RECURRENCE FORMULAE FOR THE REFLECTANCE AND TRANSMITTANCE OF MULTILAYER FILMS WITH APPLICATIONS*

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Summary

Formulae are developed which, by repeated application, give the changes in amplitude and phase when light passes normally through any number of layers of homogeneous absorbing or transparent materials. These formulae are suitable for numerical computation and provide a direct route to the analytical expressions for the reflectance and transmittance in particular cases. Special attention is given to the case in which the layers are a quarter wave or multiples of a quarter wave in thickness.

I. INTRODUCTION

The theory of reflection and transmission of light by multilayer films has been studied by a number of workers, mainly in the last 15 years. These studies lead to results which are equivalent but differ somewhat in their complexity and in the ease with which they may be applied. The results obtained by the various authors may be classified, as regards their method of derivation, in several ways. Starting with Fresnel's well-known expressions for the reflectance and transmittance at the boundary between two transparent isotropic media, the corresponding quantities may be derived for a single layer, taking into account internal multiple reflections. Using these results the effect of adding another layer may be obtained by the same method. It is then possible to obtain the law by which the reflectance and transmission from a film of nlayers is obtained from those of a film of n-1 layers, so that the final result may be established in terms of the known constants of the film by a step-by-step process. A second procedure is to set down the 2(n+1) electromagnetic boundary conditions for the (n+1) surfaces involved. These provide solutions for 2(n+1)unknowns, two of which yield the reflectance and transmittance. Another much favoured procedure is to apply the method of "optical impedance".

Again, the results may be classified according to the form in which they are ultimately expressed. Thus they may be given as a recurrence formula, they may be expressed in matrix form, or the solution may be graphical.

In the present paper a new method, particularly applicable to the design of filters and of high or low reflectance films, is developed for discussing the optical properties of multilayer films. It is based on a generalized form of

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Stokes's relations between the reflectance and transmittance at a boundary, derivable from Maxwell's equations. Expressions are obtained for the amplitude, reflectance, and transmittance of a film consisting of n layers of arbitrary thickness in terms of a quantity for which the name "film index " is suggested. In the formula derived from the reflectance of a composite film at normal incidence this quantity is the complete analogue of the refractive index appearing in the corresponding expression for the reflectance at the boundary between two media. The film index for n layers is given in terms of that for n-1 layers and thus its value in terms of the film constants may be obtained by repeated applications. In the case of transparent layers the successive steps of the process are self-checking at each stage. In particular, the film index and hence the optical properties receive very simple expression in the important case in which the thickness of each layer is any multiple of a quarter wavelength. This and its applications are treated in some detail.

II. FUNDAMENTAL EQUATIONS

Consider two parallel layers X and Y (which may themselves be compounded of a number of layers of transparent or absorbing materials) separated by a medium of thickness d and bounded externally by media A and B respectively.

If a plane light wave of length λ (*in vacuo*) travelling in A is incident normally on X with amplitude unity just prior to incidence part of this light will be transmitted through X and will suffer multiple reflections between X and Y.

Denote by K the amplitude of the resultant wave leaving X and travelling towards Y, and by Δ its phase with respect to that of the incident wave.

Let the amplitude reflectance and transmittance of X for light incident internally (i.e. directed from Y to X) be r_x and t_x and the corresponding phase changes φ_x and ψ_x and let $r'_x > t_y$, φ'_x , and ψ'_x be the corresponding quantities for light incident externally on Y. Suppose also a similar notation, r_y , t_y , etc. to be used with regard to Y. Finally let r, t, φ , and ψ be the corresponding quantities for the complete film in respect of light incident on X. Then, denoting the refractive index and the absorption index of the medium between X and Y by n and k respectively,

$$t \exp (i\psi) = K \exp (i\Delta) \exp (-2\pi n k d/\lambda) \exp (-i2\pi n d/\lambda) t_y \exp (i\psi_y)$$
$$= K t_y \exp \{i(\Delta + \psi_y - p')\}, \qquad (1)$$

where

$$p' = (1 - ik)p = (1 - ik)2\pi nd/\lambda, \qquad \dots \qquad (2)$$

and

$$r \exp (\mathbf{i}\varphi) = K \exp (\mathbf{i}\Delta) \exp (-4\pi nkd/\lambda) \exp (-i4\pi nd/\lambda)t_x r_y \exp \{\mathbf{i}(\varphi_y + \psi_x)\} + r'_x \exp (\mathbf{i}\varphi'_x) = K t_x r_y \exp \{\mathbf{i}(\Delta + \varphi_y + \psi_x - 2p')\} + r'_x \exp (\mathbf{i}\varphi'_x). \quad \dots \dots \quad (3)$$

The resultant wave train $Ke^{i\Delta}$ is immediately obtained as the sum of the infinite series

$$K \exp (i\Delta) = t_x \exp (i\psi_x) + t_x r_x r_y \exp \{i(\psi_x + \varphi_x + \varphi_y - 2p')\} + t_x r_x^2 r_y^2 \exp \{i(\psi_x + 2\varphi_x + 2\varphi_y - 4p')\} + \dots$$

that is,

$$K \exp (\mathrm{i}\Delta) = \frac{\dot{t_x} \exp (\mathrm{i}\psi_x)}{1 - r_x r_y \exp (\mathrm{i}\theta')}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

where

$$\theta' = \varphi_x + \varphi_y - 2p'. \qquad (5)$$

By means of (4) and (5), (1) and (3) may be written respectively in the form

and

$$r \exp(i\varphi) = \frac{t_x t_x r_y \exp\{i(\theta' - \varphi_x + \psi_x' + \psi_x)\}}{1 - r_x r_y \exp(i\theta')} + r_x \exp(i\varphi_x'), \quad \dots \dots \quad (7)$$

which are basic formulae for the calculation for the transmittance, reflectance, and phase changes for light incident normally on a Fabry and Perot etalon when the corresponding quantities for the components X and Y are known and absorption is taken into account. Consider now the case in which X is the boundary between two media which may be absorbing. Fresnel's fundamental equations (as modified to take account of absorption) yield for such a boundary

$$\begin{array}{c} t_{x}t_{x}^{'}\exp\left\{i(\psi_{x}+\psi_{x}^{'})\right\}+r_{x}^{2}\exp\left(2i\varphi_{x}\right)=1, \\ r_{x}=r_{x}^{'}, \\ \varphi_{x}-\varphi_{x}^{'}=(2m+1)\pi, \end{array} \right\} \quad \dots \quad (8)$$

where *m* is any integer, positive or negative. These equations degenerate for the case of no absorption ($\varphi_x=0$ or π , $\psi_x=\psi'_x=0$) into the well-known Stokes's relations. By using (8), (7) may be written for this case

$$r \exp(i\varphi) = \frac{r_y \exp\{i(\theta' - \varphi_x)\} - r_x \exp(i\varphi_x)}{1 - r_x r_y \exp(i\theta')}. \quad \dots \qquad (9)$$

Suppose now that Y consists of (s-1) layers so that the complete etalon constitutes a film of s layers of refractive indices n_1, n_2, \ldots, n_s , absorption indices k_1, k_2, \ldots, k_s , and thicknesses d_1, d_2, \ldots, d_s , bounded by media B and A of refractive and absorption indices n_0, k_0 and n_{s+1}, k_{s+1} respectively. Then we may write (9) in the form

$$r_s \exp(\mathrm{i}\varphi_s) = \frac{r_{s-1} \exp\left\{\mathrm{i}(\theta'_s - \varphi_s)\right\} - r_s \exp(\mathrm{i}\varphi_s)}{1 - r_s r_{s-1} \exp\left(\mathrm{i}\theta'_s\right)}, \quad \dots \quad (10)$$

in which r_s and φ_s represent the amplitude reflectance and phase change of a film of s layers,

$$\begin{array}{c} \theta_{s}^{'} = \varphi_{s} + \varphi_{s-1} - 2p_{s}^{'}, \\ p_{s}^{'} = (1 - ik_{s})p_{s} = 2\pi n_{s}^{'}d_{s}/\lambda, \end{array} \right\} \qquad (11)$$

 n'_s being the complex refractive index $n_s(1-ik_s)$. From the electromagnetic theory (Drude 1933) we may write (treating the electric vector as the light vector)

and from (10) and (12) we obtain

$$\frac{1 - r_s \exp(i\varphi_s)}{1 + r_s \exp(i\varphi_s)} = \frac{n'_s [1 - r_{s-1} \exp\{i(\theta'_s - \varphi_s)\}]}{n'_{s+1} [1 + r_{s-1} \exp\{i(\theta'_s - \varphi_s)\}]}.$$
 (13)

Now let N_q be a complex function of n'_0, n'_1, \ldots, n'_q ; k_0, k_1, \ldots, k_q ; d_1, d_2, \ldots, d_q satisfying the relation

$$r_q \exp(i\varphi_q) = \frac{n'_{q+1} - N_q}{n'_{q+1} + N_q},$$
 (14)

in which r_q and φ_q are the amplitude reflectance and phase change for a part of the complete film formed by the q layers bounded by B and the medium of refractive and absorption indices n_{q+1} and k_{q+1} . It is assumed that the relation holds for any value of q between zero and s. Then by means of (13) and (14), after putting q=s, we obtain

$$\frac{N_{s}}{n_{s}'} = \frac{1 - r_{s-1} \exp\{i(\theta_{s}' - \varphi_{s})\}}{1 + r_{s-1} \exp\{i(\theta_{s}' - \varphi_{s})\}},$$

$$\frac{N_{s} - n'_{s}}{N_{s} + n'_{s}} = -r_{s-1} \exp \{i(\theta'_{s} - \varphi_{s})\} = -r_{s-1} \exp \{i(\varphi_{s-1} - 2p'_{s})\}, \quad .. \quad (15)$$

and finally by (14) after putting q=s-1

Thus, N_s is known if N_{s-1} is known. Now (14) is satisfied for q=0 by putting N_0 equal to the known quantity n'_o and (16) is similarly satisfied for s=1 so that N_q and N_s may be obtained by repeated applications of (16). As N_q has the same role in the calculation of the reflectance of a film by (14) as refractive index in the calculation of the light reflected at the boundary of an isotropic medium, we shall refer to N_q as the film index for a film of q layers. It is a quantity independent of the medium of incidence.

It is of interest to note that in discussing three-layer non-absorbing films King and Lockhart (1946) introduced a related quantity X which is the reciprocal of the film index. Returning now to equation (6) and assuming again, as in the

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or

derivation of (13), that Y consists of (s-1) layers of a complete film and that X is the boundary between the medium of incidence, A, and the adjacent layer, we may write this equation in the form

$$t_s \exp(i\psi_s) = \frac{t_x t_{s-1} \exp\{i(\psi_{s-1} + \psi_x - p_s')\}}{1 - r_x r_{s-1} \exp(i\theta_s')}, \quad \dots \dots \quad (17)$$

in which the significance of t_s , ψ_s , t_{s-1} , and ψ_{s-1} is obvious. Again from electromagnetic theory

$$t_x' \exp(i\psi_x) = \frac{2n_{s+1}}{n_{s+1}' + n_s'},$$
 (18)

so that substituting in (17) for t_x , r_x , and r_{s-1} from (18), (12), and (15) respectively we obtain

By repeated applications of (19)

$$\frac{t_s \exp (i\psi_s)}{t_0 \exp (i\psi_0)} = \exp \left\{ -i(p'_1 + p'_2 + \dots + p'_s) \right\} \frac{n'_{s+1}}{n_0} \prod_{1}^{s} \frac{N_q + n'_q}{N_q + n'_{q+1}}, \dots (20)$$

in which

$$t_0 \exp(i\psi_0) = 2n'_0/(n'_1 + n'_0),$$

so that

$$t_{s} \exp (i\psi_{s}) = 2 \exp \{-i(p_{1}' + p_{2}' + \dots + p_{s}')\} \frac{n_{s+1}'}{n_{1}' + n_{0}'} \prod_{1}^{s} \frac{N_{q} + n_{q}'}{N_{q} + n_{q+1}'}.$$

When there is no absorption equations (16), (14), and (21) take the form (with s replacing q in (14)):

$$\frac{N_s - n_s}{N_s + n_s} = \frac{N_{s-1} - n_s}{N_{s-1} + n_s} \exp((-2ip_s), \qquad (22)$$

$$r_s \exp(i\varphi_s) = \frac{n_{s+1} - N_s}{n_{s+1} + N_s}, \qquad (23)$$

$$t_{s} \exp (i\psi_{s}) = 2 \exp \{-i(p_{1}+p_{2}+\ldots+p_{s})\} \frac{n_{s+1}}{n_{1}+n_{0}} \prod_{1}^{s} \frac{N_{q}+n_{q}}{N_{q}+n_{q+1}},$$
(24)

in which n_s , n_q , and p_s are now real but N_s and N_q are still complex.

The reflectance R_s is thus given, for this case, by

and the transmittance T_s by

$$T_{s} = \frac{n_{0}}{n_{s+1}} t_{s}^{2} = \frac{4n_{0}n_{s+1}}{(n_{1}+n_{0})^{2}} \prod_{1}^{s} \left| \frac{N_{q}+n_{q}}{N_{q}+n_{q+1}} \right|^{2}.$$

From

$$\left. \frac{n_q - N_{q-1}}{n_q + N_{q-1}} \right|^2 = \left| \begin{array}{c} \frac{n_q - N_q}{n_q + N_q} \right|^2,$$

which follows from (22), and from the identities

$$1 - \left| \begin{array}{c} \frac{n_{q+1} - N_q}{n_{q+1} + N_q} \end{array} \right|^2 = \frac{4n_{q+1}A_q}{\mid n_{q+1} + N_q \mid 2}, \quad 1 - \left| \begin{array}{c} \frac{n_q - N_q}{n_q + N_q} \end{array} \right|^2 = \frac{4n_qA_q}{\mid n_q + N_q \mid 2}$$

in which A_q is the real part of N_q , it follows that

$$\frac{n_{q+1}}{n_q} \left| \begin{array}{c} n_q + N_q \\ n_{q+1} + N_q \end{array} \right|^2 = \frac{1 - |(n_{q+1} - N_q)/(n_{q+1} + N_q)|^2}{1 - |(n_q - N_{q-1})/(n_q + N_{q-1})|^2}.$$

Since $N_0 = n_0$, this result leads to

$$rac{n_{s+1}}{n_1}\prod\limits_1^s \left| \; rac{n_q\!+\!N_q}{n_{q+1}\!+\!N_q} \;
ight|^2 \!=\! rac{(n_1\!+\!n_0)^2}{4n_1n_0} \!\left\{ 1\!-\!\left| \; rac{n_{s+1}\!-\!N_s}{n_{s+1}\!+\!N_s} \;
ight|^2
ight\}\!\!,$$

which, on substitution in the expression for T_s above, yields

$$T_{s}=1-\left|\frac{n_{s+1}-N_{s}}{n_{s+1}+N_{s}}\right|^{2}$$
. (26)

Thus, (25) and (26) are in agreement with the energy principle, namely,

$$R_{s} + T_{s} = 1.$$

Returning now to equation (9) and considering the case of no absorption so that θ' is replaced by θ and $\varphi_x=0$ or π we have, on multiplying by the conjugate quantity $re^{-i\varphi}$,

$$R = r^{2} = \frac{r_{x}^{2} + r_{y}^{2} - 2r_{x}r_{y}\cos\theta}{1 + r_{x}^{2}r_{y}^{2} - 2r_{x}r_{y}\cos\theta}, \quad \dots \dots \dots \dots \dots (27)$$

which gives the reflectance in the absence of absorption with the additional restriction which applies to (9), namely, that X is the boundary between two isotropic media. It will be shown now that this expression for R is true in the more general case in which Y is any plane reflector (which may be partly or completely absorbing) and X any non-absorbing reflector, the separating medium being non-absorbing.

Let R, T, and A be the reflectance, transmittance, and absorptance of the complete combination and R_y , T_y , and A_y the corresponding quantities for Y. Then, if C is the radiant flux for unit area reaching Y for unit incident radiant flux on X, $T=CT_y$, $A=CA_y$, A=1-T-R, $A_y=1-T_y-R_y$, so that

$$T/A = T_u/A_u$$
 or $T/(1-T-R) = T_u/(1-T_u-R_u)$,

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that is (compare Dufour 1948),

$$T/(1-R) = T_y/(1-R_y).$$
 (28)

Now the well-known expression for the transmittance of a Fabry and Perot etalon gives

$$T = \frac{T_{x}T_{y}}{1 + R_{x}R_{y} - 2(R_{x}R_{y})^{\frac{1}{2}}\cos\theta}, \quad \dots \dots \dots \dots \dots (29)$$

in which R_x and T_x are the reflectance and transmittance of X and θ has the same significance as in (5) for the special case of no absorption. On substituting this value for T in (28) and replacing T_x by $(1-R_x)$ we obtain

$$R = \frac{R_x + R_y - 2(R_x R_y)^{\frac{1}{2}} \cos \theta}{1 + R_x R_y - 2(R_x R_y)^{\frac{1}{2}} \cos \theta}, \quad \dots \dots \dots \dots \quad (30)$$

which is of the same form as (27).

The application of these formulae to multilayer films is obvious. For example, Y could be a metallic layer on a backing medium, for which the quantities R_y , φ_y , and T_y are known, and X a non-absorbing film of s layers for which R_x and φ_x may be calculated from (22) and (23). R and T are then obtained from (30) and (28) respectively.

III. METHOD OF APPLYING THE FORMULAE

Consider first formula (22) for non-absorbing material. If used in the form shown it will lead to an expression for N_s in terms of the quantities n_q , $\cos 2p_q$, and $\sin 2p_q$ (q=1, 2, ..., s). However, (22) may be expressed in the form

$$\frac{N_s}{n_s} = \frac{N_{s-1} + in_s \tan p_s}{n_s + iN_{s-1} \tan p_s}, \quad \dots \quad (31)$$

which gives a value for N_s in terms of the quantities n_q and $\tan p_q$. For the purposes of reduction write

$$\frac{N_q}{n_q} = \frac{A_q + iB_q}{C_q + iD_q}, \quad \dots \quad (32)$$

with the condition that

 $A_0 = C_0 = 1, B_0 = D_0 = 0.$ (33)

Thus in conformity with (31) and (32) (with s replaced by q+1) we may write

$$\begin{array}{c} A_{q+1} = n_q A_q - n_{q+1} D_q \tan p_{q+1}, \\ B_{q+1} = n_q B_q + n_{q+1} C_q \tan p_{q+1}, \\ C_{q+1} = n_{q+1} C_q - n_q B_q \tan p_{q+1}, \\ D_{q+1} = n_{q+1} D_q + n_q A_q \tan p_{q+1}, \end{array} \right) \qquad (34)$$

and from (34)

$$\begin{array}{c} A_{q+1} \tan p_{q+1} = D_{q+1} - n_{q+1} D_q (1 + \tan^2 p_{q+1}), \\ B_{q+1} \tan p_{q+1} = n_{q+1} C_q (1 + \tan^2 p_{q+1}) - C_{q+1}. \end{array} \right\} \quad \dots \quad (35)$$

Equations (34) combined with conditions (33) are very suitable for numerical computation of N_s and (35) are useful checking formulae.

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In particular, the first few values of A_s , B_s , C_s , and D_s , obtained from (33) and (34), are

$$A_{1}=n_{0}, B_{1}=n_{1} \tan p_{1}, C_{1}=n_{1}, D_{1}=n_{0} \tan p_{1}; \dots (36)$$

$$A_{2}=n_{0}(n_{1}-n_{2} \tan p_{1} \tan p_{2}), B_{2}=n_{1}(n_{1} \tan p_{1}+n_{2} \tan p_{2}), \\C_{2}=n_{1}(n_{2}-n_{1} \tan p_{1} \tan p_{2}), D_{2}=n_{0}(n_{2} \tan p_{1}+n_{1} \tan p_{2}); \\\dots (37)$$

$$A_{3}=n_{0}n_{2}(n_{1}-n_{2} \tan p_{1} \tan p_{2})-n_{0}n_{3} \tan p_{3}(n_{2} \tan p_{1}+n_{1} \tan p_{2}), \\B_{3}=n_{1}n_{2}(n_{1} \tan p_{1}+n_{2} \tan p_{2})+n_{1}n_{3} \tan p_{3}(n_{2}-n_{1} \tan p_{1} \tan p_{3}), \\\dots$$

$$C_{3} = n_{1}n_{3}(n_{2} - n_{1} \tan p_{1} \tan p_{2}) - n_{1}n_{3} \tan p_{3}(n_{1} \tan p_{1} + n_{2} \tan p_{2}),$$

$$D_{3} = n_{0}n_{3}(n_{2} \tan p_{1} + n_{1} \tan p_{2}) + n_{0}n_{2} \tan p_{3}(n_{1} - n_{2} \tan p_{1} \tan p_{2}).$$
(38)

Allowing for a change of notation the values of N_2 and N_3 obtained from (32) by substitution from (37) and (38) respectively agree with the values of 1/X given by King and Lockhart (1946, 1947).

In the case of absorbing material it is easily shown by means of (16) and using the symbol h_s for the quantity $\tanh(k_s p_s)$ that (31) is replaced by

$$\frac{N_s}{n_s(1-ik_s)} = \frac{N_{s-1} + n_s(h_s + k_s \tan p_s) + i\{N_{s-1}h_s \tan p_s + n_s(\tan p_s - k_sh_s)\}}{n_s + h_s(k_s \tan p_s + N_{s-1}) + i\{N_{s-1} \tan p_s + n_s(h_s \tan p_s - k_s)\}}.$$
(39)

Thus, for a single-layer absorbing film of refractive index n and absorption index k deposited on non-absorbing material of refractive index n_0 , the film index N is obtained directly from (39) by setting $N_{s-1}=n_0$, $n_s=n$, $k_s=k$, $p_s=2\pi nd/\lambda$, $h_s=\tanh(2\pi nkd/\lambda)$. Alternatively the expression for N_s in the case of absorption may be obtained by developing the quantities A_q , B_q , C_q , and D_q as indicated above in (34) and replacing n_q and p_q in the final result by the complex quantities n'_q and p'_q . Other forms of the expression for N_s are obtained, involving terms of the type exp $(-2k_qp_q)$ by using (16) directly.

IV. QUARTER-WAVE LAYERS

When there is no absorption and all the layers have equivalent thicknesses of one or more quarter-waves, $k_s=0$ and $n_s d_s=\lambda/4$ for all values of s. Then (22) takes the form

$$\frac{N_{s} - n_{s}}{N_{s} + n_{s}} = \frac{n_{s} - N_{s-1}}{n_{s} + N_{s-1}},$$

 \mathbf{or}

$$N_s N_{s-1} = n_s^2$$
. (40)

Since N_0 is the real quantity n_0 , N_s is real. Equation (23) then gives

and

$$R_{s} = \left(\frac{n_{s+1} - N_{s}}{n_{s+1} + N_{s}}\right)^{2}.$$
 (42)

Also (24) and (26) give

$$T_{s} = \frac{n_{0}}{n_{s+1}} t_{s}^{2} = 1 - \left(\frac{n_{s+1} - N_{s}}{n_{s+1} + N_{s}}\right)^{2}, \\ \psi_{s} = -(p_{1} + p_{2} + \ldots + p_{s}). \qquad (43)$$

The film index N_s is immediately obtained from (40). Thus for the cases when s=2m and s=2m+1 we have respectively

The application of these formulae to high-reflectance films and interference filters on the one hand and to anti-reflection films will be considered separately.

V. HIGH-REFLECTANCE FILMS

From equation (42) it is obvious that the reflectance of a film may be made as close to unity as required by making the film index N_s either sufficiently large or sufficiently small and from (44) and (45) it will be seen that either of these alternatives can be satisfied by a proper selection of the refractive indices n_1, n_2 , etc. and the number of layers. The method of making high-reflecting films composed of quarter-wave layers of non-absorbing materials, the alternate layers having the same refractive index, is well known. The usual practice is to have an odd number of layers of two materials of high and low index respectively, the even layers being of low and the odd of high refractive index. From (45) it is evident that this corresponds to a high value of the film index. However, it follows from (44) that a high value of the film index can also be obtained by having an even number of layers with the reverse sequence of refractive indices. Let us denote the high and low refractive indices by h and l. Then on comparing the two cases just mentioned it is easily shown that the fractional increase in film index on increasing the number of layers from 2mto (2m+1) exceeds that on increasing the number from (2m-1) to 2m if hl is greater than n_0^2 . Also, if we write

$$f = (h/l)^{2m}$$
,

equations (42), (44), and (45) give, with the sequence of layers $l, h, l, h, l, \ldots, l, h$,

 $\simeq 1 - \frac{4n_{s+1}}{fn_0}, \quad \dots \quad \dots \quad \dots \quad (47)$

$$R_{s} = R_{2m+1} = \left(\frac{fh^{2}/n_{0} - n_{s+1}}{fh^{2}/n_{0} + n_{s+1}}\right)^{2} \qquad (48)$$
$$\simeq 1 - \frac{4n_{0}n_{s+1}}{fh^{2}} \qquad (49)$$

Consider now the conditions for producing a high reflectance by making the film index very small with the use of the same materials for the layers. Clearly this requires an even number of layers with the sequence h, l, h, l, \ldots, h, l , or an odd number with the sequence $l, h, l, h, \ldots, l, h, l$. These alternatives give respectively

$$\simeq 1 - \frac{4n_0}{fn_{s+1}}, \quad \dots \quad (51)$$

and

$$R_{s} = R_{2m+1} = \left(\frac{n_{s+1} - l^{2}/n_{0}f}{n_{s+1} + l^{2}/n_{0}f}\right)^{2} \qquad (52)$$

$$\simeq 1 - 4l^2 / n_{s+1} n_0 f. \quad \dots \quad \dots \quad \dots \quad (53)$$

In the case of a multilayer interference filter consisting of two composite high-reflectance components separated by a spacing layer, the properties of either of the high-reflectance components are obtained on taking n_{s+1} to be the refractive index of the spacing layer. Thus equations (47) and (49) indicate that this refractive index should have as low a value as possible when the arrangement of the layers is that described in connexion with these equations. On the other hand when the arrangement is that described in connexion with (51) and (53) the separating medium should have as high a refractive index \sim s possible.

In the foregoing discussion of high-reflectance films it has been assumed that all materials involved are non-absorbing. However, the results obtained may be made to apply to the case in which the backing medium is partially or wholly absorbing, e.g. glass coated with a layer of silver. The reason for this is that the reflectance R_y and the phase change produced by reflection are the only properties of the boundary of this medium which are involved. Thus it is only necessary when applying equations (44) or (53) to such a case to write

$$n_0 = n_1 \frac{1 - R_y^{\frac{1}{2}}}{1 + R_y^{\frac{1}{2}}}$$
 (54)

(in which $R_{y^{\frac{1}{2}}}$ is taken as positive) and allow for the phase change by increasing the thickness of the first layer to a value given by

$$n_1 d_1 = \frac{\lambda}{4} + \frac{\varphi_y \lambda}{4\pi}.$$
 (55)

As an example of the use of formulae (46)-(53) consider the problem of estimating the number of layers required to make a multilayer film of specified reflectance.

REFLECTANCE AND TRANSMITTANCE OF MULTILAYER FILMS

If this is of the usual type to which equation (49) applies and if the reflectance is to differ from unity by a small quantity ε , then (49) gives

$$m \log (h/l) = \log 2 - \log h + \frac{1}{2} (\log n_{s+1} + \log n_0 - \log \varepsilon), \quad \dots \quad (56)$$

from which the number of layers, 2m+1, can be estimated. Suppose two films of this type are to be used to make a symmetrical filter, the materials employed being :

Glass		$n_0 = 1.51$
Cryolite		$l{=}1{\cdot}35$
Zinc sulphide		$h = 2 \cdot 30$
Separating medium.	crvolite 4	$n_{-1} = 1.35$

Then (56) gives

 $2m+1=2-4:3 \log \varepsilon, \ldots \ldots \ldots (57)$

so that seven layers are required for $\varepsilon = 0.06$. Also a filter which transmits one order only in the visible range corresponding to a wavelength λ has a band width d λ (at half peak intensity) given by (compare Tolansky 1948)

$$d\lambda/\lambda = (1-R)/\pi R^{\frac{1}{2}} \simeq \varepsilon/\pi,$$

where R is the reflectance of both components. Equation (57) then gives, for light near the middle of the spectrum,

$$2m+1=16-4\cdot 3 \log A, \ldots (58)$$

where A is the value of $d\lambda$ in angstroms.

Thus, for the bandwidth to be 40 Å nine layers would be required in each of the two components giving 19 layers in all. By making the refractive indices of two or more adjacent layers equal it is obvious that equations (44) and (45) apply to films of which some of the layers have a thickness which is any multiple of a quarter-wave. Thus the filter just discussed may be considered as having two components separated by a medium of half-wave thickness and applying (44) to each component, or as one symmetrical film of 2m layers in which $n_1 = n_m$, $n_2 = n_{2m-1}, \ldots, n_m = n_{m+1}$ and the medium of incidence and the backing medium have the same refractive index, i.e. $n_{s+1} = n_0$. Equation (44) then gives, in the case of this alternative treatment, $N_{2m} = n_0$ so that the transmittance is unity as required. The fact that these equations may be applied in the case when some of the layers have a thickness of a half-wavelength has some application in connexion with anti-reflection films now to be discussed.

VI. ANTI-REFLECTION FILMS

The first requirement of an anti-reflection film is that it should have zero reflectance for a specified wavelength λ_0 . Thus if quarter-wave layers are used the film index as given by (44) or (45) should be unity. Then, for a single layer of refractive index n_1 , equation (45) yields the well-known condition

A second requirement is that in this spectral region the curve of reflectance against wavelength should be as flat as possible.

With a single layer this condition is not fulfilled particularly well and, where n_0 refers to low index glass, practical considerations prevent (59) from being satisfied exactly. The use of multiple layers gives greater flexibility as regards both these requirements. Trial and error methods seem to have been used mainly in selecting materials to satisfy the condition of a flat reflectance curve, so that some consideration of an analytical approach is justified. It is possible to obtain some idea of the relative flatness of the reflectance curves for different films from the value of the curvature, $\partial^2 R/\partial\lambda^2$, at the point $\lambda = \lambda_0$. This is, of course, only a guide to the flatness in general. In evaluating the curvature it is convenient to write, after putting $n_{s+1}=1$ in (25),

$$R_{s} = \frac{W_{s}^{2} + X_{s}^{2}}{Y_{s}^{2} + Z_{s}^{2}}, \qquad (60)$$

in which

$$W_s = n_s A_s - C_s, X_s = n_s B_s - D_s, Y_s = n_s A_s + C_s, Z_s = n_s B_s + D_s, \dots$$
 (61)

 A_s , B_s , C_s , and D_s being derived from (34).

Applying this to the case of a two-layer film in which each layer has an effective thickness of $\frac{1}{4}\lambda_0$ so that

$$p_1 = p_2 = p = \pi \lambda_0 / 2 \lambda, \ldots \ldots \ldots \ldots$$
 (62)

the condition of zero reflectance requires, as seen from equation (44),

and then, after some reduction, using (60), (62), and (63) and neglecting dispersion (see also equation (74)),

$$\left(\frac{\partial^2 R_2}{\partial \lambda^2}\right)_{\lambda=\lambda_0} = \frac{\pi^2}{4\lambda_0^2} \left(\frac{\partial^2 R_2}{\partial p^2}\right)_{\lambda=\lambda_0} = \frac{\pi^2}{8\lambda_0^2} \{n_1(n_0^{-\frac{1}{2}}+1)(n_0^{-\frac{1}{2}}-n_0n_1^{-2})\}^2. \dots (64)$$

The minimum (zero) value of this expression occurs when

but this cannot be realized on account of the low value of n_2 required. The best that can be done is to make the expression as small as possible using practicable values for n_1 and n_2 . King and Lockhart (1947) have drawn attention to the advantage of using a three-layer film in which n_1 and n_3 are of quarterwave thickness and n_2 of half-wave thickness. More general conclusions may be derived as follows. Suppose the first and third layers of the film are of quarter-wave thickness but the middle layer is of thickness $\frac{1}{2}m\lambda_0$ where m is any integer. Equation (44) shows that the film satisfies the condition of zero reflectance (or unit film index) for $\lambda = \lambda_0$ if

$$(n_1/n_3)^2 = n_0.$$
 (66)

Writing

a similar treatment to that used in deducing (64) yields, after eliminating n_3 by means of (66),

The case of m=1 corresponds to that discussed by King and Lockhart, who have given the reflectance curves in four cases. In all of these $n_0=1\cdot52$, $n_1=1\cdot80$, and $n_3=1\cdot44$, thus approximately satisfying (66); but n_2 is different in each case, having in turn the values $2\cdot4$, $2\cdot15$, $2\cdot00$, and $1\cdot80$. They show that the higher the value of n_2 the flatter the curve, the case of $n_2=2\cdot4$ giving a



Fig. 1.—Reflectance curves for films on glass, consisting of two layers of thickness $\frac{1}{4}\lambda$ separated by a layer of thickness $\frac{1}{2}m\lambda$, where m=0, 1, 2, 3, or 5. The refractive indices are: glass, 1.53; adjacent $\frac{1}{4}\lambda$ layer, 1.80; outer $\frac{1}{4}\lambda$ layer, 1.44; separating layer, 2.40.

particularly flat curve. It is of interest to note that for these four cases the curvature at $\lambda = \lambda_0$ is proportional to 0.16, 0.50, 0.84, and 1.43, in conformity with the results of King and Lockhart. Using the same values for the indices but setting m=0 in (68), the corresponding figure for the curvature is 0.56 which shows that of the four cases mentioned above only the first two give a flatter curve at $\lambda = \lambda_0$ than the corresponding two-layer curve. Again, on giving *m* the values 1, 2, 3, and 5 in succession and using for the indices in all

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cases the values $n_1=1.80$, $n_2=2.4$, $n_0=1.53$, the curvatures for the four values of *n* are proportional to 0.16, 0.0018, 0.097, 1.0. Thus optimum results are achieved with m=2, i.e. a full-wave layer separating the two quarter-wave layers.

Figure 1 gives reflectance curves centred on $\lambda_0 = 0.55 \text{ m}\mu$ for the above index sequence and for various values of m, including the corresponding two-layer case (m=0). It will be noted that the curve m=2 gives a smaller reflectance than the curve m=1 over a large part of the spectrum, though this no longer applies when λ is sufficiently less than λ_0 .

VII. A DIFFERENTIAL FORM OF THE FILM INDEX AND ITS APPLICATIONS

If we assume that dispersion is negligible equation (16) yields, after taking logarithms of both sides,

$$\frac{n'_{s}}{N_{s}^{2}-(n'_{s})^{2}} \frac{dN_{s}}{d\lambda} = \frac{n'_{s}}{N_{s-1}^{2}-(n'_{s})^{2}} \frac{dN_{s-1}}{d\lambda} - i\frac{dp'_{s}}{d\lambda}, \quad \dots \dots \quad (69)$$

so that the value of $dN_s/d\lambda$ may be obtained by repeated applications of (69) and (16). When there is no absorption, so that n'_s and p'_s are replaced by n_s and p_s , and all the layers are of quarter-wave thickness for a wavelength λ_0 , equation (69) reduces, with the aid of (40), to

$$\frac{\mathrm{d}N_s}{\mathrm{d}\lambda} = \frac{n_s}{N_{s-1}^2} \left\{ \frac{\mathrm{i}\pi\lambda_0}{2\lambda^2} (n_s^2 - N_{s-1}^2) - n_s \frac{\mathrm{d}N_{s-1}}{\mathrm{d}\lambda} \right\}, \quad \dots \dots \quad (70)$$

so that

$$\left(\frac{\mathrm{d}N_s}{\mathrm{d}\lambda}\right)_{\lambda=\lambda_0} = \left[\frac{n_s}{N_{s-1}^2} \left\{\frac{\mathrm{i}\pi}{2\lambda_0} (n_s^2 - N_{s-1}^2) - n_s \frac{\mathrm{d}N_{s-1}}{\mathrm{d}\lambda}\right\}\right]_{\lambda=\lambda_0} \dots (71)$$

Repeated applications of (71) and (40) (noting that $N_0 = n_0$) enable $(dN_s/d\lambda)_{\lambda=\lambda_0}$ to be determined for any value of s. Examples are

$$\begin{pmatrix} \frac{\mathrm{d}N_1}{\mathrm{d}\lambda} \end{pmatrix}_{\lambda=\lambda_0} = \frac{\mathrm{i}\pi n_1}{2\lambda_0 n_0^2} (n_1^2 - n_0^2), \qquad (72)$$

$$\begin{pmatrix} \frac{\mathrm{d}N_2}{\mathrm{d}\lambda} \end{pmatrix}_{\lambda=\lambda_0} = \frac{\mathrm{i}\pi n_2}{2\lambda_0 n_1^4} (n_0^2 n_2 - n_1^3) (n_1 + n_2). \qquad (73)$$

A useful application of (71) is the determination of the curvature of the reflectancewavelength curve of an anti-reflection film or a filter at the point of zero reflectance $(\lambda = \lambda_0)$. For this purpose the following supplementary analysis is required. From equation (25)

$$\frac{\mathrm{d}R_s}{\mathrm{d}\lambda} = 2 \left| \begin{array}{c} \frac{n_{s+1} - N_s}{n_{s+1} + N_s} \end{array} \right| \cdot \begin{array}{c} \frac{\mathrm{d}}{\mathrm{d}\lambda} \\ \end{array} \left| \begin{array}{c} \frac{n_{s+1} - N_s}{n_{s+1} + N_s} \end{array} \right|,$$

so that if

$$(R_s)_{\lambda=\lambda_0}=0,$$

 $\left(\frac{\mathrm{d}R_s}{\mathrm{d}\lambda}\right)_{\lambda=\lambda_0}=0,$

 then

and

$$\left(\frac{\mathrm{d}^2 R_s}{\mathrm{d}\lambda^2}\right)_{\lambda=\lambda_0} = 2 \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} \left| \frac{n_{s+1} - N_s}{n_{s+1} + N_s} \right| \right]_{\lambda=\lambda_0}^2.$$

But the first derivative of the modulus of a complex expression for a value of the variable that makes the modulus zero equals the modulus of the derivative of the expression for the same value of the variable. From this result and the condition of zero reflectance $(N_s = n_{s+1} = 1)$

$$\left(\frac{\mathrm{d}^2 R_s}{\mathrm{d}\lambda^2}\right)_{\lambda=\lambda_0} = 2 \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{n_{s+1}-N_s}{n_{s+1}+N_s}\right) \right|_{\lambda=\lambda_0}^2 = \frac{1}{2} \left| \frac{\mathrm{d}N_s}{\mathrm{d}\lambda} \right|_{\lambda=\lambda_0}^2. \quad \dots \quad (74)$$

Equation (64), which applies to a two-layer anti-reflection film, may be derived immediately by substituting from (73) in (74) and eliminating n_2 by means of (63).

VIII. References

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