AN APPLICATION OF THE DIFFUSION EQUATION TO VISCOUS MOTION WITH A FREE SURFACE*

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The recent development of methods for the numerical solution of the diffusion equation when the diffusivity is concentration-dependent enables certain phenomena, previously not amenable to quantitative analysis or ineffectively described by empirical functions, to be studied more fully. This communication points out an example of concentration-dependent diffusion which does not appear to have previously received attention.

The steady one-dimensional viscous flow of a liquid over any plane surface in a gravitational field may be expressed as

$$q=-D(\mathrm{d}y/\mathrm{d}x-\tan\alpha), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where q is the discharge per unit width normal to the direction of the motion, y is the depth of the fluid, x is the horizontal ordinate, α is the angle the plane

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makes with the horizontal, and D is a function of y depending on the properties of the surface. Thus for a smooth surface $D=(gy^3 \cos \alpha)/3\nu$, where g is the acceleration due to gravity and ν is the kinematic viscosity of the liquid; for a densely vegetated surface (and depth of flow rather less than the height of the vegetation) with the permeability at height y' equal to K,

$$D=\frac{g \cos \alpha}{\nu}\int_0^y K\mathrm{d}y'.$$

For the motions we study here the flow velocity is so small that we may neglect velocity head. If we also make the reasonable, but not exact, assumption

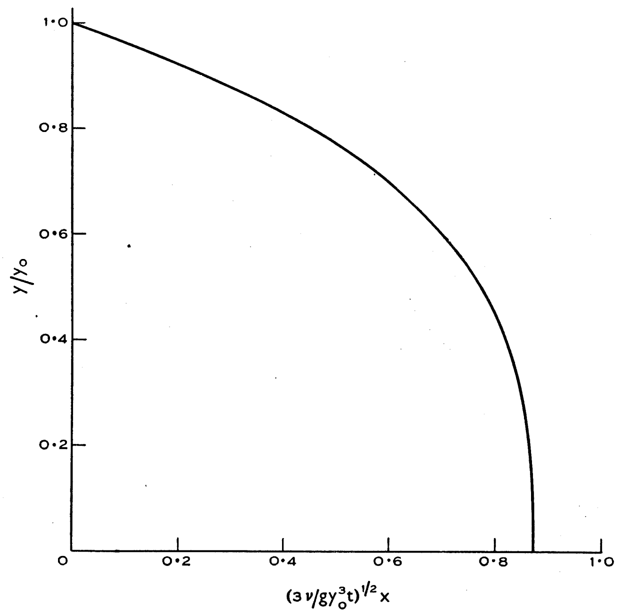

Fig. 1.—The advance of a viscous translatory wave over a smooth dry horizontal surface.

that all motion is parallel to the plane surface, (**1**) is valid for unsteady motion. Then, applying the requirement of continuity to (**1**), we obtain

$$\frac{\partial y}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial y}{\partial x}\right)-\tan\,\alpha\frac{\partial D}{\partial x}. \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{2})$$

For the case of a horizontal surface, $\tan \alpha=0$, and (**2**) becomes the well-known diffusion equation with diffusivity concentration-dependent. A rapid method of solving this equation for certain simple but important conditions has been given by Philip (1955). For the case of a sloping plane (**2**) is a generalization of the Fokker-Planck equation. Philip (1957) gives a quick method of solving this equation—for the same simple conditions. These are

$$t=0,\ x\geqslant 0,\ y=y_n,\quad t\geqslant 0,\ x=0,\ y=y_0. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{3})$$

Conditions (**3**) correspond to the sudden imposition of a liquid depth y_0 at $x=0$ on a surface originally covered by a depth of liquid y_n. We confine the remainder of this communication to flow over the horizontal surface.

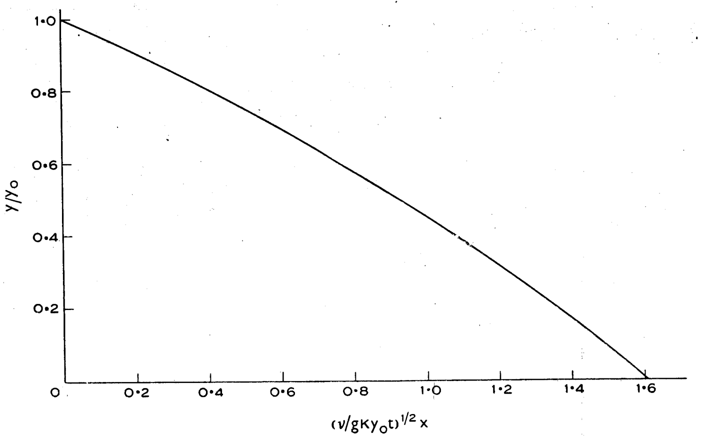

Fig. 2.—The advance of a viscous translatory wave over a vegetated dry horizontal surface with K, the permeability of the vegetation, constant.

An interesting feature of the solution of (**2**) subject to (**3**) is that, provided $D(y_n)=0$, the infinite " tail " common in the mathematics of diffusion does not occur, and $y=y_n$ for some finite value of $xt^{-\frac{1}{2}}$. For the important case

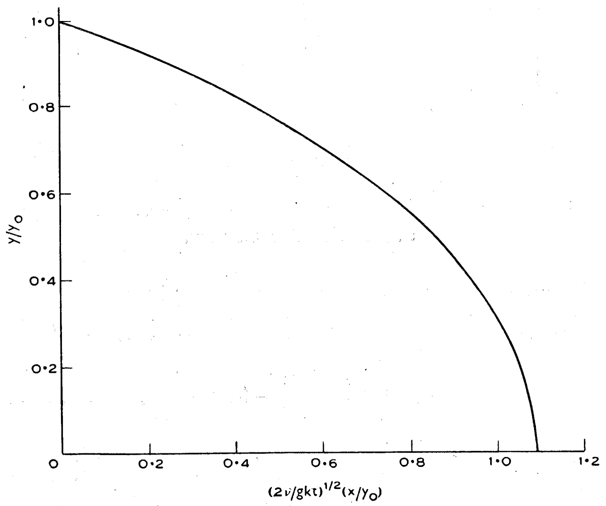

Fig. 3.—The advance of a viscous translatory wave over a vegetated dry horizontal surface with $K=ky$.

$y_n=0$, this condition is always met, so that the analysis predicts the advance of a wave front over an initially dry surface. However, for $y_n>0$, $D(y_n)\neq 0$, so that $y\to y_n$ as $x\to\infty$, i.e. the usual infinite tail occurs. Thus a blind observer at x equipped with very sensitive depth-measuring apparatus would detect the change in boundary conditions *instantaneously* for the case of $y_n > 0$, but for $y_n = 0$ the disturbance would not be detected until a finite time (proportional to x^2 when $\alpha = 0$) after $t = 0$.

Solution of three cases of (2) subject to (3) (for $\alpha = 0$, $y_n = 0$) are given in Figures 1–3. Figure 1 is for flow over a smooth surface. Figure 2 depicts flow over a densely vegetated surface with the permeability of the vegetation, K, constant between depths $y = 0$ and $y = y_0$. Figure 3 gives the result of an intermediate case in which the permeability increases linearly with y from zero at the surface to a value ky_0 at $y = y_0$. This case is relevant where the density of vegetation decreases with height. It should be pointed out that the numerical method employed to obtain these solutions is equally applicable to any form of variation of permeability with y.

This communication gives the results of a preliminary study of the problem. It is hoped that it will be possible later to treat these matters in more detail. However, certain conclusions, important in fluid mechanics and hydrology, may be presented at this stage:

(i) The advance of a viscous translatory wave over a dry or initially submerged surface may be analysed by treating the problem as one in diffusion. Since this is a class of motion which has hitherto eluded analysis, this appears to be a significant advance in this sector of fluid mechanics.

(ii) Study of the hydraulics of overland flow, important in storm hydrology and the hydraulics of surface methods of irrigation, has long been retarded by the need for a suitable measure of the hydraulic properties of vegetative cover. The concept of a height-variable permeability seems an acceptable means of representing these properties. Then, if conditions (3) are realized on the surface to be measured and the liquid profile observed at some time t, it is a simple matter to deduce D as a function of y. Then differentiating D with respect to y will give the required $K = K(y)$.

References

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