GROWING ELECTRIC SPACE–CHARGE WAVES

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Summary

A theory of growing space-charge electric waves, due to Pierce, Haeff, and others, is thought to explain the operation of Haeff's electron-wave tube and other amplifying devices and perhaps the origin of some solar radio emission. The theory is shown to be untenable, the growth predicted being spurious and due to misinterpretation of the dispersion equations.

Two rules are given which should be observed when interpreting dispersion equations :

(*a*) The frame of reference in which the dispersion equation is developed should be stationary in the gas in which the waves are propagated.

(*b*) When choosing real or imaginary parts of frequency or propagation constant for the dispersion equation, the choice must be consistent with physically realizable and relevant conditions.

An alternative theory of operation of the electron-wave and other growing-wave tubes is given and some design factors are briefly discussed.

I. INTRODUCTION

The results described in this paper may be applicable to any type of wave but refer specifically to longitudinal electron oscillations in a neutral ion-electron plasma. These are called space-charge, electron pressure, or plasma oscillations. They were first investigated as standing waves (Tonks and Langmuir 1929), but a more general theory of travelling space-charge waves is now available (Thomson and Thomson 1933 ; Bohm and Gross 1949*a*).

A theory of spontaneous " growth " of space-charge waves in interpenetrating electron streams or mixed ion and electron streams has been developed by Haeff (1948, 1949*a*, 1949*b*), Nergaard (1948), Pierce (1948, 1949, 1950), Pierce and Herbenstreit (1948), Bohm and Gross (1949*b*), Feinstein and Sen (1951), Rydbeck and Forsgren (1951), and others. The theory purports to show how the waves " grow " or steadily increase in amplitude as they propagate along the composite electron stream. It is believed to explain the operation of Haeff's electron-wave tube and (in varied forms) other growing-wave amplifying devices and perhaps also the origin of some solar radio emission.

It is the purpose of the present paper to show that the theory is not tenable, the growth predicted being spurious and due to misinterpretation of the dispersion equations. It is possible that the same criticism may apply to the theory of operation of the travelling-wave tube (Pierce 1950) although here the position

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is complicated by the presence of a metal helix in the electron stream. A similar criticism applies (Piddington 1955) to a theory of growing transverse electromagnetic (radio) waves advanced by Bailey (1948, 1950, 1951, and other references). These errors indicate the desirability of some rules to use in the interpretation of dispersion equations when dealing with waves in moving gas; two rules are suggested.

Since the electron-wave tube demonstrably *does* work, it behoves a critic of the theory to suggest an alternative mechanism. Possible modes of growth of space-charge waves are briefly discussed and a theory of the electron-wave tube, and perhaps other amplifying devices, is suggested.

II. The Substitution Analysis

The theory of the electron-wave tube depends on a substitution analysis in which certain types of solution are sought of the relevant equations: Maxwell's electromagnetic field relationships and the equations of motions of electrons in two or more interpenetrating streams having different drift velocities. The solutions have the form $\exp \mathrm{i}\,(\omega t-kx)$ and describe plane waves moving in time t and space x. In general both the frequency ω and propagation constant k may be complex quantities: $\omega=\omega_r+\mathrm{i}\omega_i$, $k=k_r+\mathrm{i}k_i$, where ω_r,ω_i, k_r,k_i are real. The wave is then given by $\exp\,(k_ix-\omega_it)\,.\,\exp\,\mathrm{i}(\omega_rt-k_rx)$ and so may grow or decay in both time and space. The result of the substitution analysis is a dispersion equation relating ω and k but not giving either directly. The derivation of this equation is, apart from occasional mathematical complexities, generally not difficult. It is in its interpretation that trouble has been met.

By equating real and imaginary parts, the dispersion equation provides two relationships between the four quantities concerned. Hence two of the quantities must be chosen more or less arbitrarily. The procedure from this point has been somewhat a matter of physical intuition guided by mathematical checks. In many cases, notably the magneto-ionic theory of radio waves due to Appleton, Hartree, and others, the results have proved eminently successful. However, when dealing with an electron gas which moves relative to the observer some serious errors have been made and definite rules of procedure are desirable.

In the magneto-ionic theory it is usual to assume $\omega_i=0$ and assign some particular value to ω_r. This means, physically, that the emitter, operating at a given frequency, is steady and stationary relative to the observer; otherwise the observed wave amplitude would change with time ($\omega_i=0$). Within one frequency range the value of k is then found to be complex with real and imaginary parts of opposite sign, indicating a wave which decays in space. The decay is associated with a scattering of electrons by collisions with heavy ions, atoms, or molecules, thus destroying their ordered motion. Within another frequency range k is imaginary, indicating stationary or reflected waves. These are described more fully below.

A similar procedure has been followed in the electron-wave tube theory but here values of k are found within certain wave bands which indicate wave *growth*. This is interpreted as indicating an increase in wave energy at the expense of the kinetic energy of the electron streams. Pierce (1950) has called it an electromechanical process. A somewhat similar theory is used to explain the operation of the travelling-wave tube (Pierce 1950).

Twiss (1951) has sensed a danger in this interpretation of the dispersion equation. For a double electron stream he has shown that if k (rather than ω) is assumed real, then the equation leads to a complex value of ω, which might be interpreted as a wave growing in time. Such growth is not observed experimentally, so that doubt is cast on the theory. He concludes that a theory of growing waves may only be developed in relation to the boundaries of the medium which are essential factors in promoting growth. It is shown below that neither Twiss's criticism of the theory nor his emphasis of the role of boundaries is justified.

However, there is a fundamental error in the theory of the electron-wave tube, due to misinterpretation of the dispersion equation. When correctly interpreted no real wave growth is found under any conditions. The main reason for the error is that the wave equations are derived in a system of axes moving relative to the electron gas.

III. The Moving Observer

Space-charge electric waves comprise perturbations of an electron gas together with an (electric) potential field due to the perturbations. They are propagated *relative to the gas itself* and, if this drifts relative to the observer, it carries the wave with it so that the latter assumes different apparent properties. It is easy to see that a steady, spatially attenuated wave might, in a suitable reference system, appear to grow in time. It is less obvious, but will now be shown, that other waves may appear to grow in space when no real growth is present.

To illustrate spurious wave growth, the simplest and best-known example of space-charge waves is chosen: waves in a gas whose electrons have thermal random motions but no mass drift. The dispersion equation (Thomson and Thomson 1933; Bohm and Gross 1949*a*) is

$$v_t^2k^2=\omega^2-\omega_0^2, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where v_t is of the order of the electron thermal velocity and ω_0 is the resonance frequency of the plasma, and collisions of electrons with heavy ions or atoms have been neglected. When $\omega>\omega_0$ loss-free travelling waves propagate with velocity $v_t\omega(\omega^2-\omega_0^2)^{-\frac{1}{2}}$. When $\omega<\omega_0$, k is imaginary and a pair of waves of the form $\exp k_ix\,.\,\exp i\omega_rt$ occur near the emitter. These are standing, exponentially spatially attenuated or "evanescent" waves and are, in effect, waves which are rejected by the medium and are being reflected back into the emitter. It is with these waves that we are concerned; a pair of them is shown schematically in Figure 1 (*a*).

At no frequency do any of these waves grow as they propagate. The only spatial intensity change is due to a process of reflection.

Now consider these same waves as seen by an observer moving relative to the gas with velocity U along the x axis. The waves are now described by the constants ω_1, k_1 given by the Lorentz transformation :

$$\omega=\beta(\omega_1+Uk_1) \quad \text{and} \quad k=\beta\left(k_1+\frac{U}{c^2}\omega_1\right), \quad \ldots\ldots\ldots \quad (2)$$

where $\beta=(1-U^2/c^2)^{-\frac{1}{2}}$. When $U \ll c$ and the wave velocity is of the order c or less, the Newtonian transformation

$$\omega=\omega_1+Uk_1, \qquad k=k_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

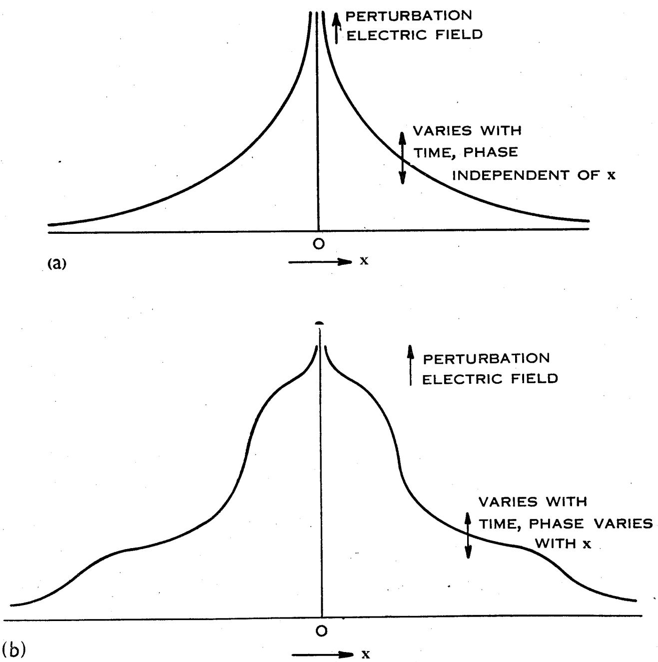

Fig. 1.—Schematic diagrams of pairs of evanescent waves radiated from the emitting surface $x=0$. (*a*) The emitter stationary in the gas : waves have the form $\exp k_i x . \exp \mathrm{i}\, \omega_r t$ (k_i positive or negative). (*b*) The emitter moving relative to the gas : waves have the form $\exp (k_i x-\omega_i t). \exp \mathrm{i}(\omega_r t-k_r x)$.

may be used. For simplicity this form is adopted in the present discussion, which mainly concerns low gas and wave velocities. This is quite justified, since Newtonian mechanics are assumed by Haeff and by the other authors referred to in Section I in determining electron motions. The Lorentz transformation leads to essentially similar, but more complicated, results. The new dispersion equation, found by combining equations (**1**) and (**3**), is

$$k_1=\frac{U\omega_1}{v_t^2-U^2}\left[1\pm\left\{1-\frac{(v_t^2-U^2)(\omega_0^2-\omega_1^2)}{U^2\omega_1^2}\right\}^{\frac{1}{2}}\right]. \quad \ldots . \quad (4)$$

When $v_t^2 > U^2$ this equation yields complex values of k_1 for real values of ω_1 in the frequency range

$$\omega_1 < \omega_0\left(1-\frac{U^2}{v_t^2}\right)^{\frac{1}{2}}.$$

Hence, if it were interpreted in the manner of the electron-wave tube theory, it would show travelling waves, one of which grows in space, while remaining steady in time.

This wave growth is spurious ; it does not correspond to an increase of wave energy at the expense of the electron kinetic energy but rather to a process of reflection. As would be expected, the frequency range within which the effect occurs is the same, except for a Doppler shift effect, as that in which a stationary observer sees ordinary evanescent waves.

IV. Travelling Evanescent Waves

It is desirable to consider in more detail the physical nature of the " growing " waves described by equation (**4**). Later they will be compared with the waves of the electron-wave tube theory. In this theory the emitter moves relative to the gas and it is convenient to make this assumption here. However, contrary to the travelling-wave tube theory, we initially choose an observer stationary in the gas so that equation (**1**) is applicable.* Since the observer now moves relative to the (steady) emitter he sees a wave whose intensity changes with time so that ω is complex.

Equation (**1**) may be rewritten

$$v_t^2(k_r+\mathrm{i}k_i)^2=(\omega_r+\mathrm{i}\omega_i)^2-\omega_0^2,$$

and when real and imaginary parts are equated separately :

$$v_t^2(k_r^2-k_i^2)=\omega_r^2-\omega_i^2-\omega_0^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

$$v_t^2k_rk_i=\omega_r\omega_i. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Solutions are required of the form $\omega_i=Uk_i$ so that equation (**6**) reduces to

$$v_t^2k_r=U\omega_r, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

and equation (**5**) to

$$k_i^2(v_t^2-U^2)=\omega_0^2-\omega_r^2(1-U^2/v_t^2). \quad \ldots\ldots\ldots\ldots \quad (8)$$

Since all quantities are assumed real, necessary conditions for this form of solution are

$$v_t^2>U^2 \quad \text{and} \quad \omega_r^2<\omega_0^2(1-U^2/v_t^2).$$

We now have the wave fully specified in terms of the (assumed) quantities ω_r and ω_i.

* Clearly the form of the dispersion equation for any given set of circumstances is determined by the movement of the *observer* relative to the gas and is not affected by movement of the *emitter*. Having found the dispersion equation the movement of the emitter is taken account of when more or less arbitrarily choosing two of the four variables ω_r, ω_i, k_r, and k_i. This choice is discussed in Section VI.

The wave is an evanescent wave being reflected back into the emitter. It differs from the stationary evanescent waves illustrated in Figure 1 (*a*) because the emitter is moving. This has two effects: since each successive (in time) intensity maximum occurs in a different place the wave is changed to a travelling wave and also, due to the changing separation of emitter and observer, it grows or decays in time. It may be regarded as a wave packet travelling with *group* velocity $\omega_i/k_i=U$.

The type of wave in which we are particularly interested grows in space while remaining steady in time. Our travelling evanescent wave acquires this property when viewed by an observer moving with velocity U. From equation (3) we determine its parameters* as:

$$\begin{aligned}\omega_1&=\omega_r+\mathrm{i}\omega_i-U(k_r+\mathrm{i}k_i)\\&=\omega_r+Uk_r,\\k_1&=k_r+\mathrm{i}k_i.\end{aligned}$$

Thus the wave has the form:

$$\exp\left(\frac{\omega_0^2}{v_t^2-U^2}-\frac{\omega_r^2}{v_t^2}\right)^{\frac{1}{2}}x\ .\ \exp\mathrm{i}\left\{\left(1+\frac{U^2}{v_t^2}\right)\omega_r t-\frac{U\omega_r}{v_t^2}\ .\ x\right\}.$$

If the rate of growth is not too rapid this wave has a phase velocity

$$\sim\frac{v_t^2+U^2}{U}.$$

A travelling evanescent wave is illustrated schematically in Figure 1 (*b*). For numerical examples we might choose $\omega_r=10^8$ rad sec^{-1}, $v_t=10^7$ cm sec^{-1}, and $U=10^6$ cm sec^{-1}, and two values of ω_0, say 10^8 and 10^9. The two waves are approximately:

$$\exp x\ .\ \exp\mathrm{i}(10^8t-x),$$

and

$$\exp 100x\ .\ \exp\mathrm{i}(10^8t-x).$$

V. The Double Electron Stream

The theory of the electron-wave tube is most fully developed for the case of a double electron stream, the velocities and resonant frequencies of the individual streams being v_a, v_b, ω_a, and ω_b. Haeff (1949*b*) and others have shown that the dispersion equation is

$$\frac{\omega_a^2}{(\omega-v_ak)^2}+\frac{\omega_b^2}{(\omega-v_bk)^2}=1. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (9)$$

They show that within certain, rather wide limits k has complex values when ω is real and that some of these correspond to waves whose amplitude is increasing

* It will be seen that the transformation of a given wave from one system of axes to another may be effected in either of two ways. The dispersion equation for the new system may be found by a Newtonian transformation and the wave then determined by a suitable substitution of two of the four variables, ω_r, ω_i, k_r, k_i, in this equation. Alternatively the wave parameters, ω, k, may be transformed directly to the new parameters, ω_1, k_1.

in the direction of propagation. Maximum amplification occurs when $\omega_a=\omega_b$ or when $\omega_a/v_a=\omega_b/v_b$.

As shown above, apparent growth of a wave (complex values of k) may be seen by an observer moving relative to gas in which an evanescent wave is present. The observer for whom the above dispersion equation was derived is moving relative to the gas and so is likely to see such spurious growing waves. To see whether real growing waves are present the dispersion equation for an observer sharing the " mean " velocity of the gas must be derived. It is not, in general, clear what is the " mean " velocity but when the electron streams have equal density ($\omega_a=\omega_b=\omega_0$) then clearly the mean velocity is $\frac{1}{2}(v_a+v_b)$. On transforming to a system with this velocity, equation (9) becomes

$$\frac{\omega_0{}^2}{(\omega_1-vk_1)^2}+\frac{\omega_0{}^2}{(\omega_1+vk_1)^2}=1, \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

where $v=\frac{1}{2}(v_b-v_a)$. The same result is obtained, of course, by initially assuming two electron streams with velocities $\pm v$. If ω_1 is now assumed real, the values of k_1 are given by

$$v^2k_1{}^2=\omega_1{}^2+\omega_0{}^2\pm(4\omega_1{}^2\omega_0{}^2+\omega_0{}^4)^{\frac{1}{2}}. \quad \ldots\ldots\ldots\ldots \quad (11)$$

The upper sign corresponds to travelling waves which do not grow or decay and need not be considered further. The lower corresponds, when $\omega_1{}^2<2\omega_0{}^2$, to imaginary values of k_1 indicating waves of the form $\exp k_ix \,.\, \exp i\omega_rt$. These are evanescent waves of the type illustrated in Figure 1 (*a*). At no frequency do they grow or decay, so that at no frequency can any electromechanical process occur of the sort envisaged in the electron-wave tube theory.

In the more general case when $\omega_a\neq\omega_b$ it is not clear what is the " mean " velocity of the gas. However, Feinstein and Sen (1951), in their analysis of the two-beam dispersion equation, have shown that amplification (that is, complex values of k) cannot occur when $v_a=-v_b$. This means that if the observer assumes the simple arithmetic mean velocity $\frac{1}{2}(v_a+v_b)$ of the two streams he can never see growing waves. There may be other velocities in which he sees no apparent growth but it is sufficient for our purposes that there should be one. In addition it should be remembered that the case we have analysed ($\omega_a=\omega_b$) is one for which, according to Haeff, growth is strongest.

The real nature of the waves whose growth is predicted by the theory concerned is now fairly clear. They are derived in the same way as the spurious growing waves of Sections III and IV. Starting with a stationary observer, frequency bands are chosen so that evanescent waves are emitted. Observer and emitter are then given a drift velocity and the waves change to a type of travelling wave (since ω_r and k_r are both finite) which is still strongly exponentially attenuated. Of the two modified evanescent waves one then appears to grow and one to decay in space. However, there is no real growth, the observed spatial change in intensity being due to a process of reflection and the wave travel to the fact that the gas carries the wave past the observer.

It is concluded that wave growth in the electron-wave tube by the process hitherto invoked cannot occur. Doubts are raised regarding the somewhat similar theory of the travelling-wave tube and other like electronic devices.

VI. Interpretation of the Dispersion Equation

It would seem that when studying space-charge wave properties that a system of axes should be chosen stationary in the gas. However, even when this is done there may still remain difficulties in the interpretation of the dispersion equation since this does not give either ω or k directly. Two of the four quantities, ω_r, ω_i, k_r, k_i, may be chosen arbitrarily but *the choice must be consistent with physically realizable and relevant conditions.*

The choice really amounts to specifying the method of introduction of the wave into the medium. For example, consider a dispersion equation which is satisfied by complex values of k for real values of ω or alternatively by complex values of ω for real values of k. Both interpretations are reasonable. The first means that the wave is introduced by a steady emitter fixed relative to the observer; the wave then grows (or decays) in space but not in time. The second interpretation is not as obvious but is physically realizable. The wave has the form $\exp -\omega_i t \, . \exp \mathrm{i}(\omega_r t - k_r x)$ and so grows (if ω_i is negative) in time. Assume that prior to $t=0$ the properties of the medium causing wave growth were non-existent so that the wave was propagated with uniform intensity. When the wave emitted by a distant radiator had permeated the whole gas the growth would then commence. The wave would then appear to grow in time since wave crests passing the observer at later times would have travelled further and so have grown more. An alternative, and perhaps more realistic, way of introducing this wave would be by an emitter whose output increased with time at a rate sufficient to compensate for the growth in space. Again the medium would be permeated by a wave of uniform amplitude which grew in time; again the growth would really be a spatial growth. Thus the alternative interpretations of the dispersion equation are self-consistent, indicating a medium in which the waves grow as they propagate. This would appear to answer Twiss's (1951) main criticism of the electron-wave tube theory.

Not all dispersion equations are as unambiguous as that discussed above. The case of the double electron stream with the observer stationary in the gas raises a difficulty. When ω_1 is assumed real the equation takes the form given in (**11**) and the waves are those discussed above. The alternative assumption that k_1 is real requires consideration; the dispersion equation may be written as

$$\omega_1^2 = \omega_0^2[1 + v^2k_1^2/\omega_0^2 \pm (1 + 4v^2k_1^2/\omega_0^2)^{\frac{1}{2}}]. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{12})$$

When $v^2k_1^2 > 2\omega_0^2$ both waves are travelling waves of constant amplitude. When $v^2k_1^2 < 2\omega_0^2$ the lower sign results in imaginary values of ω_1 so that the waves have the form $\exp -\omega_i t \, . \exp -\mathrm{i}k_r x$. This is a new type of wave: a non-oscillatory standing wave. It comprises a spatial distribution of electric charge varying sinusoidally throughout the whole of the medium. No oscillations occur, only exponential growth or decay (in time) of the whole pattern.

Bohm and Gross (1949*b*) have used Boltzmann's equation instead of Maxwell's momentum transfer equation to derive the dispersion equation for a pair of equal-density electron beams. One pair of roots of their equation is

$$\omega_1^2 \simeq -Ak_1^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{13})$$

where A is a real, positive quantity. This equation is similar to (**12**) (the lower sign chosen) except that ω_1 is imaginary for *all* real values of k_1, that is, for all wavelengths. Bohm and Gross conclude that this dispersion equation indicates an instability of the system and give a " physical reason " for this instability.

The waves might be regarded as limiting cases of travelling growing (or decaying) waves having the form $\exp -\omega_i t \,.\, \exp i(\omega_r t - k_r x)$. As shown above the dispersion equation for such waves may be interpreted by assuming either ω_1 or k_1 real, corresponding to two different assumed methods of injection each of which is physically realizable. In the limiting case when $\omega_r \rightarrow 0$ the wave velocity approaches zero and the wave grows without appreciable movement. There are two objections to the conclusion of Bohm and Gross that the limiting case indicates instability. First, the waves seem physically unrealizable: the limit of zero wave velocity is also a limit of infinite difficulty in injecting the waves. They cannot be propagated into the medium nor instantaneously brought into existence with uniform intensity throughout the medium. In the case of travelling growing waves the alternative assumptions concerning the dispersion equation correspond to alternative (assumed) injection mechanisms but lead to identical wave growths. This raises the second objection to the interpretation of Bohm and Gross: when ω_1 is made real, equations (**12**) and (**13**) provide no evidence of wave growth or medium instability. On the contrary they yield, in the frequency bands concerned, ordinary evanescent waves. Such waves are physically realizable and so provide a satisfactory explanation of the equations. The alternative explanation appears untenable.

The " physical explanation " of the instability is at first sight suggestive but on closer inspection is quite unconvincing. It is not reasonable to assume, as Bohm and Gross have done, that the two beams are velocity modulated (by an initial small disturbance) more or less independently of one another and that the effect is fed back in an amplified form. The electron distribution in each beam contributes equally to a total electric field which, in turn, determines changes in the electron distribution in each beam.

VII. Real Growing Waves

If the mechanism of wave growth of Haeff, Pierce, and other authors is abandoned it is desirable to find an alternative. The proposal depends on electron thermal motions which have been neglected in most of the analyses mentioned but have been included in that of Bohm and Gross (1949*a*, 1949*b*). Before discussing the mechanism of growth it is desirable to mention other effects of thermal motion.

When neither thermal nor drift electron motions are present, then travelling space-charge waves are not possible; the medium may only support stationary oscillations of frequency ω_0. With the introduction of thermal motions a dispersion equation of the form given by equation (**1**) results, and travelling waves are supported at frequencies above ω_0. As seen above, an additional drift motion of the electrons may then cause spurious growing waves to appear at frequencies below ω_0. If differential drift, instead of thermal, motions are introduced then travelling waves are also possible so that the differential drift serves the same purpose as the thermal motions : it diffuses the stationary oscillations and turns them into travelling waves.

The discussion of the previous sections seems to suggest that mass drifts of electrons are never, in themselves, sufficient to cause real wave growth. They may be important, however, in modifying the dispersion equation so as to take better advantage of some primary growth mechanism. There is one such general mechanism which appeals on physically intuitive grounds and may well be the only mechanism capable of causing growth. It depends on the trapping of electrons between potential troughs of the wave and on the extraction of the electron kinetic energy by the wave. The simplest way of extracting the energy is for the wave to slow down, that is, for the phase velocity to decrease. Such an effect requires a medium of variable density and is not likely to be important in the case of the electron-wave tube, although it may play a part in the operation of certain other tubes and is worth considering as a method of amplification.

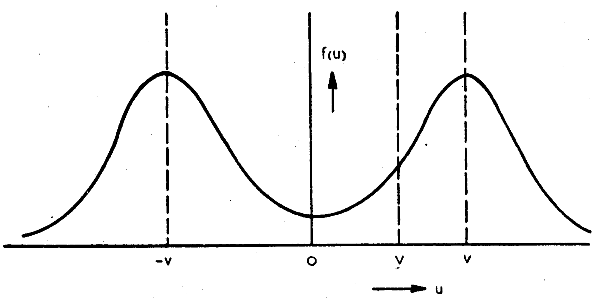

Fig. 2.—A hypothetical velocity distribution in a two-beam electron-wave tube. Velocities are measured relative to the mean electron velocity.

There is another way of extracting the energy, which depends on collisions between electrons, a factor not considered in the simple electron-wave tube theory. It requires that there be electrons close to the phase velocity of the wave ; those within a narrow range of velocities will be captured by the wave and oscillate between potential troughs. After a time they will suffer a collision and be scattered back into the statistically steady velocity distribution $f(u)$ of all the electrons. Electrons with initial velocities slightly above the wave velocity will, on an average, lose energy to the wave. Those with velocities slightly below will gain energy from the wave. If there are more of the former than the latter the wave will gain energy and grow, the criterion for growth being $\partial f(u)/\partial u > 0$. The process has been discussed in some detail by Bohm and Gross (1949*b*).*

The possibility of this method of wave growth in the electron-wave tube may be seen by reference to Figure 2. This shows a hypothetical electron

* They express the growth as an imaginary part of ω. For the present purposes at least, it would be better to express it as an imaginary part of k. This is only a matter of interpreting the dispersion equation appropriately.

velocity distribution relative to the " mean " velocity of all the electrons. The two beams are spread by thermal and other effects but have peaks at velocities $\pm v$. Let the wave velocity be denoted by V; it is given by the possible values of ω_1/k_1 in equation (**10**). Thus we have

$$V = v\{1 + X \pm (4X + X^2)\}^{-\frac{1}{2}}, \qquad \text{(14)}$$

where $X = \omega_0^2/\omega_1^2$. Obviously then values of V may occur lying between zero and v and so we have $\partial f(u)/\partial u > 0$ when $u = V$, as shown in the diagram. It will also be clear that the rate of capture of electrons, and hence of wave growth, will be proportional to the intensity of the wave so that the growth is logarithmic as observed.

It is of interest to consider this method of growth for the single-beam electron-wave tube developed by Haeff (1949*b*) and sometimes called the " whistle " tube. The theory of operation of this tube developed by Haeff, along the same lines as for the double-stream tube, has been criticized by Pierce

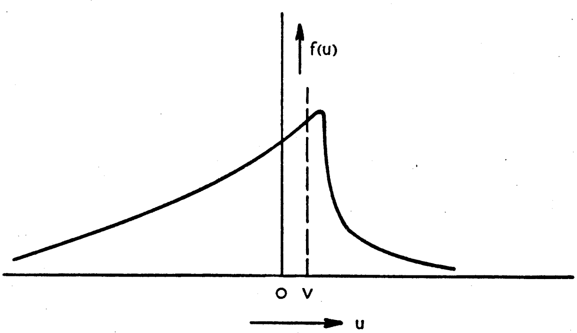

Fig. 3.—A hypothetical velocity distribution (about the mean) in a single-beam electron-wave tube.

(1950, Ch. 16) who concluded that the fact that the tube worked at all was " something of a mystery ". The tube would work by the new mechanism, provided the electron velocity distribution (about the " mean " velocity) were asymmetrical, as shown in Figure 3. It is then possible for $f(u)$ to have a positive slope in regions corresponding to the velocity of real travelling waves (V).

Finally, and tentatively, the mechanism is suggested as applying to Pierce's " travelling-wave tube ". The same requirement concerning $f(u)$ is necessary and the function of the helix is to slow down the wave, presumably close to or even below the mean electron velocity, so that the requirement is satisfied.

VIII. Design of Amplifying Tubes

If the mechanism of space-charge wave growth just described is accepted, there are several rather obvious factors which should be considered when designing actual amplifying devices.

In the first place, the energy gain of the wave increases with the slope of the $f(u)$ curve at the point where $u = V$. The slope should, therefore, be made as steep as possible. The use of two separate beams allows steep slopes to be attained, particularly if the random electron velocities are kept small.

There is a second reason why low electron thermal velocities may be necessary. As Bohm and Gross (1949*a*) have shown, organized oscillations cannot exist at all when the wave phase velocity (ω/k) is below mean thermal velocities. The wave velocity is confined within certain limits by the $f(u)$ distribution so that thermal velocities must be kept within closer limits.*

A second factor promoting growth is the frequency of collisions of electrons with electrons. This should be as high as possible so that the rate of circulation of electrons between the trapped and untrapped conditions is a maximum. Once again a reduction in thermal motions would probably help since the effect of higher electron speed on collision frequency is more than offset by the change in collision cross section. The effect of electron collisions with heavy ions or atoms is to damp the wave by a scattering process.

A desirable factor in addition to gain is bandwidth and these requirements will inevitably clash with one another. Thus change of frequency will change X in equation (**14**) and hence V. A large bandwidth requires a large velocity range over which $f(u)$ is large and positive. The combination of a steep and long slope of the $f(u)$ curve implies a large value of total electron density and this in turn is limited by the minimum frequency to be used.

It seems likely that the general form of the dispersion equation for several beams of different densities will be

$$\omega/k=F(v_1, v_2, \ldots, X_1, X_2, \ldots). \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{15})$$

By similar choice of the various parameters an optimum balance between the different requirements may be effected. This illustrates the importance of introducing a second electron beam (or alternatively, perhaps, a helix); in effect it provides another degree of freedom which can be used to obtain optimum results.

Finally, the possibility suggests itself of using a wave whose velocity (relative to the mean electron velocity) decreases as it propagates. This would allow a large proportion of the kinetic energy of the electrons (within the range of variation of V) to be abstracted. A method of achieving this effect might be by using a magnetic field which is not constant in direction. This would not only change the X terms in equation (**15**) but introduce fresh terms depending on the strength and direction of the magnetic field.

IX. Conclusions

(1) The current theory of the electron-wave tube is untenable; the wave growth predicted is spurious, owing to the movement of the observer relative to the gas which carries the wave. The criticism applies to some other theories of growing waves including, perhaps, that of the travelling-wave tube.

* The converse argument is that, having a certain, inevitable spread of thermal velocities the spread of *ordered* velocities must be greater than a certain minimum value if coherent amplified waves are to occur. This may explain the observed wide spread of ordered velocities in practical amplifying devices.

(2) Theories of the type concerned may be correctly developed without reference to boundaries, although if these are present their effects must be included as additional factors.

(3) Two rules must be observed in using the method of substitution analysis to study waves in gases. First, the frame of reference in which the equations are developed should be stationary relative to the gas. Second, when (more or less arbitrarily) choosing real or imaginary parts of frequency or propagation constant in the dispersion equation, the choice must be consistent with physically realizable and relevant conditions.

(4) The only known method of growth of space-charge waves is by the trapped electron mechanism, by which electrons are trapped between potential troughs and their kinetic energy abstracted. This seems the most likely explanation of the amplification which undoubtedly does occur in the electron-wave and other growing-wave tubes.

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