TWO-DIMENSIONAL AERIAL SMOOTHING IN RADIO ASTRONOMY

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Summary

The *visibility* of a Fourier component of a two-dimensional temperature distribution which is scanned by certain kinds of rigid aerial is given by the normalized complex autocorrelation function of the field distribution over the aerial aperture (assuming that turning the aerial in its own plane is not allowed). Hence, for finite aerials, the visibility of the Fourier components falls to zero at finite values of spatial frequency. Consequently observations need only be made at certain *peculiar intervals* whose size is worked out. *Interpolation* between observations so spaced can be carried out by a method which then, by a simple extension, permits *filtering* of data which are to be freed from high spatial frequencies. Both interpolation and filtering are basic processes in the handling of two-dimensional data and contour maps in radio astronomy. The *restoration* of smoothed data is discussed from the viewpoint that only the simplest operations on extensive two-dimensional data are feasible, and details of a suitable technique of restoration are summarized. Application of *further smoothing* to existing data is shown to be important, and a method for doing it is given, again under the restriction to simple operations. The *flux density* of a source is shown to be given exactly by summing one in four of the isolated values observed at the peculiar intervals.

I. Introduction

The essential mathematical phenomena of aerial smoothing have been set out by Bracewell and Roberts (1954) in a study (henceforth referred to as paper I) of the one-dimensional case ; the aim of the present paper is to give the generalization to two dimensions. On advancing from one to two dimensions, one meets a new phenomenon which is connected with the rotation of celestial objects in the field of view of some optical instruments (e.g. sextants, theodolites, siderostats) but not others (equatorially mounted telescopes, coelostats). Identical aerials pointed in the same direction in space, but mounted differently, may " see " different temperatures as a result of their " fields of view " being differently oriented. Thus, in specifying how an aerial beam is pointed, one must give the two direction coordinates and also the " position angle " ψ. Section II of this paper develops the general equations which form the basis of two-dimensional aerial smoothing but the remainder of the paper is restricted to the case where ψ is constant over the sky. When ψ is not constant we obtain a branch of the theory which includes the important problem of strip integration, a development which is covered in another paper (Bracewell 1956).

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If we suppose a two-dimensional temperature distribution $T(x,y)$ analysed into components of the form

$$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\,(2\pi ux-\varphi_u)\ \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\,(2\pi vy-\varphi_v),$$

then the effect of aerial smoothing is to eliminate the component if u or v exceeds certain limits, and to modify its strength relative to other components if it is not eliminated. On the uv-plane the points representing eliminated components lie outside a central area, usually circular or rectangular. This is proved in Section III and the consequences are worked out in Section IV, which establishes the discrete interval theorem according to which an observed distribution is fully determined by spot values lying on a rectangular lattice.

Section V discusses the construction of contour diagrams from data spaced at discrete intervals (interpolation) and a procedure for ensuring that contour diagrams are free from detail which is not warranted.

In Section VI the question of correcting for aerial smoothing is taken up and emphasis is laid on the necessity for simplicity in any operation that is proposed for application to an extensive two-dimensional array of data.

Section VII deals with the application of aerial smoothing, an operation which is called for in the reduction of data. Again the approach is from the direction of simple operations. In Section VIII it is shown that the integral over a true temperature distribution is given correctly by the integral over the observed distribution and that this integral is exactly evaluated by summing only one in four of the spot values just sufficient to determine the distribution.

II. The Basic Equation

Let $T(\theta, \varphi)$ be the distribution of brightness temperature for the frequency and polarization accepted by the aerial, θ and φ being the co-declination and Right Ascension respectively.

To specify the orientation of the aerial it is necessary to give not only (θ_0, φ_0), the co-declination and Right Ascension towards which the aerial is pointed, but also a position angle which determines the rotation of the aerial about this main axis. The position angle ψ gives the direction of a transverse axis, fixed in the aerial, measured eastwards from north (Fig. 1).

$A_1(\alpha, \beta)$ is the directional diagram of the aerial, α and β being spherical polar coordinates relative to axes fixed in the aerial. The polar angle α is measured from the main axis ; β from the great circle containing the main and transverse axes. $A_1(\alpha, \beta)$ is supposed normalized so that

$$\int_0^{2\pi}\int_0^{\pi} A_1(\alpha, \beta) \sin\alpha \, d\alpha d\beta=1. \qquad \ldots\ldots\ldots\ldots \quad (1)$$

When the aerial is pointed towards (θ_0, φ_0) with position angle ψ, the temperature measured is a weighted mean of $T(\theta, \varphi)$:

$$T_a(\theta_0, \varphi_0, \psi)=\int_0^{2\pi}\int_0^{\pi} A_1(\alpha, \beta)T(\theta, \varphi) \sin\alpha \, d\alpha d\beta. \quad \ldots\ldots \quad (2)$$

The co-declination and Right Ascension (θ, φ) appearing in the integral are related to α, β, θ_0, φ_0, and ψ as in the spherical triangle of Figure 1, from which we have the relations

$$\cos \theta = \cos \theta_0 \cos \alpha + \sin \theta_0 \sin \alpha \cos (\beta - \psi),$$
$$\cot (\varphi_0 - \varphi) = -\cos \theta_0 \cot (\beta - \psi) + \sin \theta_0 \cot \alpha \operatorname{cosec} (\beta - \psi).$$

The basic formula does not appear to have been given explicitly before. Among the conditions for its validity are two of interest. Firstly, it has been assumed that powers from different directions add, which would not be the

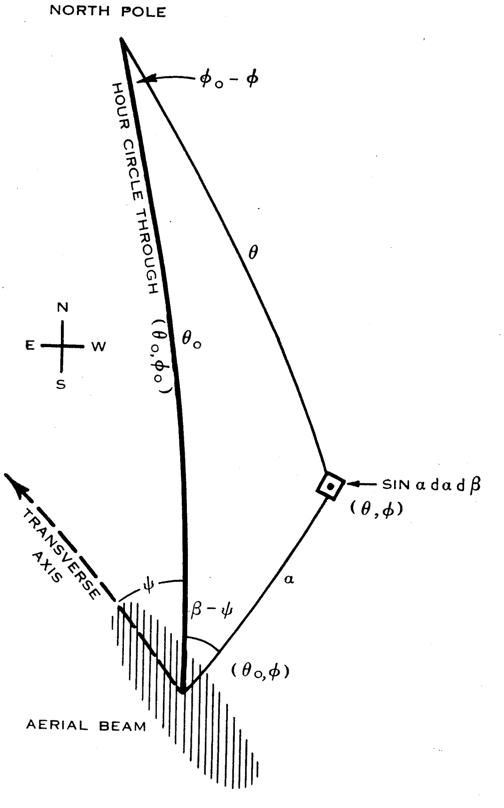

Fig. 1.—Spherical triangle showing the relation between the celestial coordinates θ, φ and the aerial coordinates α, β, ψ, θ_0, φ_0.

case if there were any coherence in phase. Such coherence can arise from ground reflection and it is actually invoked in the case of the sea interferometer, to which the present theory is not directly applicable. Coherence between radiations from different parts of the sky have so far been detected only as atmospheric scintillations. Secondly, it should be noted that the equation applies only to rigid aerials for which the aerial pattern does not change with different positions of the aerial. Interferometers whose parts may be tilted or rotated form instances of deformable aerials, as do equatorially mounted aerials which receive any radiation (via the main beam or side lobes) from the ground.

For the present purposes it is desirable to specialize equation (2) to a simpler form which still includes essential phenomena of two-dimensional smoothing. To do this we note that $\theta-\theta_0$ and $\varphi-\varphi_0$ will be approximately rectangular coordinates if attention is limited to a small fraction of the celestial sphere (not near the poles). Let x and y be the approximate rectangular co-ordinates of the point at which the aerial is directed, assume ψ to be constant, and use the fact that with a sufficiently directional aerial the area of sky contributing to the received power is so minute that the limits of integration are unimportant and may for convenience be replaced by infinite limits. Equation (2) can then be replaced by one of the form

$$T_a(x, y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x-\xi,\ y-\eta)T(\xi, \eta)\mathrm{d}\xi\mathrm{d}\eta, \quad \ldots\ldots\ldots\ldots \quad (3)$$

where ξ and η are the coordinates of the element $\mathrm{d}\xi\mathrm{d}\eta$, and $T(x, y)$ and $T_a(x, y)$ are respectively the true and measured brightness temperatures at (x, y). The function $A(\xi-x, \eta-y)$ representing the directional diagram is related to the definitive directional pattern $A_1(\alpha, \beta)$ in a way which involves the declination of the area where the approximate rectangular coordinates are to be drawn; therefore, in all that follows, the further assumption is implied that attention is confined within a zone of declination (or other appropriate coordinate, such as galactic latitude) narrow enough for the function A to have the same form throughout.

III. The Spectral Visibility Theorem

The two-dimensional Fourier transform of $T_a(x, y)$ is defined by

$$\bar{T}_a(u, v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi(ux+vy)}T_a(x, y)\mathrm{d}x\mathrm{d}y. \quad \ldots\ldots \quad (4)$$

Then, by the two-dimensional convolution theorem (Sneddon 1951),

$$\bar{T}_a(u, v)=\bar{A}(u, v)\bar{T}(u, v). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Equation (3), which takes the form of a two-dimensional convolution, we shall write as

$$T_a=A*T$$

following a convenient notation† used in paper I.

The important quantity $\bar{A}$, which will be called the spectral visibility function, has a simple relationship to the aerial aperture distribution of electric field $E(x/\lambda, y/\lambda)$, where x and y are rectangular coordinates in the plane of the aperture. We know that a two-dimensional aperture distribution and the angular spectrum of its accompanying field, P ($\sin\gamma$, $\sin\delta$) are Fourier transforms one of the other, provided we adopt as independent variables x/λ, y/λ,

† For the sake of simplicity it will be assumed that $A(x,y)$ is symmetrical about both the x-and y-axes. The asterisk notation for convolution is then not complicated by the need to draw a distinction between " aerial directional diagram " and " response to a point source ".

$\sin\gamma$, $\sin\delta$, where the angles γ and δ are measured from the planes, normal to the plane of the aperture, which contain respectively the y- and x-axis. Thus

$$P\,(\sin\gamma,\,\sin\delta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)\exp\left[\mathrm{i}2\pi\left(\frac{x}{\lambda}\sin\gamma+\frac{y}{\lambda}\sin\delta\right)\right]\mathrm{d}\left(\frac{x}{\lambda}\right)\mathrm{d}\left(\frac{y}{\lambda}\right),$$

and

$$E\left(\frac{x}{\lambda},\frac{y}{\lambda}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P\,(\sin\gamma,\,\sin\delta)\exp\left[-\mathrm{i}2\pi\left(\frac{x}{\lambda}\sin\gamma+\frac{y}{\lambda}\sin\delta\right)\right]\times\mathrm{d}\,(\sin\gamma)\mathrm{d}\,(\sin\delta).$$

We may write this latter equation

$$E=\bar{P},$$

where the bar indicates, here and below, the two-dimensional Fourier transform. For real values of γ and δ, P is proportional to the field directional diagram. Hence the power directional diagram A is given by

$$A=\text{const. }|\,P\,|^2,$$

or if P is not complex, by

$$A=\text{const. }P^2.$$

Hence, by the convolution theorem,

$$\begin{aligned}\bar{A}&=\text{const. }\bar{P}*\bar{P},\\&=\text{const. }E*E.\end{aligned} \qquad (6)$$

Thus the spectral visibility function $\bar{A}$ is proportional to the self convolution of the electric field distribution.† The constant of proportionality is fixed by the requirement that $\bar{A}\,(0)$ should be unity, a consequence of the normalization of A. Finally we have

$$\bar{T}_a=\text{const. }(E*E)\bar{T}. \qquad (7)$$

The spectral visibility theorem (**7**) is useful both as a mental aid to seeing the general form of $\bar{A}$ and also as a means for computing it.

A most important property of $\bar{A}$ is that, for aerial apertures of finite extent, it falls to zero outside a central region in the uv-plane. The shape of this region can readily be determined graphically by performing the self convolution indicated in equation (**6**). In general the shape is complicated, but for circular apertures it is circular, and for rectangular apertures, rectangular.

An observed distribution taken with an aerial consisting of a finite aperture will, by equation (**7**), have the same property as $\bar{A}$, namely, that its transform

† In the more general case of P complex and not symmetrical, $\bar{A}$ is the (normalized) complex autocorrelation function of E,

$$\bar{A}(u,v)=\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E(x/\lambda-u,\;y/\lambda-v)E^*(x/\lambda,\;y/\lambda)\mathrm{d}(x/\lambda)\mathrm{d}(y/\lambda)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E(x/\lambda,\;y/\lambda)E^*(x/\lambda,\;y/\lambda)\mathrm{d}(x/\lambda)\mathrm{d}(y/\lambda)}.$$

is zero outside a central region. The consequences of this property are given in the next section.

In what follows it is occasionally convenient to say that a function is "cut off" if there exists a central region outside which it is everywhere zero. Also the term "spectrum" is occasionally used as a synonym for Fourier transform.

IV. The Discrete Interval Theorem

A function $T_a(x, y)$ such that $\bar{T}_a(u, v)$ is zero for $|u| \geqslant u_c$ or $|v| \geqslant v_c$, is completely determined by its values at the points $(m/2u_c'+a,\ n/2v_c'+b)$, where m and n assume all integral values, a and b are arbitrary constants, and the spacing between points may be as wide as is compatible with $u_c' \geqslant u_c$ and $v_c' \geqslant v_c$.

The condition on $\bar{T}_a(u, v)$ may be expressed by saying that it is zero on and outside a rectangle which is centred on the origin of the uv-plane and set with its sides parallel to the axes; and we may note that this covers the case of $\bar{T}_a(u, v)$ zero on and outside a circle or other region provided the rectangle is chosen sufficiently large.

It is sufficient to give a proof for the case where a and b are zero, i.e. where the origin of x and y is one of the sampling points. For, if the transform of $T_a(x, y)$ is zero on and outside a given rectangle, so also is that of $T_a(x+a, y+b)$ by virtue of the two-dimensional shift theorem, according to which the Fourier transform of $T_a(x+a, y+b)$ is $\bar{T}_a(u, v) \exp\{-i2\pi(au+bv)\}$, which must be zero where $\bar{T}_a(u, v)$ is zero. Therefore, if the theorem is true for $T_a(x, y)$, it is also true for $T_a(x+a, y+b)$; but values of $T_a(x+a, y+b)$ at points of an array which includes the origin are values of $T_a(x, y)$ taken over an offset array.

To prove the theorem we use the bed-of-nails function ${}^2\mathrm{III}(x, y)$ consisting of a two-dimensional array of unit impulses separated by unit distance. Thus

$${}^2\mathrm{III}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} {}^2\delta(x-m, y-n),$$

where the two-dimensional impulse function ${}^2\delta(x, y)=0$ for $x \neq 0$ or $y \neq 0$, but

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} {}^2\delta(x, y)\,\mathrm{d}x\mathrm{d}y = 1.$$

The bed-of-nails function is shown in Appendix I to be its own two-dimensional Fourier transform.

Proof of Theorem

Let $\bar{F}(u, v) \equiv (4u_c'v_c')^{-1}\ {}^2\mathrm{III}(u/2u_c',\ v/2v_c') * \bar{T}_a$, a function which may be pictured as an array of islands in the uv-plane, each the same as $\bar{T}_a$, spaced at intervals $2u_c'$ in the u direction and $2v_c'$ in the v direction. The islands will not overlap (but may touch) if $u_c' \geqslant u_c$ and $v_c' \geqslant v_c$. Under this condition, in the central region where $|u| < u_c$ and $|v| < v_c$, we have

$$\bar{F}(u, v) = \bar{T}_a.$$

Hence $\bar{T}_a$ may be recovered from $\bar{F}(u, v)$, and consequently T_a may be recovered from $F(x, y)$, the two-dimensional Fourier transform of $\bar{F}(u, v)$. But, by the two-dimensional convolution theorem,

$$F(x, y)=4u_c'v_c'\ {}^2\mathrm{III}(2u_c'x,\ 2v_c'y)T_a(x, y),$$

which contains values of T_a only at discrete intervals $(2u_c')^{-1}$ and $(2v_c')^{-1}$ of x and y. Hence T_a is completely determined by its values at discrete intervals of x and y which are equal to or less than $(2u_c)^{-1}$ and $(2v_c)^{-1}$. Since these intervals are peculiar to each aerial they will be referred to as the peculiar intervals.

V. Interpolation and Filtering

If the distribution T_a is measured at suitable discrete intervals of x and y, the numerical problem of interpolating will often arise. Let the scale factors be changed so that $(2u_c)^{-1}$ and $(2v_c)^{-1}$ both become unity. Then it is required to calculate $T_a(x, y)$ for any x and y, given ${}^2\mathrm{III}T_a$. From the preceding section we can say that

$$\bar{T}_a=\bar{M}\ \overline{{}^2\mathrm{III}T_a},$$

where

$$\bar{M}(u, v)=\begin{cases}1, & (|\,u\,|<\tfrac{1}{2}\text{ and }|\,v\,|<\tfrac{1}{2}),\\ 0, & (|\,u\,|>\tfrac{1}{2}\text{ or }|\,v\,|>\tfrac{1}{2}).\end{cases}$$

In other words, we formally express the recovery of $\bar{T}_a$ from the array of islands $\bar{F}$ (written now as $\overline{{}^2\mathrm{III}T_a}$) as the operation of multiplying by $\bar{M}$, the parallelopiped function of unit height, length, and breadth. Then, by the two-dimensional convolution theorem,

$$T_a=M*({}^2\mathrm{III}T_a). \qquad \text{(8)}$$

Now it can be shown that

$$M=\frac{\sin \pi x \sin \pi y}{\pi^2 xy},$$

but because of the character of ${}^2\mathrm{III}$ the convolution integral (8) reduces exactly to the sum of products of the array of values of T_a at discrete intervals with the array of values of M taken at the same intervals. In calculating any particular value, therefore, we need only know a discrete set of values of M. There are two principal interpolations for which it is useful to have the M array recorded for reference. The points A and B (Fig. 2) show the relation of the required points to the lattice points where T_a is known. For interpolation at A, the centre of the square, we use

$${}^2\mathrm{III}(x-\tfrac{1}{2},\ y-\tfrac{1}{2})M,$$

and for interpolation at B, the mid point of a lattice line, we use

$${}^2\mathrm{III}(x-\tfrac{1}{2},\ y)M.$$

These two arrays are given in Table 1 and Table 2. For interpolation at C we use the B array with rows and columns interchanged. The reason that interpolation at B needs only the values of T_a in the row through B is that a cross section of a surface whose (two-dimensional) spectrum is cut off, itself has a cut-off (one-dimensional) spectrum. The amount of work needed to perform

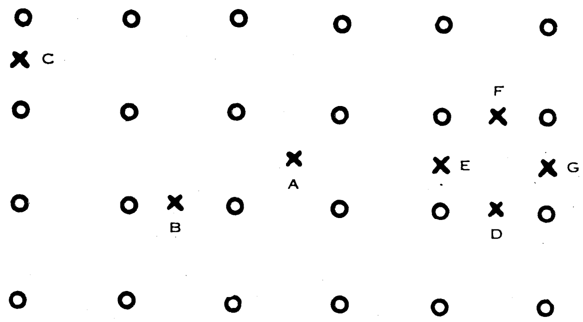

Fig. 2.—Known values of T_a are marked ○. Interpolated values are required at A, B, and C.

A-interpolation considerably exceeds that for B-interpolation, so that in practice it will often be preferable to use only B-interpolation, proceeding by steps such as D, E, F, G (Fig. 2). Table 2 is given at greater length in Table 1 of paper I.

TABLE 1

ARRAY FOR A-INTERPOLATION

(The first octant is shown at greater length on the right)

		0·08	0·08		
	0·05	$\overline{0\cdot14}$	$\overline{0\cdot14}$	0·05	
0·08	$\overline{0\cdot14}$	0·41	0·41	$\overline{0\cdot14}$	0·08
0·08	$\overline{0\cdot14}$	0·41	0·41	$\overline{0\cdot14}$	0·08
	0·05	$\overline{0\cdot14}$	$\overline{0\cdot14}$	0·05	
		0·08	0·08		

				0·005
			0·008	$\overline{0\cdot006}$
		0·016	$\overline{0\cdot011}$	0·009
	0·045	$\overline{0\cdot027}$	0·019	$\overline{0\cdot015}$
0·405	$\overline{0\cdot135}$	0·081	$\overline{0\cdot058}$	0·045

We have so far assumed that $\overline{T}_a$ is cut off outside the rectangle determined by $|u|=u_c$, $|v|=v_c$, without enquiring whether it is also zero anywhere within the rectangle; but if the aerial has circular symmetry, as is often the case, the cut-off boundary will be circular.

Let $\bar{A}$ fall to zero on the circle $u^2+v^2=R^2$. Then

$$\overline{T}_a=\overline{N}\ {}^2\overline{\mathrm{III}T_a},$$

where

$$\bar{N}(u, v)=\begin{cases}1, & (u^2+v^2<R),\\ 0, & (u^2+v^2>R).\end{cases}$$

Hence

$$T_a=N*({}^2\mathrm{III}\,T_a),$$

and we know that

$$N=\frac{R\mathrm{J}_1\{2\pi R\sqrt{(x^2+y^2)}\}}{\sqrt{(x^2+y^2)}}.$$

TABLE 2

ARRAY FOR B-INTERPOLATION

			0	0			
		0	0	0	0		
———	0·127	$\overline{0\cdot212}$	0·637 —	— 0·637	$\overline{0\cdot212}$	0·127	———
		0	0	0	0		
			0	0			

The array for A-interpolation in this case is shown in Table 3. When the aerial is elongated and produces a " fan beam ", then as long as the position angle of the beam is kept fixed, this case is covered by the scale change making $(2u_c)^{-1}$ and $(2v_c)^{-1}$ unity. In practice a rectangular lattice of points is used to define the data.

TABLE 3

ARRAY FOR INTERPOLATION WHEN THE AERIAL HAS CIRCULAR SYMMETRY

(Only the first octant is shown)

				0·005
			0·016	$\overline{0\cdot016}$
		$\overline{0\cdot027}$	0·005	0·006
	$\overline{0\cdot025}$	0·038	$\overline{0\cdot030}$	0·022
0·391	$\overline{0\cdot102}$	0·046	$\overline{0\cdot027}$	0·018

The operation of filtering is closely related to interpolation, but the independent variables, instead of assuming discrete values, vary continuously. It is required to remove the spectral components outside a certain cut-off boundary—which is precisely the means by which interpolation was effected. However, the integral does not reduce to a summation as in that case. The filtered value T_a corresponding to an unfiltered value T is given by

$$T_a = M * T,$$

in the case of a rectangular cut-off boundary. To evaluate this integral we would calculate a summation

$$T_{\xi\eta} = \left\{ {}^2\mathrm{III}\left(\frac{x}{\xi}, \frac{y}{\eta}\right) M \right\} * T,$$

which would approach the desired integral as the spacings ξ and η approached zero. Just how small they need be we shall now enquire.

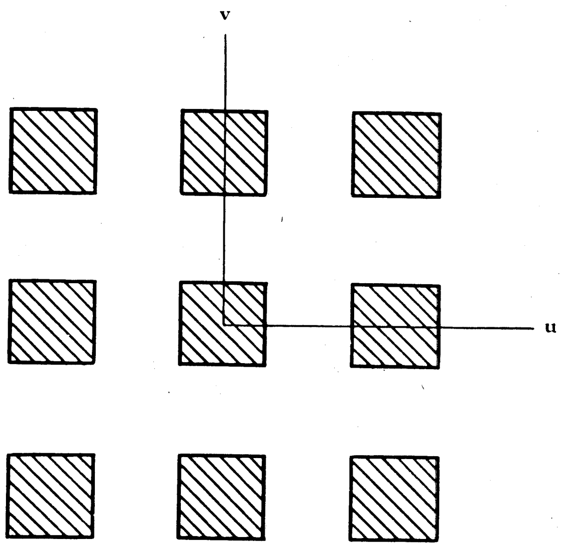

Fig. 3.—To illustrate the function $\frac{1}{4}{}^2\mathrm{III}(\frac{1}{2}u, \frac{1}{2}v) * \overline{M}$.

Beginning with the coarsest interval $\xi = \eta = 1$, we find $T_{11} = T$, that is, no filtering has been achieved; but with $\xi = \eta = \frac{1}{2}$, we find

$$T_{\frac{1}{2}\frac{1}{2}} = \{ {}^2\mathrm{III}(2x, 2y) M \} * T,$$

whence

$$\overline{T}_{\frac{1}{2}\frac{1}{2}} = \tfrac{1}{4} \left\{ {}^2\mathrm{III}\left(\frac{u}{2}, \frac{v}{2}\right) * \overline{M} \right\} \overline{T}.$$

Thus $\overline{T}_{\frac{1}{2}\frac{1}{2}}$ consists of a central part $\overline{M}\,\overline{T} = \overline{T}_a$ plus islands in the uv-plane covering only one-quarter of the plane, as shown in Figure 3. For many purposes this would be sufficient filtering, since the components to be rejected would often lie chiefly just beyond the central region.

The test for adequacy of filtering is to compare B-interpolations with T in the most unfavourable areas. Where further filtering is required, further application of the same process with $\xi=\eta=\frac{1}{4}$ will remove the eight peripheral islands of Figure 3 as well as others further out. Table 4 gives for permanent reference the array needed for the summations to which filtering is reduced by the suggested procedure. Subsequent stages of filtering, where indicated, use exactly the same array.

A further comment may be made on a procedure which appears at first sight to be equivalent to filtering. If T is read off at discrete intervals corresponding to the desired cut-off, the set of values so obtained defines a function

TABLE 4

ARRAY FOR FILTERING

						0
					0·016	0
				0	0	0
			0·045	0	$\overline{0\cdot 027}$	0
		0	0	0	0	0
	0·101	0	$\overline{0\cdot 034}$	0	0·020	0
0·250	0·159	0	$\overline{0\cdot 053}$	0	0·032	0

with cut-off spectrum which may then be recovered by interpolation. But this function, $M*({}^2\mathrm{III}T)$, is not the same as T_a since it is contributed to by high frequency components of T. A use for it is suggested in Section VII.

Both interpolation and filtering have application to the handling of contour diagrams of brightness distribution. When data have been obtained at the peculiar intervals, it is not always easy to draw the contours in by eye, especially in places where the contours are tightly bunched or strongly curved. Interpolation of extra values is here useful. On the other hand, when data have been obtained continuously or semi-continuously high frequencies can appear accidentally, and may be revealed by interpolation and removed by filtering. One source of spurious high frequencies is undoubtedly the freehand interpolation of contours—the apparent detail so generated should be ignored during subsequent interpretation. In fact, before any spatial detail near the limit of resolution is discussed, it should be filtered to ensure that no spurious feature is included which is unjustified on the known aerial resolution.

It is unlikely that any of the published surveys of galactic noise have been filtered and it is certain that many of them contain unwarranted detail. It is suggested that when publishing a contour diagram one should make clear how the interpolation was done and whether filtering was done. As an alternative, or adjunct, to a contour diagram, a table of values at the peculiar interval would have merits.

VI. Restoration

The possibilities of restoration have been studied in paper I for one dimension and they are the same in the present case. In brief, nothing can be done about spectral components near or beyond cut-off, but the correct relative proportions of the remaining components can be restored. Whether a given T_a can be gainfully restored depends on $\bar{T}$—whether it falls to an unimportant level before the aerial cuts it off. If it does so, the distribution is fully resolved and may be confidently restored. More usually, however, $\bar{T}$ is such that the justifiable degree of restoration is a nice compromise between improvement due to the restored balance of the low frequency components and deterioration due to overshoot and magnification of errors.

The method of successive substitutions is applicable in two dimensions but has the disadvantage of requiring an amount of computation that renders it infeasible in much of the recent extensive high resolution work. There therefore appears to be scope for a less elaborate procedure which would at least indicate the order of magnitude of the corrections and enable one to see where fuller attention to restoration might be wise. The special case of a Gaussian aerial has been approached on this basis (Bracewell 1955*a*) with encouraging results. The method is a generalization of one worked out for the one-dimensional case (Bracewell 1955*b*) in which the correction is given by the amount by which the distribution lies above the mid point of a certain chord. In two dimensions it is useful to imagine the corresponding plane, but the calculation must be done numerically, not graphically. The amount of the correction is

$$-\tfrac{1}{4}(\Delta_{xx}+\Delta_{yy})T_a,$$

where Δ_{xx} is the second difference of T_a when y is kept constant, and the interval over which the differencing is done is equal to $\sqrt{2}$ times the standard deviation of the Gaussian beam. In the case of beams other than Gaussian the correction becomes

$$-(\chi\Delta_{xx}+\psi\Delta_{yy})T_a,$$

where χ, ψ, and the differencing intervals α and β are fixed by matching $\chi \sin^2 \pi\alpha u+\psi \sin^2 \pi\beta v$ to $\bar{A}^{-1}-1$ as previously described for the one-dimensional case (Bracewell 1955*c*).

VII. Carrying Out Aerial Smoothing

The need to carry out aerial smoothing numerically can arise in various ways :

(*a*) In comparing data obtained with different aerials but on the same wavelength, further smoothing on the better resolved data must be carried out.

(*b*) In determining spectra by comparison of data taken on different wavelengths, different resolving powers must usually be allowed for.

(*c*) In utilizing optical data presented in the form of contour diagrams or counts showing the frequency of occurrence of celestial objects, some numerical operation akin to aerial smoothing must first be applied.

One might consider restoring the less well-resolved data before making comparison, and in some cases this might be desirable, but more often the sacrifice of detail in the procedure discussed here will be more than compensated for by the reliability of the comparison; for a smoothed distribution can be uniquely determined whereas there is uncertainty in the determination of a restored distribution.

Case (*c*) differs from (*a*) and (*b*) through the presence of high frequency components resulting from the high resolution of optical data, and such data should first be roughly filtered by taking running means, smoothing by eye, or some other appropriate method.

Since the present problem has arisen in connexion with aerials having Gaussian beams, Gaussian smoothing will be used to illustrate the proposed methods.

Let

$$A(x, y)=\frac{1}{2\pi\sigma^2} \exp \left(-\frac{x^2+y^2}{2\sigma^2}\right),$$

where σ is the standard deviation of any section through $x=0$, $y=0$. Then

$$\bar{A}(u, v)=\exp \{-2\pi\sigma^2(u^2+v^2)\}.$$

We now take T to stand for a distribution which is to be smoothed and let T_a represent the result of the smoothing. The operation by which T_a is obtained from T must be such that

$$\begin{aligned}\bar{T}_a&=\bar{A}\,\bar{T}\\&=\exp \{-2\pi\sigma^2(u^2+v^2)\}\bar{T}.\end{aligned}$$

We shall take as an approximation to this exact formula, for reasons which follow later,

$$\bar{T}_a=\tfrac{1}{2}\{1+\cos 2\sqrt{(2\pi)}\sigma u \cos 2\sqrt{(2\pi)}\sigma v\}\bar{T}. \quad \ldots\ldots\ldots\ldots \quad (9)$$

The degree of agreement with the accurate Gaussian function is shown in Figure 4. Applying the Fourier transformation to equation (9) we have

$$\begin{aligned}T_a(x, y)=\tfrac{1}{2}T(x, y)+ \tfrac{1}{8}\{&T(x-x_1,\ y-x_1)+T(x+x_1,\ y-x_1)\\&+T(x+x_1,\ y+x_1)+T(x-x_1,\ y+x_1)\}, \quad \ldots\ldots\ldots\ldots \quad (10)\end{aligned}$$

where

$$x_1=\sqrt{(2/\pi)}\sigma.$$

Equation (10) is interpreted diagrammatically in Figure 5 which shows a square grid ruled at intervals x_1. First take the mean of the four values surrounding the value (100) to be corrected and then average the result with 100. Alternatively, average the differences shown in smaller figures and subtract half the amount from 100. The value for T_a is thus $98 \cdot 5$. The possibility of reducing the numerical operations to this simple procedure of averaging was the reason for selecting the approximating function in equation (**9**).

The amount of work involved in applying the operation here described is consistent with the economy of effort desirable in any two-dimensional operation, and is much less than the work involved in carrying out smoothing by convolution with a Gaussian distribution. A little further study is, however, necessary for two reasons. Firstly, the function $\bar{A}$ cannot be strictly Gaussian since it

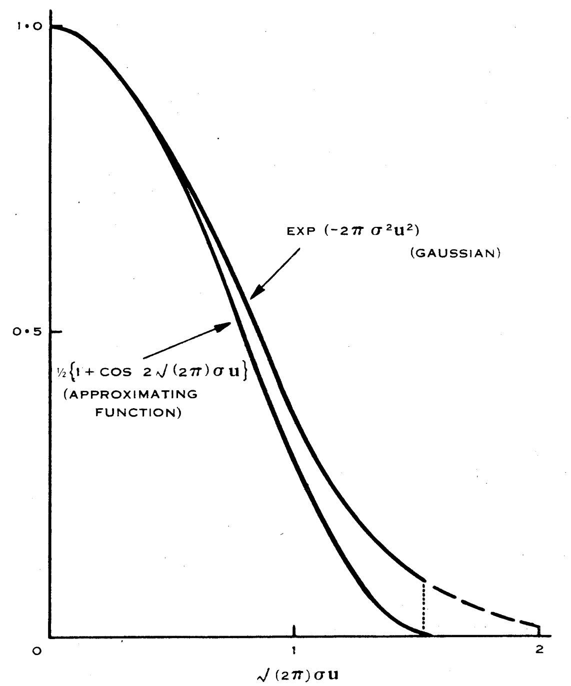

Fig. 4.—Agreement of the approximating function with the Gaussian distribution.

must fall to zero outside an interval set by the finite extent of the aerial and secondly, the approximating function is periodic.

The periodic character of the approximating function

$$\tfrac{1}{2}\{1+\cos 2\sqrt{(2\pi)}\sigma u \cos 2\sqrt{(2\pi)}\sigma v\}$$

is obviously immaterial if $\bar{T}$ is zero for values of σu and σv greater than $(\tfrac{1}{8}\pi)^{\frac{1}{2}}$. Any method of ensuring that this is so would therefore be sufficient to render the periodic character inoffensive. One approach is to filter the data numerically, but a simpler procedure would often be sufficient. The values of T taken at critical intervals of u and v define a function T' whose transform is zero beyond the stated limits. If T' represents T adequately, within the limits of accuracy of the observations, then one may proceed immediately to operate on these values in the manner described in connexion with Figure 5.

The fact that the function $\bar{A}$ cannot be strictly Gaussian in practice leads to a step such as is shown by the dotted line in Figure 4. Possibly this is too small to be important in many cases, but circumstances will vary from case to case and it may be desirable at least to investigate the effect. An excellent fit to the stepped distribution can be obtained by adding a constant to the approximating function and altering the coefficient $\frac{1}{2}$ to compensate. The result will

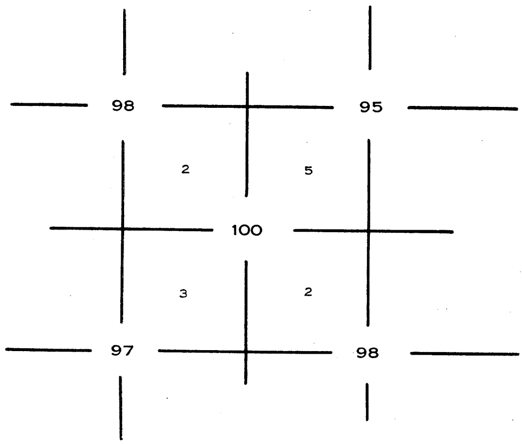

Fig. 5.—Illustrating the numerical procedure for smoothing.

depend on the particular aerial considered, but for the 1500-ft Mills Cross it becomes, on the $v=0$ axis,

$$0\cdot45\{1+\cos 2\sqrt{(2\pi)}\sigma u\}+0\cdot1.$$

The agreement of this function with $\exp(-2\pi\sigma^2u^2)$ is extremely close. The net effect is to weight the central value (100) when it is being averaged with the mean of the surrounding values.

VIII. The Calculation of Flux Density

As the result of a survey with a pencil beam, the distribution of observed brightness temperature T_a over a discrete source has been established and subtraction of the background is assumed to have been attended to. The flux density S of the source is given by integrating T over the solid angle occupied by the source, thus

$$S=\frac{2k}{\lambda^2}\int\int T\mathrm{d}\Omega.$$

We now show how to evaluate S from knowledge of T_a, first proving that

$$\int\int T\mathrm{d}\Omega=\int\int T_a\mathrm{d}\Omega,$$

The integral of a function between infinite limits is equal to the value of its Fourier transform at the origin, e.g. put $u=v=0$ in equation (4). Hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_a \mathrm{d}x\mathrm{d}y=[\bar{A}\,\bar{T}]_{u=v=0}$$
$$=[\bar{T}]_{u=v=0},$$

provided A is normalized so that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\mathrm{d}x\mathrm{d}y=1$, when $[\bar{A}]_{u=v=0}=1$.

But

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T\mathrm{d}x\mathrm{d}y=[\bar{T}]_{u=v=0},$$

therefore

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T\mathrm{d}x\mathrm{d}y=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_a\mathrm{d}x\mathrm{d}y.$$

It is consequently sufficient to calculate the flux density of a source by integrating T_a; nothing is lost through unawareness of the fine detail in T.

We now show that the integral can be replaced by a summation. If $\bar{T}_a=0$ for $|u|\geqslant\frac{1}{2}$ and for $|v|\geqslant\frac{1}{2}$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_a\mathrm{d}x\mathrm{d}y=4[{}^2\mathrm{III}(2u,\ 2v)*\bar{T}_a]_{u=v=0}$$
$$=[\overline{{}^2\mathrm{III}(\tfrac{1}{2}x,\ \tfrac{1}{2}y)T_a}]_{u=v=0}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} {}^2\mathrm{III}(\tfrac{1}{2}x,\ \tfrac{1}{2}y)T_a\mathrm{d}x\mathrm{d}y$$
$$=\sum_{\mu=-\infty}^{\infty}\ \sum_{\nu=-\infty}^{\infty} 4T_a(x-2\mu,\ y-2\nu).$$

Alternatively, *a fortiori*,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_a\mathrm{d}x\mathrm{d}y=\alpha\beta\Sigma\Sigma T_a(x-\alpha\mu,\ y-\beta\nu),$$

provided α and β are both less than 2. This result permits a great time saving: to integrate T_a, it suffices to sum only one in four of the discrete values necessary to specify T_a. The proviso that $\bar{T}_a=0$ for $|u|>\frac{1}{2}$ and for $|v|>\frac{1}{2}$ is automatically met when the units of x and y are equal to the peculiar intervals associated with $A(x, y)$, but if $\bar{T}_a(0, \pm\frac{1}{2})$ or $\bar{T}_a(\pm\frac{1}{2}, 0)$ are not all zero, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_a\mathrm{d}x\mathrm{d}y=\frac{\Sigma\Sigma 4T_a(x-2\mu,\ y-2\nu)}{1+\frac{1}{4}\{\bar{T}_a(0,\ \frac{1}{2})+\bar{T}_a(0,\ -\frac{1}{2})+\bar{T}_a(-\frac{1}{2},\ 0)+\bar{T}_a(\frac{1}{2},\ 0)\}}.$$

If $\bar{T}_a$ is discontinuous at any of the points $u=0$, $v=\pm\frac{1}{2}$ or $u=\pm\frac{1}{2}$, $v=0$, the fact would be revealed as inequality between the four possible summations based on the points $(x, y)=(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ (their mean being correct). There are different ways of regarding this phenomenon. One can avoid it entirely by taking α and β less than 2. On the other hand one can use it for testing data supposed to be free of components of semi-period equal to the interval of tabulation.

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Appendix I

The Two-dimensional Fourier Transform of the Function $^2\mathrm{III}$

The integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} {}^2\mathrm{III}(x, y)\mathrm{d}x\mathrm{d}y$ not being convergent, the function $^2\mathrm{III}(x, y)$ does not possess a regular two-dimensional Fourier transform: we are concerned here with showing that it has a transform-in-the-limit, and that this transform is identical with $^2\mathrm{III}$ itself. We prove in full the corresponding result for the one-dimensional function $\mathrm{III}(x)$, which has not been given before, to the author's knowledge, and indicate the method of proof for $^2\mathrm{III}$.

Consider the function†

$$F(\alpha, x)=\alpha^{-1}\exp(-\pi\alpha^2x^2)\sum_{n=-\infty}^{\infty}\exp\{-\pi\alpha^{-2}(x-n)^2\}.$$

For each α, $F(\alpha, x)$ represents a row of Gaussian spikes of width α, the whole multiplied by a Gaussian curve of width α^{-1}. We shall consider the limit of this function as $\alpha\to 0$. It is easy to show that $\lim F(\alpha, x)=0$ for $x\neq n$, and that, for every n, $\lim\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}F(\alpha, x)\mathrm{d}x=1$. It follows that‡

$$\lim_{\alpha\to 0} F(\alpha, x)=\mathrm{III}(x).$$

We now consider the transform of $F(\alpha, x)$, and show that as α tends to zero this transform approaches a limit which is $\mathrm{III}(s)$. In determining the transform

† In following this proof it will be helpful to notice that, in our $*$ and III notation, $F(\alpha, x)=\exp(-\pi\alpha^2x^2)\{\alpha^{-1}\exp(-\pi\alpha^{-2}x^2)*\mathrm{III}(x)\}$. Furthermore, $\exp(-\pi\alpha^2s^2)$ is the Fourier transform of $\alpha^{-1}\exp(-\pi\alpha^{-2}x^2)$; and the factors are so chosen that $\int_{-\infty}^{\infty}\alpha^{-1}\exp(-\pi\alpha^{-2}x^2)\mathrm{d}x=1$, irrespective of the value of α, thus ensuring that $\lim\alpha^{-1}\exp(-\pi\alpha^{-2}x^2)=\delta(x)$, whence, in due course, $\lim F(\alpha, x)=\mathrm{III}(x)$. The line of proof is to deduce, by conventional means, that $F(\alpha, x)$ has a regular transform, viz.

$$\alpha^{-1}\exp(-\pi\alpha^{-2}x^2)*\{\exp(-\pi\alpha^2s^2)\mathrm{III}(s)\},$$

and that both $F(\alpha, x)$ and its transform approach III as $\alpha\to 0$.

‡ This equation is to be interpreted in the same sense as one interprets $\lim\alpha^{-1}\Pi(x/\alpha)=\delta(x)$, namely, as permitting convenient notation as integrals for expressions which, when rigorously expressed, are limits of integrals.

of $F(\alpha, x)$ we first notice that the factor $\alpha^{-1}\Sigma \exp\{-\pi\alpha^{-2}(x-n)^2\}$ is periodic in x and may therefore be expressed as the Fourier series

$$\sum_{m=-\infty}^{\infty} \exp(-\pi\alpha^2 m^2) \cos 2\pi m x.$$

Hence

$$F(\alpha, x) = \sum_{m=-\infty}^{\infty} \exp(-\pi\alpha^2 m^2) \exp(-\pi\alpha^2 x^2) \cos 2\pi m x.$$

Applying the Fourier shift theorem we find for the transform of $F(\alpha, x)$

$$\overline{F}(\alpha, s) = \alpha^{-1} \sum_{m=-\infty}^{\infty} \exp(-\pi\alpha^2 m^2) \exp\{-\pi\alpha^{-2}(s-m)^2\}.$$

This function is a row of Gaussian spikes of width α with peaks lying on a Gaussian curve of width α^{-1}, and clearly,

$$\lim_{\alpha\to 0} \overline{F}(\alpha, s) = \mathrm{III}(s).$$

Hence the transform-in-the-limit of III is identical with itself.

A similar proof can be given for $^2\mathrm{III}$ by showing that the regular two-dimensional transform of $G(\Gamma * {}^2\mathrm{III})$ is $\Gamma * (G\,{}^2\mathrm{III})$, where

$$G(x, y) = \exp\{-2\pi^2\alpha^2(x^2+y^2)\},$$

and

$$\Gamma(x, y) = (2\pi\alpha^2)^{-1} \exp\{-(2\alpha^2)^{-1}(x^2+y^2)\}.$$

Here G and Γ are a transform pair, and Γ has the property $\lim \Gamma = {}^2\delta$. The doubly periodic function $\Gamma * {}^2\mathrm{III}$ can be expressed as a double Fourier series and the two-dimensional shift theorem can be applied term by term. Both $G(\Gamma * {}^2\mathrm{III})$ and $\Gamma * (G\,{}^2\mathrm{III})$ approach $^2\mathrm{III}$ as $\alpha \to 0$.