BOUNDS FOR THE COLLAPSE LOAD OF A BEAM COMPRESSED BY THREE DIES

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Summary

This paper deals with the problem of the plastic deformation of a beam under the action of three perfectly rough rigid dies, two dies applied to one side, one die to the other side of the beam, the single die being situated between the two others. It is treated as a problem of plane plastic flow. Discontinuous stress and velocity fields are assumed and upper and lower bounds for the pressure sufficient to cause pronounced plastic yielding determined by limit analysis.

The treatment is approximate and subject to experimental verification.

I. INTRODUCTION

This problem had its origin in an enquiry regarding the straightening of the spar-beams for an aircraft. The beams were received bent and it was required to straighten them without introducing severe tensile stresses. It was suggested that by applying pressure successively along the beam by three rigid dies, two on one, one on the opposite side of the beam (Fig. 1), it could be straightened by introducing, mainly, shear deformation.

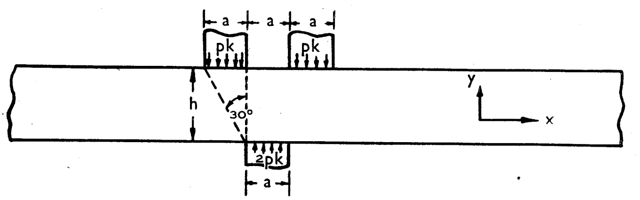

Fig. 1.—Beam under the action of three rigid, perfectly rough dies.

The present paper deals with this problem in a highly idealized form. The approximate solution presented here enables an estimate to be formed of the die pressure necessary to cause overall plastic deformation and also suggests a picture of the change of shape of the boundary when the dies are pressed together with given velocity.

II. GENERAL THEORY

The solution of a problem in plane plastic flow consists of the determination of the three stress components σ_x, σ_y, and τ_{xy}, and the two velocity components u and v. The first and second slip lines are defined as lines in the directions

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(the first and second shear direction) obtained by 45° counterclockwise rotations of the directions (first and second principal direction) across which the tensile stress is a maximum and minimum, respectively. These lines are orthogonal to each other and it can be shown (Prager and Hodge 1951) that

$$\left.\begin{array}{l}\omega-\theta=\text{constant along a first slip line,}\\ \omega+\theta=\text{constant along a second slip line,}\end{array}\right\} \quad \ldots\ldots\ldots \quad (1)$$

where $2k\omega$ is the mean normal stress (k is the yield stress in pure shear and so $2k$ the yield stress in pure tension), and θ is the angle between the first slip line and the negative y-axis. In terms of ω and θ, the stress components are given by

$$\left.\begin{array}{l}\sigma_x=2k\omega+k\sin 2\theta,\\ \sigma_y=2k\omega-k\sin 2\theta,\\ \tau_{xy}=-k\cos 2\theta.\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

These satisfy the equilibrium equations and the yield condition

$$(\sigma_x-\sigma_y)^2+4\tau_{xy}^2=4k^2.$$

It has been shown (Prager 1948; Freiberger 1956) that in cases where it is impossible to satisfy all the boundary conditions, discontinuities may be introduced to give a solution. Moreover, in some cases which admit continuous solutions, it is possible to construct discontinuous solutions as a device to obtain approximate solutions more easily (Hodge 1950; Prager and Hodge 1951). If a line of discontinuity (" shock ") separates two regions of perfectly plastic material, then the following relation must be valid (Prager 1948):

$$\left.\begin{array}{l}\theta_1+\theta_2=2\alpha_{12},\\ \omega_2-\omega_1=\sin(\theta_1-\theta_2).\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Here, subscripts 1 and 2 refer to opposite sides of the shock, α_{12} is the angle between the shock and the negative y-axis. The first equation (1) means that the shock must bisect the intersecting slip lines of either family. Conservation of mass requires the velocity component normal to the shock to be continuous across the shock.

To estimate the load at which sufficient plastic flow has taken place in the beam for pronounced overall yield to take place (" collapse ") we make use of the theorems of limit analysis (Hodge 1950; Prager and Hodge 1951) which are true for continuous and discontinuous fields of stress and velocity. These theorems are:

Theorem 1.—Collapse will not occur until the largest values of the surface tractions are reached for which it is possible to find a statically admissible stress field.

Theorem 2.—Collapse will occur under the smallest values of the surface tractions for which it is possible to find a kinematically admissible velocity field.

A stress field is called statically admissible if it satisfies the equations of equilibrium and the boundary conditions and nowhere violates the yield condition.

A velocity field is called kinematically admissible if it satisfies the incompressibility condition and the boundary conditions and if the rate at which the applied tractions do work on the velocities of their points of application equals (or is greater than) the rate of internal energy dissipation. For a surface of discontinuity, the rate of internal energy dissipation per unit area at any point of the surface is (Shield and Drucker 1953)

$$D=kv, \qquad (4)$$

where v is the change in the tangential velocity component across the surface of discontinuity at the point.

III. Velocity Field—Upper Bound for Limit Pressure

The slip line field of the right-hand side of Figure 2 will be assumed for the velocity distribution; the surfaces of the dies being perfectly rough, the slip lines meet the dies at right angles. It is also seen that the first of equations (3) is satisfied across each shock.

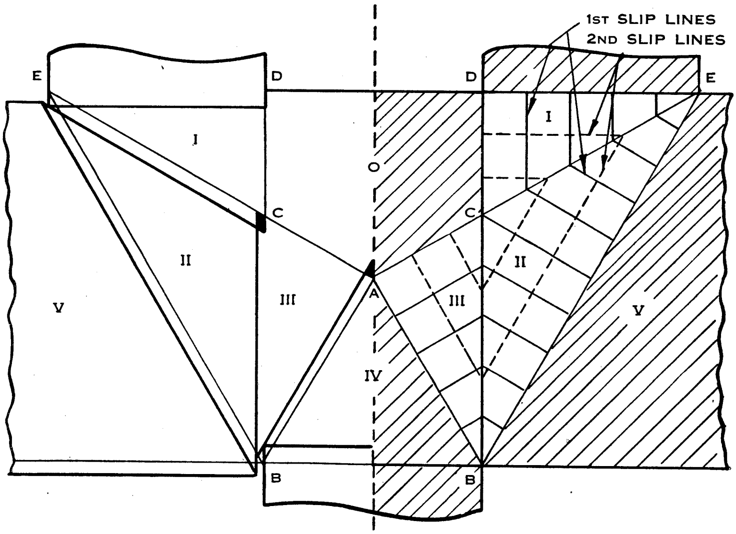

Fig. 2.—Left-hand side: boundaries after incipient plastic flow. Right-hand side: slip line field at incipient plastic flow, shaded areas rigid.

Let the pressure per unit length on each of the upper dies be pk; for the lower die it is therefore $2pk$, by equilibrium. Since the assumed relative velocity between upper and lower dies is irrelevant, let the dies move with unit velocity, the upper dies downwards, the lower die upwards.

Let U and V be the velocities along first and second slip lines, respectively, and denote the region to which they refer by a subscript 1, 2, or 3. With reference to Figure 2, applying the condition that normal velocity components must be continuous across lines of discontinuity and applying the boundary condition that dies move with unit velocity, we obtain :

across EC : $U_1 \cos 30 + V_1 \sin 30 = U_2 \cos 30 + V_2 \sin 30$,
across CB : $U_3 \cos 30 - V_3 \sin 30 = U_2 \cos 30 - V_2 \sin 30$,
across AB : $U_3 = v_4 \sin 30 = 1 \sin 30$,
across CA : $V_3 = 0$, across DE : $U_1 = 1$, across DC : $V_1 = 0$.

It follows that $U_1=1,\ U_2=1,\ U_3=\frac{1}{2}$,
$V_1=0,\ V_2=\frac{1}{2}\sqrt{3},\ V_3=0$,

or, in terms of velocities along coordinate directions, with subscripts referring to regions :

$$u_0=0,\ u_1=0,\ u_2=-\tfrac{1}{4}\sqrt{3},\ u\ =-\tfrac{1}{4}\sqrt{3},\ u_4=0,\ u_5=u_2=-\tfrac{1}{4}\sqrt{3},$$
$$v_0=0,\ v_1=-1,\ v_2=\tfrac{3}{4},\quad v_3=\tfrac{1}{4},\quad v_4=1,\ v_5=-v_2=-\tfrac{3}{4}.$$

These values for the velocities are used in the left-hand side of Figure 3 to indicate the deformation of the boundaries at the moment of pronounced plastic flow. For this velocity field, clearly, $\theta_1=180°$, $\theta_2=60°$, $\theta_3=-60°$.

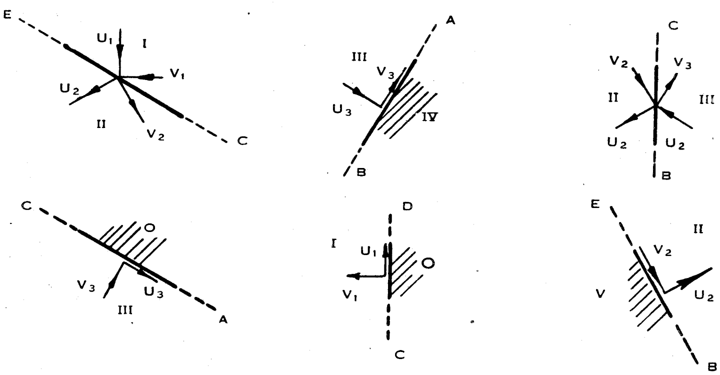

Fig. 3.—Conditions at boundaries of plastic regions.

There will be a value of $p=p^*$ such that if $p \geqslant p^*$ the velocity field of Figure 3 is a kinematically admissible velocity field and will yield an upper bound for the limit pressure. Denoting by $\overline{AB}$ etc., the lengths of the boundaries AB etc., and by v_{AB} etc., the change in the tangential velocity components across the lines of discontinuity, we have, for the rate of internal energy dissipation,

$$D=2k(\overline{AB}\,.\,v_{AB}+\overline{AC}\,.\,v_{AC}+\overline{BC}\,.\,v_{BC}+\overline{CD}\,.\,v_{CD}+\overline{EC}\,.\,v_{EC})$$
$$=2k\left(1\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}\cdot\frac{1}{2}+\frac{2}{\sqrt{3}}\cdot 1+\frac{2}{\sqrt{3}}\cdot\frac{1}{2}\right)=4(\sqrt{3})k. \quad . (5)$$

The rate of external work done by the dies is

$$W=4apk. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Equating (5) and (6) it is found that the rate of external work done will not be less than the rate of dissipation of internal energy if the value of p is not less than

$$p^* = \sqrt{3}, \qquad (7)$$

which is an upper bound for the limit pressure, by theorem 2.

The same upper bound is found by assuming the following displacement mode: the single die doubly shears a section of the beam and pushes it a small distance between the dies.†

IV. Stress Field—Lower Bound for Limit Pressure

To find, with the help of theorem 1, a lower bound for the limit pressure, a statically admissible stress field must be found. This can be done by using a discontinuous stress field for a truncated wedge (see Winzer and Carrier 1948), employed by Shield and Drucker (1953). This pattern is reproduced in Figure 4 (a) and the modified pattern for the present problem in Figure 4 (b). The latter is obtained from the former by superimposing a tension of value k in the region vertically below what is line DD in Figure 4 (b).

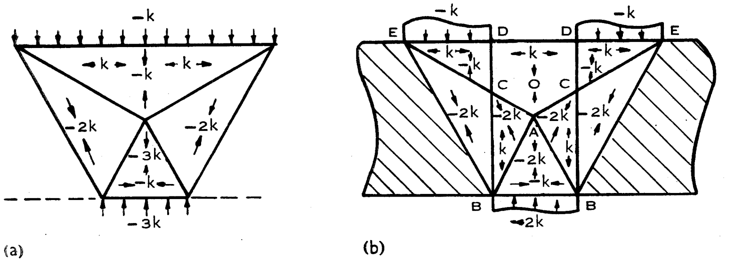

Fig. 4 (a).—Stress pattern for truncated wedge (Winzer and Carrier 1951).

Fig. 4 (b).—Modified stress pattern for beam. Compression negative, tension positive.

This stress field, statically admissible for the present problem, at once gives for p the value

$$p^{**} = k \qquad (8)$$

as a lower bound for the limit pressure on the upper dies.

Hence, combining the results of this section and those of Section III, we find, if P_L is the limit pressure,

$$k < P_L < k\sqrt{3}. \qquad (9)$$

V. Conclusion

Under the assumption of perfect plasticity (flow under constant maximum shear stress) and plane strain, bounds are obtained for the collapse load of a beam subjected to compression by three rough rigid dies.

A discontinuous velocity field, satisfying velocity boundary conditions, is used to determine an upper bound to the limit pressure, i.e. the pressure per

† This comment is due to Dr. J. M. Alexander, English Electric Company, England.

unit length on the upper dies sufficient to cause plastic flow throughout the cross section of the beam. This is found to have the value $k\sqrt{3}$. The velocity field is also used to give an indication of the deformation of the boundaries at the moment of pronounced plastic flow.

A discontinuous stress field, satisfying stress boundary conditions, is found to determine a lower bound for limit pressure. The value of this lower bound is k.

The difference between the upper and lower bounds is due to the different fields assumed for the velocity and stress solutions.

VI. Acknowledgment

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VII. References

Freiberger, W. (1956).—The elastic-plastic torsion of circular ring sectors. *Quart. Appl. Math.* **14**: 259.

Hodge, P. G. (1950).—Approximate solutions of problems of plane plastic flow. *J. Appl. Mech.* **17**: 257.

Prager, W. (1948).—"Discontinuous Solutions in the Theory of Plasticity." Courant Anniversary Vol. (Interscience Publishers: New York.)

Prager, W., and Hodge, D. G. (1951).—"Theory of Perfectly Plastic Solids." (John Wiley and Sons: New York.)

Shield, R. T., and Drucker, D. C. (1953).—The application of limit analysis to punch-indentation problems. *J. Appl. Mech.* **20**: 453.

Winzer, A., and Carrier, G. F. (1948).—The interaction of discontinuity surfaces in plastic fields of stress. *J. Appl. Mech., Trans. A.S.M.E.* **70**: 261.