of the image which are thus seen to be present in the case of all the images reproduced by the hologram.

(*g*) *Other Applications*

Zernike (1948) has described the Fraunhofer diffraction pattern due to a slit covered with a transparent layer of strongly absorbing metal with a narrow scratch in the centre of the slit. Obviously equation (**8**) contains the complete theory of this experiment, the amplitude factors A and A_1 referring respectively to the slit and scratch as simple apertures. The effect of varying the slit width and the position of the scratch may easily be observed with the aid of an optical bench.

Zernike (1948) has also described the existence of fringes within the geometrical shadow of a straight edge when the diffracting screen is slightly transparent. Again referring to equation (**8**), A and Δ relate in this case to the unimpeded wave from the source and A_1 and Δ_1 to the pattern produced by the corresponding opaque screen. Thus within the shadow the term in A_1^2 is negligible and the fringes referred to arise from the variations in the coefficient of AA_1. For example, if the effect of φ is neglected this coefficient is proportional to $\cos(\Delta - \Delta_1)$ or $\sin \Delta_1$ since Δ has the value $\pi/2$ (Preston 1928). The variations of $\sin \Delta_1$ are readily obtainable from Cornu's spiral or a table of Fresnel's integrals.

Similar but modified effects to those just mentioned are produced by diffraction at the edge of a transparent lamina when phase changes are introduced (Wood 1911, p. 250) and the same equation gives the analysis.

III. References

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Appendix I

Equivalent Diffraction Apertures

Suppose light converges on the point O (Fig. 1) in the direction $B'O$ and is diffracted by an aperture AB. It will be shown that, if MX is a spherical surface concentric with O and further from AB than is O, the diffraction pattern produced on MX is the same as would be produced by an aperture $A'B'$ which is the mirror image of AB in respect of the surface MX.

Consider two conjugate points P and P'. Since all optical paths from P to P' must be equivalent we have for any point X

$$PX + P'X = PM + P'M,$$

or

$$P'M - P'X = PX - PM. \qquad \text{(A1)}$$

Thus light from P' reaches X in a phase which is as much ahead of the phase in which it reaches M as the phase with which light from P reaching X is behind the phase with which it reaches M. Let the absolute value of this difference be Δ_n. Further, let the phase of light on reaching M along the path $POP'M$ lag behind that at O by an amount Δ. Then the phase with which the light diffracted by an infinitesimal area at P will reach X is $\Delta - \Delta_n$, whilst the phase

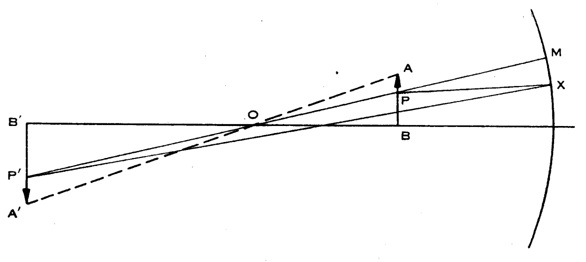

Fig. 1

with which the light diffracted by the conjugate element at P' will reach X is $\Delta + \Delta_n$, Δ being a constant for all positions of P and P'. The amplitudes of the diffracted light reaching X from the two conjugate elements at P and P' respectively are equal since the flux through them is the same. Denoting these amplitudes by a_n, the resultant wave functions at X due to the apertures AB and $A'B'$ are represented by

$$\Omega = \Sigma a_n e^{-i(\Delta - \Delta_n)} = e^{-i\Delta} \Sigma a_n e^{i\Delta_n}, \qquad \text{(A2)}$$
$$\Omega' = \Sigma a_n e^{-i(\Delta + \Delta_n)} = e^{-i\Delta} \Sigma a_n e^{-i\Delta_n}. \qquad \text{(A3)}$$

Therefore

$$\Omega' = e^{-2i\Delta} \Omega^*, \qquad \text{(A4)}$$

where Ω^* is the conjugate of Ω.

Thus the amplitudes of Ω' and Ω are the same at all points and the diffraction patterns due to the two apertures are identical in shape and position. Obviously these relations are reciprocal, that is, equation (**A4**) holds when AB and its image are interchanged.

When the aperture AB is closer to MX than to O as in Figure 2, a similar conclusion may be drawn but with the following modification. The aperture AB gives the same diffraction pattern on MX, for light diverging from O, as its image $A'B'$ for light converging on O. To prove this we note that equation (**A1**) is replaced by

$$P'X - P'M = PX - PM,$$

which follows from the fact that waves from P reach M and X respectively with the same phase difference as if they had originated from the image P'. Thus, if we consider two wave trains, one converging on O and the other diverging from it, then, at the instant when both trains are in phase at O, the phase of both at M will be the same (Δ say); but at the same instant the phase at X of the waves diffracted by P and P' will be $(\Delta - \Delta_n)$ and $(\Delta + \Delta_n)$ where Δ_n corresponds to the common path difference.

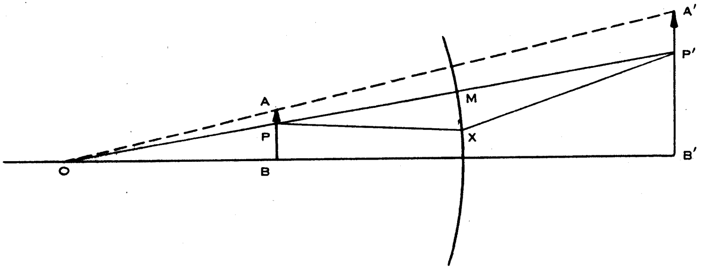

Fig. 2

Hence in place of (**A2**) and (**A3**) we have for the diverging wave

$$\Omega = \Sigma a_n e^{-i(\Delta - \Delta_n)} = e^{-i\Delta} \Sigma a_n e^{i\Delta_n}, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{A5})$$

and for the converging wave

$$\Omega' = \Sigma a_n e^{i(\Delta + \Delta_n)} = e^{i\Delta} \Sigma a_n e^{i\Delta_n}, \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{A6})$$

or

$$\Omega' = e^{2i\Delta}\Omega. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{A7})$$

It is obvious from (**A7**) that $A'B'$ and AB produce the same diffraction pattern on MX.

It may be pointed out that the conjugate of Ω' as defined by (**A7**) would represent a wave front converging on $A'B'$ to form a real image of AB. If a device could be placed in position MX which would impart to a beam from O transmitted through it a wave front defined by $(\Omega')^*$ such a real image would be formed. The hologram succeeds partially in producing this effect.

When MX is sufficiently distant from O the diffraction screen MX may be plane. On the other hand when the diffraction screen passes through O (Fraunhofer diffraction) the above principles cease to apply.