THE SCATTERING OF 121 keV ELECTRONS AND POSITRONS

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Summary

The angular distribution of the single scattering of 121 keV positrons in gold and of 121 keV electrons and positrons in argon is calculated for an exponentially screened field. The single scattering distributions are integrated numerically to give mean square angles of multiple scattering, and hence the percentage difference in the r.m.s. angles of multiple scattering of electrons and positrons.

I. Introduction

Calculations have previously been carried out by Mohr and Tassie (1954*a*)† of the effect of screening on the elastic single scattering of electrons by gold, and these results were applied to the problem of multiple scattering (Mohr and Tassie 1954*b*)‡. In the latter work, the single scattering of 121 keV positrons by gold was also calculated making approximate estimates for the phase shifts. This calculation of positron scattering is repeated using more accurate phase shifts. The single scattering distribution is then integrated numerically to give the mean square angle of multiple scattering, and the difference in the multiple scattering of electrons and positrons obtained by comparison with the results for electron scattering (paper B).

As the available experimental results are for light elements, a similar calculation is carried out for scattering by argon. Although experimental results are for somewhat higher energies, 150 keV upwards, the calculation is made for incident particles of energy 121 keV, the same energy as in the calculation for gold. Any effects of screening by the atomic electrons would be expected to be smaller for higher energies, and any effects of the electron spin to be larger. The Hartree field of argon (Hartree and Hartree 1938) was approximated by a one-term field (**5**). Although a one-term field is a good approximation to the Hartree field of gold, in the case of argon additional terms of the form $br \exp(-\beta r)$ are really necessary (Ibers and Hoerni 1954). However, the use of a one-term field greatly simplifies the calculation and is sufficient to determine the nature of the effects of screening on the scattering.

II. General Theory

The differential cross section for the scattering by a potential $V(r)$ is given by (Mott and Massey 1949)

$$I(\theta)=|f|^2+|g|^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{1})$$

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† Hereafter referred to as paper A.

‡ Hereafter referred to as paper B.

where

$$2ikf(\theta)=\Sigma[(l+1)\{\exp(2i\eta_l)-1\}+l\{\exp(2i\eta_{-l-1})-1\}]P_l(\cos\theta),$$
$$2ikg(\theta)=\Sigma\{\exp(2i\eta_{-l-1})-\exp(2i\eta_l)\}P_l^1(\cos\theta), \quad \ldots\ldots\ldots\ldots \quad (2)$$

$k^2=4\pi^2(W^2-m_0^2c^4)/h^2c^2$, W being the total energy. η_l is the phase shift such that $\sin(kr-\frac{1}{2}l\pi+\eta_l)$ is the asymptotic form of that solution G_l of the equation

$$G_l''+\{k^2-l(l+1)r^{-2}-U_l(r)\}G_l=0, \quad \ldots\ldots\ldots\ldots \quad (3)$$

which vanishes at the origin.

U_l is the modified Dirac potential

$$U_l(r)=8\pi^2WV/h^2c^2-4\pi^2V^2/h^2c^2-(l+1)\alpha'/\alpha r+3\alpha'^2/4\alpha^2-\alpha''/2\alpha, \quad \ldots \quad (4)$$

where $\alpha=2\pi(W-V+m_0c^2)/hc$.

An exponentially screened field was used for the atomic field V

$$V(r)=Z_p\varepsilon^2/r,$$
$$Z_p=Z\exp(-\chi r/a_0), \quad \ldots\ldots\ldots\ldots \quad (5)$$

where $a_0=h^2/4\pi^2m_0\varepsilon^2$. The same field was used for gold as in paper A, namely, (5), with $\chi=3$. The Hartree field of argon (Hartree and Hartree 1938) was

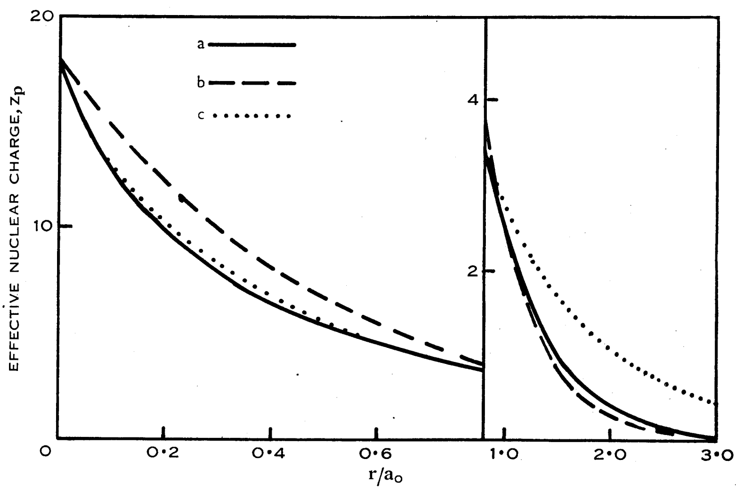

Fig. 1.—Variation with distance of the effective nuclear charge Z_p for argon. *a*, Hartree field; *b*, one-term field, equation (5); *c*, Molière's three-term field approximating to the Thomas-Fermi field.

fitted approximately by (5) with $\chi=2$. A comparison of this field with the Hartree field and with the Thomas-Fermi field as used by Molière (1947) is given in Figure 1.

III. Determination of the Phase Shifts

The same procedure was used for computing the phase shifts as in paper A. The phase shifts η_l^0 for only the first term in the effective potential (4) (i.e. the Schroedinger phase shifts) were first obtained and then corrected for the effects of spin as given by further terms in the effective potential. The region of validity of the approximations used has been previously established by comparison with numerical integration for the scattering of electrons by gold, and checks by numerical integration and by the WKB method were carried out in only a few cases. The K_0 approximation,

$$\eta_l^0 = (\gamma Z/ka_0)K_0\{\chi(l+\tfrac{1}{2})/ka_0\}, \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

where

$$\gamma = (1-v^2/c^2)^{-1}$$

was used with the correction due to Pais (1946) for $l \geqslant 8$. For $l<8$, the η_l^0 were obtained for the scattering of electrons by argon by comparison with the phases for a Coulomb field and the difference in the $l=1$ phase for electrons and positrons, $0 \cdot 006$, was determined by the WKB method. For the scattering of positrons by gold, the zero-order phase was obtained by numerical integration and the other phases by interpolation, a procedure which was checked by calculating η_5^0 by the WKB method.

The effect of spin on the phases, $\eta_l - \eta_l^0$, was obtained from the effect of spin on the Coulomb phases, corrected for screening for argon by an approximate analytical correction (Appendix I). For gold this spin effect was obtained by the WKB method for $l=0$, 5. The modification of the spin effect by screening is small, e.g. the modification is largest for $l=0$ for gold where $\eta_0 - \eta_0^0 = 0 \cdot 181$ for the Coulomb field, and $0 \cdot 148$ for the screened field, and any error introduced by the use of the above procedure is therefore small.

IV. Angular Distribution of Single Scattering

The Mott series (**2**) was summed for large angles ($\theta \geqslant 30°$) using the same procedure as in paper A, and for small angles using method (*a*) of paper B. For the small angle scattering of positrons by gold, the difference in the series (**2**) for electrons and positrons was summed, and the results of paper A for the summation for electrons were used.

The results obtained for the scattered intensity are shown in Figures 2 and 3 as the ratio R to the relativistic Rutherford scattering $Z^2\varepsilon^4 \operatorname{cosec}^4 \frac{1}{2}\theta/4\gamma^2 m_0{}^2 v^4$. The values of R for the Coulomb field of mercury are taken from Massey (1942). For light elements Curr (1955) gives R as a power series in $\alpha = Z/137$ and $q=\alpha/\beta$. For the scattering by argon at 121 keV, $\alpha = 0 \cdot 13$ and $q = 0 \cdot 22$, and the power series converges rapidly.

For large angles the scattering by the screened field is the same within the accuracy of the calculation as the scattering by a Coulomb field. The ratio of the scattered intensities for electrons and positrons is approximately the same for the screened and Coulomb fields of argon. However, for gold this ratio is larger for scattering by the screened field than by the Coulomb field.

The results for the scattering through small angles are compared in Figure 3 with the results of paper B for the scattering of 121 keV electrons by gold and with the Molière single scattering distributions. Molière (1947) gives two single scattering distributions. One is obtained from an empirical interpolation formula which interpolates between the Born approximation and the classical approximation. This is labelled M' in Figure 3. The other is a more approximate distribution of simpler form,

$$R=\theta^4/(\theta^2_{\mathrm{min.}}+\theta^2)^2, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

where $\theta_{\mathrm{min.}}$ is determined by comparison with his more accurate distribution. This is labelled M in Figure 3.

For argon, the scattering distributions given by the Born approximation for the one-term field, for the Hartree field (Hartree and Hartree 1938), and for the Thomas-Fermi field in the form used by Molière (1947) are also shown. This indicates that the difference between the results calculated here and the Molière

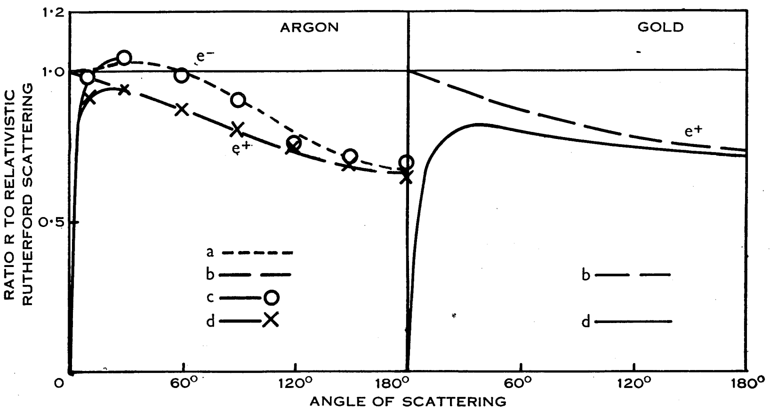

Fig. 2.—Angular distribution of single scattering of 121 keV electrons and positrons. *a*, Electrons, Coulomb field; *b*, positrons, Coulomb field; *c*, electrons, screened field; *d*, positrons, screened field.

scattering distributions is largely due to the difference in the fields used. The one-term field is a rather poor approximation to the atomic field of argon and the Thomas-Fermi field used by Molière is more appropriate for $\theta>1°$.

V. Multiple Scattering

The mean square angle of multiple scattering is given by

$$\theta^2_{\mathrm{r.m.s.}}\equiv\overline{\theta^2}=\int_0^{\theta_{\mathrm{max.}}}\theta^2P(\theta)\mathrm{d}\theta, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

where $P(\theta)=2\pi NtI(\theta)\sin\theta$; N is the number of atoms per unit volume and t is the thickness of scattering material.

Values of $\theta_{\mathrm{r.m.s.}}$ were obtained by numerical integration of (8) using the single scattering distributions for electrons and positrons. This allows the determination of the difference in $\theta_{\mathrm{r.m.s.}}$ for positrons and electrons as a function of $\theta_{\mathrm{r.m.s.}}$ for electrons. The results for this are shown in Figure 4 as $100(\theta^-_{\mathrm{r.m.s.}}-\theta^+_{\mathrm{r.m.s.}})/\theta^-_{\mathrm{r.m.s.}}$. As would be expected, this difference is greater for gold than for argon.

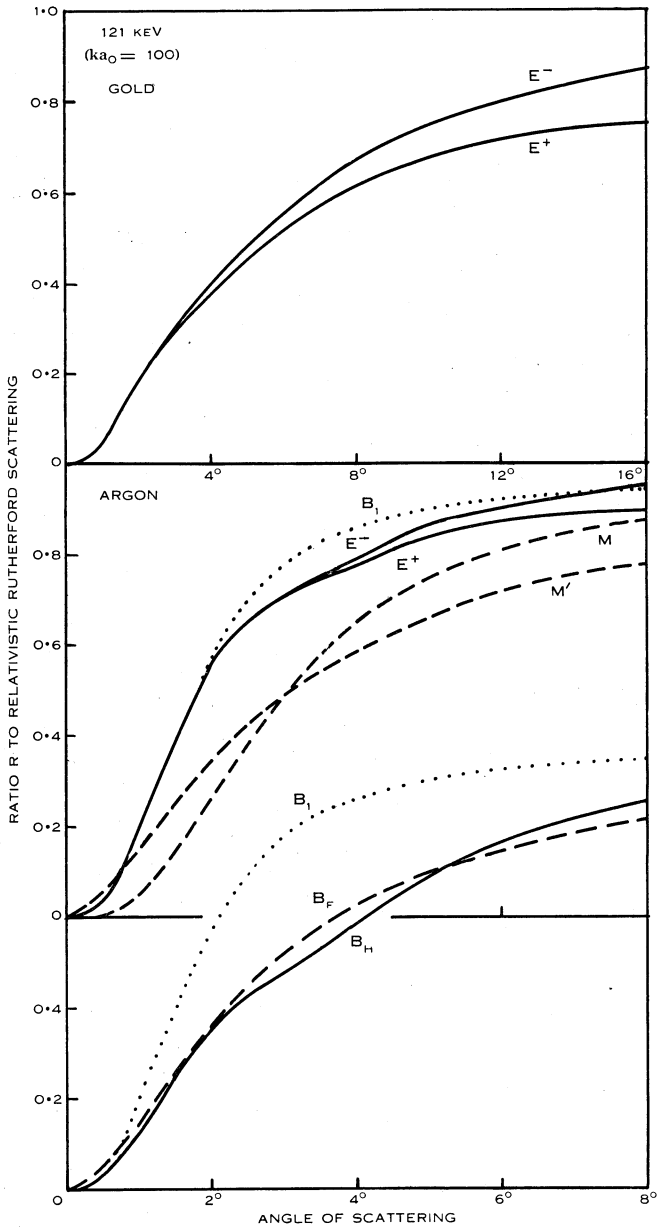

Fig. 3.—Angular distribution of single scattering of 121 keV electrons and positrons. B_1, Born approximation with one-term field, equation (**5**); B_H, Born approximation with Hartree field; B_F, Born approximation with Thomas-Fermi field. *M*, Calculated from the simplified Molière theory for application to multiple scattering, equation (**7**) (Molière 1947, equations (**9.1**) and (**9.3**)); *M′*, calculated from Molière's more accurate theory (Molière 1947, equation (**8.6**)). E^-, E^+, calculated using the exact theory with the one-term field for electrons and positrons respectively; E^- for gold is taken from Mohr and Tassie (1954*b*).

The results obtained here are in qualitative agreement with previous calculations of paper B and of Mohr (1954), in predicting that the percentage difference in $\theta^{+}_{\text{r.m.s.}}$ and $\theta^{-}_{\text{r.m.s.}}$ is small. The latter calculation using the second Born approximation gives for an energy of 1 MeV and $\theta_{\text{r.m.s.}}=20°$ a difference of $1\cdot7$ per cent. for $Z/137=0\cdot15$ and 5 per cent. for $Z/137=0\cdot58$, a difference which increases as the energy decreases. This would seem to indicate that $(\theta^{-}_{\text{r.m.s.}}-\theta^{+}_{\text{r.m.s.}})/\theta_{\text{r.m.s.}}$ has a maximum value somewhere between 1 MeV and 121 keV. However, Mohr uses the theory of Molière (1948) to obtain a mean value of $\theta_{\text{r.m.s.}}=(\theta^{-}_{\text{r.m.s.}}+\theta^{+}_{\text{r.m.s.}})/2$, a procedure which cannot safely be used for the present case.

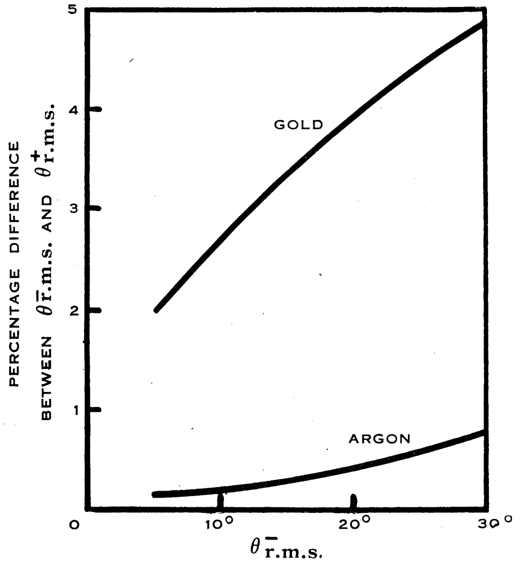

Fig. 4.—The percentage difference between the root mean square angle of multiple scattering of 121 keV electrons and positrons as a function of the r.m.s. angle for electrons, for scattering by gold and by argon.

The small values of $(\theta^{-}_{\text{r.m.s.}}-\theta^{+}_{\text{r.m.s.}})/\theta_{\text{r.m.s.}}$ predicted by theory are in disagreement with the experimental results of Groetzinger, Humphrey, and Ribe (1951), who report a difference of about 10 per cent. in $\theta^{-}_{\text{r.m.s.}}$ and $\theta^{+}_{\text{r.m.s.}}$ for argon. However, the later and more accurate experiments of Cusack and Stott (1955) indicate no significant difference in the multiple scattering of electrons and positrons in argon.

VI. Acknowledgments

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Appendix I

Effect of Spin on the Phase Shifts

We require

$$S_l=\eta_l-\eta_l^0, \quad S_{-l-1}=\eta_{-l-1}-\eta_l^0$$

for a field of the form (**5**). This can be obtained by applying the WKB method to the modified Dirac potential U_l, equation (**4**), and assuming U_l small compared with $k^2a_0^2$. The result of this procedure is then corrected to give the exact Coulomb field values of S as $\chi\to 0$, giving the final result :

$$S_l=P_1(l)C_l^{(1)}+P_2(l)C_l^{(2)}+C_l^{(3)},$$
$$S_{-l-1}=P_1(l)C_{-l-1}^{(1)}+P_2(l)C_{-l-1}^{(2)}+C_{-l-1}^{(3)},$$

where

$$C_{-l-1}^{(1)}=\tfrac{1}{2}\pi(l-\rho_l),$$
$$C_{-l-1}^{(3)}=\arg\Gamma(\rho_l+1-iq)-\arg\Gamma(l+1-iq),$$

($C_l^{(1)}$ and $C_l^{(3)}$ are obtained by replacing l by $l+1$) ;

$$-C_l^{(2)}=\arctan\{q/(l+1)\}-\tfrac{1}{2}\arctan(q/\rho_{-l-1})-\tfrac{1}{2}\arctan\{q'/(l+1)\},$$
$$C_{-l-1}^{(2)}=\tfrac{1}{2}\arctan(q/\rho_l)-\tfrac{1}{2}\arctan(q'/l),$$
$$\rho_l=(l^2-\alpha^2)^{\frac{1}{2}},\quad \alpha=Z/137,\quad q=\alpha/\beta,\quad q'=(1-\beta^2)^{\frac{1}{2}};$$
$$P_1^{(l)}=\{1-2\pi^{-1}(-2b\ln 2b+2\cdot 2320b+8b^3/3)\},$$
$$P_2^{(l)}=\{1+b^2\ln b-0\cdot 3508b^2-\tfrac{1}{4}b^4\},$$

where $b=\chi(l+\frac{1}{2})/ka_0$.