LINEAR SUPERPOSITION IN VISCO-ELASTICITY AND THEORIES OF DELAYED EFFECTS

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Summary

Writers on elastic after-working or " visco-elasticity " have given a formula expressing the stress-history as an integral involving the strain-history linearly. If this is to be based directly on experiment, linear superposition is a necessary physical datum ; but it is not sufficient, and examples are given to demonstrate this. Necessary and sufficient conditions are established in this paper, and expressed in physical as well as in mathematical terms ; they may admit physical verification in some degree. Histories both with and without beginning are considered.

I. INTRODUCTION

Theories of hereditary phenomena generally use a relation between stimulus and response of the form

$$s(t)=\mu\sigma(t)+\int_0^t \varphi(t-\tau)\sigma(\tau)\mathrm{d}\tau, \qquad \text{(A)}$$

where $\sigma(t)$ measures the stimulus at time t, $s(t)$ the response at time t, μ is an instantaneous modulus, and $\varphi(\omega)$ is the intensity of memory over a time ω. Such a formula was propounded by Boltzmann (1876) for elastic after-working of metals, and further studied by Becker (1925) ; it was also developed by Volterra in studying hereditary phenomena in general. Some writers have preferred to use, in place of (**A**), the formula

$$s(t)=\int_0^t \sigma(t-\omega)\varphi(\omega)\mathrm{d}\omega. \qquad \text{(B)}$$

This is equivalent to (**A**) if $\varphi(\omega)$ is permitted to contain a δ-function.

In the context of visco-elasticity, (**A**) or (**B**) is usually based on assumptions regarding the micro-structure of the material (Gross 1947 ; Sips 1950 ; and others). However, theories of micro-structure are designed to explain the observations, and a less circuitous course would be to infer (**A**) directly from the observations. This has been suggested by I. M. Stuart in unpublished work to which he has kindly permitted me to refer.

If (**A**) holds, it is evident that linear superposition gives the response when two sets of stimuli operate together. But, if linear superposition is observed to hold, can we conversely infer (**A**) or (**B**)? Examples below show that this converse is false. The question thus arises : *What conditions, besides linear*

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superposition, are necessary and sufficient to ensure that the stress-history s(t) *and the strain-history* σ(t) *are related by a formula such as* (**A**) *or* (**B**) *?*

This paper gives two answers to this question, one for strain-histories with a beginning and the other for those without. Most of the components of these answers are to be found in mathematical literature, and I am attempting to do little more than to assemble them.

II. Linear Superposition

Linear superposition is not enough to ensure (**A**) or (**B**). With the meaning of " linear superposition " specified below, at L, the truth of this contention is demonstrated by the following examples.

(i) Suppose the material is viscous, with viscosity η, so that the strain-history $\sigma(t)$ is differentiable for all t and

$$s(t)=\eta\sigma'(t). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{1})$$

Then the sum of two separate strain-histories clearly gives rise to the sum of the corresponding stress-histories, so that linear superposition holds. But there is no integrable function $\varphi(\omega)$ such that (**B**) holds. For, if there were, the strain-histories

$$\sigma_n(t)=0 \ \ (t\leqslant 0), \qquad\qquad \sigma_n(t)=\frac{\sin^2 nt}{n} \qquad (t\geqslant 0)$$

would determine, for each positive integer n, the stress-history

$$s_n(t)=\int_0^t \varphi(\omega)\frac{\sin^2 n(t-\omega)}{n}d\omega \qquad\qquad (t\geqslant 0),$$

so that

$$|\, s_n(t)\,|\leqslant\frac{1}{n}\int_0^t|\,\varphi(\omega)\,|\,d\omega.$$

Thus $s_n(t)$ would tend to zero as $n\to\infty$, for each fixed value of t; whereas, on the contrary, (**1**) gives

$$s_n(t)=\eta\sin 2nt \qquad\qquad (t\geqslant 0).$$

If it is objected that (**1**) is inappropriate to this subject, as it makes stress independent of the previous history of strain, this objection cannot be raised against

$$s(t)=\eta\sigma'(t)+\varepsilon\int_0^t\sigma(\tau)d\tau,$$

but remarks similar to those above apply to this stress-strain relation also.

(ii) Suppose the stress-strain relation is

$$s(t)=\int_0^t\sigma(\tau)\tau d\tau. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{2})$$

Linear superposition clearly holds, but no relation (**B**). For if both held we should have, putting $\omega=t-\tau$ in (**B**),

$$\int_0^t \sigma(\tau)\{\varphi(t-\tau)-\tau\}d\tau=s(t)-s(t)=0\ ;$$

and this would hold for every strain-history, so that

$$\varphi(t-\tau)-\tau=0 \qquad (0\leqslant\tau\leqslant t),$$

that is,

$$\varphi(\omega)=t-\omega \qquad (\omega\geqslant 0, t\geqslant 0).$$

Thus $\varphi(\omega)$ would have to be a function of t as well as of ω.

Similar remarks apply to the stress-strain relation

$$s(t)=\frac{2S}{t^2}\int_0^t \sigma(\tau)\tau d\tau,$$

a plausible enough relation, if physical intuition can be trusted, since it makes the stress a weighted mean of the previous strains with the greatest weight attached to the most recent strains.

(iii) Suppose the stress-strain relation is expressed by (**4**) below, with $\psi(\omega)$ a continuous monotonic function which is not absolutely continuous (Riesz and Sz.-Nagy 1953, Section 24). Linear superposition holds for all piecewise-continuous strain-histories, but neither (**B**) nor (**A**) is capable of expressing the stress-strain relation. For if $\sigma(t)$ is the unit function defined in (**3**) below, the corresponding stress-history is $\psi(t)$ which is not absolutely continuous, but the right members of (**A**) and (**B**) are necessarily absolutely continuous.

On this account the use of a Stieltjes integral is inevitable for completeness, although in calculations which use only elementary functions it reduces to an ordinary integral. The Stieltjes integral also has the further advantage that no δ-function is necessary.

Additional conditions. In addition to linear superposition L, three other conditions, P, Q, and R, concerning stress and strain-histories are required. These are as follows.

L, linear superposition: If two strain-histories $\sigma_1(t)$ *and* $\sigma_2(t)$ *separately entail stress-histories* $s_1(t)$ *and* $s_2(t)$, *then the strain-history* $\sigma_1(t)+\sigma_2(t)$ *entails the stress-history* $s_1(t)+s_2(t)$.

Further, a history of no strain entails one of no stress.

P, approximation: If strain-histories approximate one another until a certain instant, then so do the corresponding stress-histories until that instant. More exactly :

If strain-histories $\sigma_n(t)$ approximate a strain-history $\sigma(t)$ in the sense of uniform convergence throughout $t\leqslant T$, that is,

$$\overline{\text{bnd}}_{t\leqslant T}\ |\ \sigma_n(t)-\sigma(t)\ |\to 0 \qquad \text{as } n\to\infty,$$

where the upper bound is taken for all values of $t \leqslant T$, keeping n fixed; then, for the corresponding stress-histories $s_n(t)$ and $s(t)$,

$$s_n(t) - s(t) \to 0 \qquad \text{as } n \to \infty,$$

for each $t \leqslant T$.

This excludes stress-strain relations such as (**1**). It could be expressed in the language of functional analysis: the stress at any instant is a continuous functional of the previous strain-history, the distance between strain-histories being specified as above. Such conditions are essential in a theory of this kind.

Q, invariability: If one strain-history is a replica of another except for a steady delay throughout, then the corresponding stress-histories are in the same relation. (Cf. Volterra 1931, p. 189.) More exactly:

If, for some particular time of delay δ, the relation:

$$\sigma_1(t) = \sigma_2(t-\delta) \qquad \text{for all } t,$$

connects two strain-histories, then, for the corresponding stress-histories

$$s_1(t) = s_2(t-\delta) \qquad \text{for all } t.$$

This excludes stress-strain relations such as (**2**). It is a condition which might fail if the range of deformation were extensive enough to bring on elastic fatigue.

R, relaxation behaviour: The relaxation function vanishes for $t \leqslant 0$, *is continuous-on-the-left at all times, and is bounded.* The relaxation function is here defined as the stress-history entailed by a strain-history of the unit-function continuous-on-the-left:*

$$u(t) = 0 \quad (t \leqslant 0), \qquad u(t) = 1 \quad (t > 0). \qquad \text{............ } (\mathbf{3})$$

Physically it would no doubt be impossible to verify continuity-on-the-left, and immaterial; but some such assumption is needed in the theory to ensure the existence of all integrals occurring, unless we abandon such convenient fictions as discontinuous strain-histories. Throughout the paper the discontinuities of strain-histories will be similarly restricted, for the same reason.

III. Histories beginning at $t=0$

In order that the stress-history $s(t)$ *be expressible, for every left-continuous strain-history* $\sigma(t)$ *vanishing throughout* $t \leqslant 0$, *by*

$$s(t) = \int_0^t \sigma(t-\omega) d\psi(\omega), \qquad \text{................ } (\mathbf{4})$$

in which ψ *is a function of bounded variation on every finite interval and*

$$\psi(t) = 0 \qquad \textit{for all } t \leqslant 0,$$

it is necessary and sufficient that conditions L, P, Q, R hold for all left-continuous strain-histories vanishing throughout $t \leqslant 0$.

The function $\psi(t)$ *is the relaxation function defined in R above.*

* Boundedness is only needed later, for strain-histories having no beginning.

Before proving this theorem, the following notes on the meanings of terms are necessary.

A *left-continuous function* is continuous everywhere except for an isolated set of simple discontinuities-on-the-right. It is easily seen that the sum of two left-continuous functions is left-continuous, and that a left-continuous function is bounded in any finite interval.

The integral in (**4**) is to be understood as a slight extension of the elementary Stieltjes integral (Widder 1941, Ch. I), as follows. If $f(\omega)$ is continuous except for possible simple discontinuities at $c_0, c_1, \ldots, c_{k-1}, c_k$, where

$$a=c_0<c_1<\ldots<c_{k-1}<c_k=b,$$

we define

$$\int_a^b f(\omega)\mathrm{d}g(\omega)=\sum_{i=1}^{k}\int_{c_{i-1}}^{c_i} f(\omega)\mathrm{d}g(\omega), \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{5})$$

provided all the integrals on the right exist as elementary Stieltjes integrals. This definition is consistent with the elementary definition whenever the latter applies to the left side of (**5**), but has the advantage that integrability of f with respect to g is not necessarily destroyed if both are discontinuous at the one point, unless they are discontinuous on the same side of it.

(*a*) *Proofs that* L, P, Q, R *are necessary*

We suppose there is a function ψ, of bounded variation on every finite interval and vanishing for all $t\leqslant 0$, such that (**4**) expresses the stress-history for every " permitted " strain-history, namely those which are left-continuous and vanish throughout $t\leqslant 0$. We seek to deduce R, and also L, P, Q for all permitted strain-histories to which they are applicable.

Necessity of R. By our present hypothesis, the relaxation function defined in R above must be the stress-history $s(t)$ given by

$$s(t)=\int_0^t u(t-\omega)\mathrm{d}\psi(\omega), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{6})$$

and this integral must exist for each t. By (**3**), the integrand is continuous except for a simple discontinuity-on-the-left at $\omega=t$; so the definition of the integral reduces to the elementary definition. If $t>0$, since the integral exists $\psi(\omega)$ must be continuous-on-the-left at $\omega=t$; and then

$$s(t)=\psi(t)-\psi(0)=\psi(t).$$

If $t\leqslant 0$ it follows immediately from (**3**) and (**6**) that

$$s(t)=0.$$

Thus R is established and ψ is identified with the relaxation function.

Necessity of L. For any two permitted strain-histories $\sigma_1(t)$ and $\sigma_2(t)$, the sum of the corresponding stress-histories $s_1(t)$ and $s_2(t)$ is

$$s_1(t)+s_2(t)=\int_0^t \sigma_1(t-\omega)d\psi(\omega)+\int_0^t \sigma_2(t-\omega)d\psi(\omega)$$
$$=\int_0^t \{\sigma_1(t-\omega)+\sigma_2(t-\omega)\}d\psi(\omega), \qquad \ldots\ldots\ldots\ldots (7)$$

which, by our present hypothesis, is the stress-history entailed by the strain-history $\sigma_1(t)+\sigma_2(t)$, since this is left-continuous and vanishes for $t\leqslant 0$ and is therefore one to which (4) applies.

Necessity of P. Let $T>0$ and $\sigma_1(t)$, $\sigma_2(t)$, . . ., $\sigma(t)$ be permitted strain-histories such that

$$b_n=\overline{\text{bnd}}_{t\leqslant T}\, |\, \sigma_n(t)-\sigma(t)\,|\to 0 \qquad \text{as } n\to\infty.$$

For the corresponding stress-histories, if $0\leqslant t\leqslant T$,

$$|\, s_n(t)-s(t)\,|=\left|\int_0^t \{\sigma_n(t-\omega)-\sigma(t-\omega)\}d\psi(\omega)\right|$$
$$\leqslant b_n V(\psi\,;\; 0,\; t), \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where $V(\psi\,;\; 0,\; t)$ is the total variation of ψ on $(0, t)$. Since this is independent of n, and $b_n\to 0$, P follows for $T>0$.

P is immediate for $T\leqslant 0$ since, by (4) and R,

$$s_n(t)=0=s(t) \qquad \text{for all } t\leqslant 0.$$

Necessity of Q. Let $\sigma_2(t)$ be left-continuous and vanish for $t\leqslant 0$; then $\sigma_1(t)=\sigma_2(t-\delta)$ is another such strain-history, supposing that $\delta>0$. For the corresponding stress-histories, at any particular instant $t>\delta$,

$$s_2(t-\delta)=\int_0^{t-\delta} \sigma_2(t-\delta-\omega)d\psi(\omega)$$
$$=\int_0^t \sigma_2(t-\omega-\delta)d\psi(\omega)=\int_0^t \sigma_1(t-\omega)d\psi(\omega)=s_1(t)\,;$$

while for $t\leqslant\delta$ these equations still hold, every term being zero.

(*b*) *Proof that L, P, Q, R together are sufficient*

As before, we refer to strain-histories which are left-continuous and vanish throughout $t\leqslant 0$ as " permitted " strain-histories. We need first three lemmas. Only references or indications of proofs need be given for these.

Lemma 1. *If L and P hold for all permitted strain-histories, λ is any real number, and a permitted strain-history* $\sigma(\mathrm{t})$ *entails a stress-history* $\mathrm{s}(\mathrm{t})$, *then the strain-history* $\lambda\sigma(\mathrm{t})$ *entails the stress-history* $\lambda\mathrm{s}(\mathrm{t})$.

This is proved successively for λ zero or a positive integer, positive rational, positive irrational, and negative real. Only for irrational values of λ is it necessary to use P.

Lemma 2. *If L and P hold for all permitted strain-histories, and* T *is fixed and positive, then the ratio*

$$| s(T) | \Big/ \overline{\operatorname*{bnd}_{t\leqslant T}} | \sigma(t) |$$

is bounded for all permitted strain-histories σ.

Proof as in Riesz and Sz.-Nagy (1953, p. 149), with $T\sigma$ (denoting a functional of σ) replaced by $s(T)$ and $|| \sigma ||$ replaced by $\overline{\operatorname*{bnd}_{t\leqslant T}} | \sigma(t) |$.

Lemma 3. *If* T *is fixed and positive, any permitted strain-history* $\sigma(\mathrm{t})$ *can be approximated uniformly throughout* $\mathrm{t}\leqslant\mathbf{T}$, *that is, in the sense of P, by permitted strain-histories* $\sigma_n(\mathrm{t})$ *which are also step-functions.*

In outline the proof is as follows. Referring to (**3**), the function

$$\sigma^*(t)=\sigma(t)-\sum_{j=0}^{l}\{\sigma(t_j+0)-\sigma(t_j)\}u(t-t_j)$$

is continuous in $0\leqslant t\leqslant T$, the discontinuities of $\sigma(t)$ being at the points $t=t_j$ and being cancelled by those of the step-function subtracted from it. Thus $\sigma^*(t)$ is the uniform limit, in $0\leqslant t\leqslant T$, of left-continuous step-functions $\sigma_n^*(t)$. Then

$$\sigma_n(t)=\sigma_n^*(t)+\sum_{j=0}^{l}\{\sigma(t_j+0)-\sigma(t_j)\}u(t-t_j)$$

are left-continuous step-functions converging to $\sigma(t)$ uniformly in $0\leqslant t\leqslant T$. Further, they may be defined to be zero throughout $t\leqslant 0$, and these strain-histories $\sigma_n(t)$ have the requisite properties.

We can now proceed to prove the sufficiency of our conditions. We suppose that L, P, Q hold for all permitted strain-histories, and that R holds. We seek to establish that the relaxation function ψ, defined in R, is of bounded variation on every finite interval, and that (**4**) expresses the stress-history for every permitted strain-history ; the proof is set out in these two stages. The former stage is essential to the latter ; the integral in (**4**) will not exist for all permitted strain-histories unless ψ is of bounded variation.

The relaxation function ψ *is of bounded variation on every finite interval.* The essentials of the following proof are given in Riesz and Sz.-Nagy, p. 109. This is part of the proof of Riesz's important theorem expressing linear functionals as Stieltjes integrals. It is presented below in a simpler form ; simpler because the central difficulty in Riesz's theorem is the extension of a functional outside its domain of definition, whereas in our context this extension is already provided by the relaxation function ψ.

To show that ψ is of bounded variation on an interval $(0, T)$, consider any subdivision $0=\omega_0<\omega_1<. . .<\omega_k=T$, write

$$\tau_i=T-\omega_i \qquad (i=0,\ 1, . . ., k),$$

and construct a left-continuous strain-history vanishing for $t\leqslant 0$:

$$\sigma(t)=\lambda_i \text{ in } \tau_i<t\leqslant\tau_{i-1} \ (i=1,\ 2, . . ., k),$$

where

$$\lambda_i=\begin{cases}+1 \text{ if } \psi(\omega_i)-\psi(\omega_{i-1})>0,\\ \ \ 0 \text{ if } \psi(\omega_i)-\psi(\omega_{i-1})=0,\\ -1 \text{ if } \psi(\omega_i)-\psi(\omega_{i-1})<0.\end{cases} \quad \ldots\ldots\ldots\ldots \quad (9)$$

This function is expressible in terms of the unit function (**3**):

$$\sigma(t)=\sum_{i=1}^{k}\lambda_i\{u(t-\tau_i)-u(t-\tau_{i-1})\}. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{10})$$

By R, a strain-history $u(t)$ entails stress-history $\psi(t)$.

By Q, a strain-history $u(t-\delta)$ entails stress-history $\psi(t-\delta)$.

By lemma 1, a strain-history $\lambda u(t-\delta)$ entails the stress-history $\lambda\psi(t-\delta)$; and by L it follows that the strain-history (**10**) entails the stress-history

$$s(t)=\sum_{i=1}^{k}\lambda_i\{\psi(t-\tau_i)-\psi(t-\tau_{i-1})\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{11})$$

Then

$$s(T)=\sum_{i=1}^{k}\lambda_i\{\psi(\omega_i)-\psi(\omega_{i-1})\}=\sum_{i=1}^{k}\mid\psi(\omega_i)-\psi(\omega_{i-1})\mid,$$

and

$$\mid s(T)\mid \Big/ \overline{\operatorname*{bnd}_{t\leqslant T}} \mid\sigma(t)\mid=\sum_{i=1}^{k}\mid\psi(\omega_i)-\psi(\omega_{i-1})\mid.$$

For fixed $T>0$, and any subdivision of $(0, T)$, this expression is subject to the bound established in lemma 2. Thus ψ is of bounded variation on $(0, T)$, and so on any finite interval.

The stress-history is given by (**4**) *for every permitted strain-history.* To prove this we choose a permitted strain-history $\sigma(t)$ and an instant $T>0$, and consider the approximating strain-histories $\sigma_n(t)$ provided by lemma 3.

Since each $\sigma_n(t)$ is a step-function, and left-continuous, it can be expressed in terms of the unit-function (**3**) in the form

$$\sigma_n(t)=\sum_{i=1}^{k_n}\lambda_{ni}\{u(t-t_{ni-1})-u(t-t_{ni})\},$$

where

$$0=t_{n0}<t_{n1}<\ldots<t_{nk_n}=T$$

and λ_{ni} is the value of $\sigma_n(t)$ throughout $t_{ni-1}<t\leqslant t_{ni}$. The corresponding stress-history is

$$s_n(t)=\sum_{i=1}^{k_n}\lambda_{ni}\{\psi(t-t_{ni-1})-\psi(t-t_{ni})\},$$

by applications of R, Q, lemma 1, and L analogous to those made in arguing from (**10**) to (**11**).

At time $t=T$ the above stress can be written

$$s_n(T)=-\sum_{i=1}^{k_n}\int_{t_{ni-1}}^{t_{ni}}\sigma_n(t)\mathrm{d}\psi(T-t),$$

the Stieltjes integrals existing in the elementary sense because, in $t_{ni-1} \leqslant t \leqslant t_{ni}$, $\sigma_n(t)$ is constant except for discontinuity-on-the-right at $t = t_{ni-1}$ while $\psi(T-t)$ is continuous-on-the-right by R. Using the Stieltjes integral defined in (**5**), it follows that

$$s_n(T) = -\int_0^T \sigma_n(t) \mathrm{d}\psi(T-t). \qquad \ldots\ldots\ldots\ldots\ (\mathbf{12})$$

It remains only to let $n \to \infty$ in (**12**). The left side then tends to $s(T)$, by P, where $s(t)$ denotes the stress-history entailed by the given strain-history $\sigma(t)$. The right side is expected to tend to

$$-\int_0^T \sigma(t) \mathrm{d}\psi(T-t), \qquad \ldots\ldots\ldots\ldots\ldots\ (\mathbf{13})$$

but before proving this we must know that this integral exists. It does exist in the sense of (**5**), for the component integrals between adjacent discontinuities of the integrand exist in the elementary Stieltjes sense ; this is because $\psi(T-t)$ is of bounded variation and has no discontinuity-on-the-right while $\sigma(t)$ is continuous except for a terminal discontinuity-on-the-right. Now

$$\left| \int_0^T \sigma(t) \mathrm{d}\psi(T-t) - \int_0^T \sigma_n(t) \mathrm{d}\psi(T-t) \right| \leqslant b_n V(\psi\ ;\ 0,\ T),$$

where

$$b_n = \overline{\underset{t \leqslant T}{\mathrm{bnd}}}\ |\ \sigma(t) - \sigma_n(t)\ |\ ;$$

and this tends to zero as $n \to \infty$, by lemma 3. Thus the right side of (**12**) indeed tends to (**13**), and, by P,

$$s(T) = -\int_0^T \sigma(t) \mathrm{d}\psi(T-t) = \int_0^T \sigma(T-\omega) \mathrm{d}\psi(\omega).$$

So (**4**) is established at any instant $t > 0$. To establish it for $t \leqslant 0$, it is clear that the integral is then zero, since ψ is constant in the range by R ; so we have only to ensure that the stress-history $s(t)$ entailed by any permitted strain-history $\sigma(t)$ vanishes throughout $t \leqslant 0$. This is a consequence of P and R, obtained by taking, in the former, $T = 0$ and

$$\sigma_n(t) = u(t) \qquad \text{for each } n.$$

For then

$$\overline{\underset{t \leqslant 0}{\mathrm{bnd}}}\ |\ \sigma_n(t) - \sigma(t)\ |\ = \overline{\underset{t \leqslant 0}{\mathrm{bnd}}}\ |\ \sigma(t)\ |,$$

which vanishes because $\sigma(t)$ is a permitted strain-history ; therefore, by P,

$$s_n(t) \to s(t) \text{ as } n \to \infty, \text{ throughout } t \leqslant 0.$$

But $s_n(t) = \psi(t)$, which vanishes throughout $t \leqslant 0$, by R ; consequently $s(t)$ does the same.

IV. Strain-histories having No Beginning

The preceding theorem applies only to strain-histories vanishing throughout $t \leqslant 0$, and is not properly applicable to strain-histories having no beginning, such as those of steady oscillation. An extra condition is needed, akin to Volterra's principle of dissipation of hereditary action (Volterra 1931, p. 188) but not obviously equivalent to it.

S, decay: Suppression of sufficiently ancient parts of a strain-history has practically no effect on the present stress. More exactly:

If strain-histories $\sigma_n(t)$ have agreed with a given strain-history $\sigma(t)$ since $t=-n$, so that

$$\sigma_n(t)=\sigma(t) \quad (t>-n), \qquad \sigma_n(t)=0 \quad (t\leqslant -n),$$

then, for the corresponding stress-histories $s_n(t)$ and $s(t)$,

$$s_n(t)\to s(t) \quad \text{as } n\to\infty, \text{ for each fixed } t.$$

The following theorem, analogous to that of Section III, can now be proved.

In order that the stress-history s(t) *be expressible, for every bounded left-continuous strain-history* σ(t), *by*

$$s(t)=\int_0^\infty \sigma(t-\omega)\mathrm{d}\psi(\omega), \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{14})$$

in which ψ is a function of bounded variation on $(0, \infty)$, *it is necessary and sufficient that conditions L, P, Q, R, S hold for all bounded left-continuous strain-histories.*

The function ψ(t) *is still the relaxation function defined in R.*

The infinite integral in (**14**) is to be understood as

$$\lim_{T\to\infty}\int_0^T \sigma(t-\omega)\mathrm{d}\psi(\omega),$$

where the integral on $(0, T)$ is understood in the sense of (**5**), the sense so far used in this paper.

Proofs that L, P, Q, R are necessary proceed almost as before, considering all bounded left-continuous strain-histories instead of only those previously permitted. We suppose that (**14**) expresses the stress-history corresponding to each such strain-history, the function ψ being of bounded variation on $(0, \infty)$ and not merely on each finite interval. In the course of these proofs this function ψ is again identified with the relaxation function. The only other differences are that infinite integrals occur in (**6**), (**7**), and (**8**). That in (**6**) reduces at once, if $t>0$, to the finite integral already considered; those in (**7**) and (**8**) are treated as before, the latter giving

$$|\, s_n(t)-s(t)\,| \leqslant b_n V(\psi;\ 0,\ \infty), \ \to 0 \text{ as } n\to\infty,$$

for each $t \leqslant T$ whether T is positive or not.

Proof that S necessarily holds for bounded left-continuous strain-histories. If $\sigma(t)$ and $\sigma_n(t)$ are such strain-histories, related as in the hypothesis of S, (**14**) gives

$$| s(t)-s_n(t) |=\left|\int_{t+n}^{\infty} \sigma(t-\omega)d\psi(\omega)\right| \leqslant KV\ (\psi;\ t+n,\ \infty),$$

where K is a bound of $|\sigma(t)|$. For any fixed t this expression tends to zero as $n\to\infty$, so that S holds.

Proof that L, P, Q, R, S together are sufficient, supposing that they hold for all bounded left-continuous strain-histories. These conditions hold, in particular, for all bounded left-continuous strain-histories vanishing throughout $t\leqslant 0$; so the stress-strain relation (**4**) applies in all such cases, by the preceding theorem in a form slightly modified so as to consider only bounded strain-histories.

Now suppose $\sigma(t)$ is any bounded left-continuous strain-history. Let $\sigma_n(t)$ be the strain-histories related with it as in S, and let $s(t)$ and $s_n(t)$ be the corresponding stress-histories. Since $\sigma_n(t-n)$ is a bounded left-continuous strain-history which vanishes for $t\leqslant 0$, and $s_n(t-n)$ is the corresponding stress-history by Q, it follows from (**4**) that

$$s_n(t-n)=\int_0^t \sigma_n(t-\omega-n)d\psi(\omega),$$

for all t; whence, replacing t by $t+n$,

$$s_n(t)=\int_0^{t+n} \sigma_n(t-\omega)d\psi(\omega).$$

Considering this equation with t fixed and $n>-t$, the integrand can be replaced by $\sigma(t-\omega)$, since this function agrees with it throughout the range except possibly at $\omega=t+n$, and at this terminal value $\psi(\omega)$ is continuous-on-the-left by R. Thus

$$s_n(t)=\int_0^{t+n} \sigma(t-\omega)d\psi(\omega), \qquad \text{(15)}$$

for each t and each positive integer $n>-t$.

From (**15**) we now prove that ψ is of bounded variation on $(0, \infty)$. If it were not so, there would be a succession of instants ω_i, starting with $\omega_0=0$, such that

$$\sum_{i=1}^{m} | \psi(\omega_i)-\psi(\omega_{i-1}) |\to\infty \text{ as } m\to\infty. \qquad \text{(16)}$$

Using these instants ω_i we construct a bounded left-continuous strain-history

$$\sigma(t)=\lambda_i \text{ in } -\omega_i<t\leqslant-\omega_{i-1} \qquad (i=1,\ 2,\ldots),$$

in which λ_i is 0 or ± 1 exactly as in (**9**); then, by (**15**),

$$\begin{aligned} s_n(0)&=\int_0^{n} \sigma(-\omega)d\psi(\omega)\\ &=\sum_{i=1}^{k} | \psi(\omega_i)-\psi(\omega_{i-1}) | +\sigma(-n)\{\psi(n)-\psi(\omega_k)\}, \end{aligned}$$

where k is the greatest integer i such that $\omega_i < n$; properly k should be denoted k_n, and $k_n \to \infty$ as $n \to \infty$. But the last term on the right is bounded, by R; so that, by (**16**),

$$s_n(0) \to \infty \qquad \text{as } n \to \infty.$$

This contradicts S; and so we must admit that ψ is of bounded variation on $(0, \infty)$.

We can now easily deduce that (**14**) holds for every bounded left-continuous strain-history $\sigma(t)$. In particular, the integral in (**14**) exists in the sense stated; for, if K is a bound of $|\sigma(t)|$,

$$\left| \int_T^{T'} \sigma(t-\omega) \mathrm{d}\psi(\omega) \right| \leqslant KV(\psi;\ T,\ T') \leqslant KV(\psi;\ T,\ \infty)$$

supposing $T' > T$; and this tends to zero as $T \to \infty$. Thus (**15**) gives, for each fixed t,

$$s_n(t) \to \int_0^\infty \sigma(t-\omega) \mathrm{d}\psi(\omega) \qquad \text{as } n \to \infty;$$

from which (**14**) follows by S.

V. References

Becker, R. (1925).—Elastische Nachwirkung und Plastizität. *Z. Phys.* **33**: 185.

Boltzmann, L. (1876).—Zur Theorie der elastischen Nachwirkung. *Ann. Phys., Lpz.* **7**: **624**.

Gross, B. (1947).—On creep and relaxation. *J. Appl. Phys.* **18**: 212.

Riesz, F., and Sz.-Nagy (1953).—"Leçons d'analyse fonctionnelle." (Acad. Sci. Hongrie: Budapest.)

Sips, R. (1950).—Mechanical behaviour of visco-elastic substances. *J. Polymer Sci.* **5**: 69.

Volterra, V. (1931).—"Theory of Functionals." Translation by M. Long. pp. 147, 188, 192, 217. (Blackie: London.)

Widder, D. V. (1941).—"The Laplace Transform." (Princeton Univ. Press.)