THE RESOLUTION OF THE CLOCK PARADOX

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Summary

It is shown that the so-called " clock paradox " can be resolved completely in terms of the restricted theory of relativity. Properly applied, the theory gives a unique and unambiguous value for the relative retardation of two clocks. The seeming paradox can, and does, arise only through using the " rate " of a moving clock without due regard to the exact significance of the quantity so described.

The usual hypothetical experiment, in which an observer M travels away from, and returns to, an observer R, with constant speed in a straight line, is described in terms of an accelerated reference system in which M remains at rest at the origin; but it is shown that such a description cannot be regarded as the account that would be given by M of the experiment.

The principle of equivalence is completely irrelevant to analysis and discussion of the relative retardation of clocks unless there is a real gravitational field to be taken into account and, except in such a case, the general theory of relativity can add nothing of physical significance to an analysis correctly made using the restricted theory.

I. Introduction

In discussions of the so-called clock paradox it is usual, and it suits our present purpose, to consider the following hypothetical experiment. It is supposed that two observers R and M, equipped with identical synchronized clocks, are initially at rest together, e.g. at the origin of an inertial reference system S. The observer M is sent on a journey along the x-axis of S, travelling away from R with uniform speed v for a time T, coming to rest for a time τ, and then returning with the same speed v to rejoin R after a total time $2T+\tau$ as read on R's clock.

It will, in the first instance, be supposed that the times required to accelerate, or decelerate, M are so small that they can be neglected without appreciable error. This can always be realized, even for moderate accelerations, by supposing T to be very great (McCrea 1951), or it may be justified for rapid accelerations as has been shown by Møller (1952). It is always assumed that acceleration of a clock has no direct effect on its rate (e.g. Møller 1952); this is also a basic assumption in the general theory of relativity.

According to the restricted theory of relativity, measurements made in the reference system S, e.g. by the observer R, must show that the rate of M's clock, while it is in motion with speed v, is less than that of R's clock by the factor $1/\gamma=(1-v^2/c^2)^{\frac{1}{2}}$. Therefore, when M rejoins R, at the end of his journey, his clock must be retarded, relative to R's, by the amount $2T-2T/\gamma$. This is an inescapable prediction of the theory once it is assumed that M's clock

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is not directly affected by acceleration ; it was stated by Einstein (1905) in his famous first paper on the theory.

This relative retardation effect is a surprising consequence of relativity theory ; but it is not more surprising than the reciprocity of relativistic variations of length and mass, than the relativity of simultaneity, or than the equivalence of mass and energy. There is therefore no reason to regard the effect, as such, as a paradox.

Nevertheless, the term " clock paradox " has been widely used because an appearance of paradox does arise when one considers inertial reference systems S' and S'' in which M is at rest during his outward and return journeys ; for the restricted theory also predicts that measurements made by observers in S' and S'' must show that the rate of R's clock is less than that of clocks, such as M's, at rest in their own systems. This seems incompatible with the predicted retardation of M's clock relative to R's at the end of the experiment.

The " clock paradox " can therefore be expressed as a difficulty in reconciling the asymmetry of the retardation of M's clock relative to R's with the symmetry, required by the restricted theory of relativity, between measurements in the systems S and S' and between measurements in the systems S and S''.

Alternatively, it is often expressed as a difficulty in reconciling the asymmetry of the relative retardation with the symmetry, considered from a purely kinematical point of view, between the motions of M relative to R and of R relative to M.

It is quite generally accepted that the paradox can only be resolved by denying the applicability of the restricted theory and by using the general theory of relativity.

It is, however, shown here that it can be resolved completely in terms of the restricted theory. It is also shown that the general theory can add nothing of physical significance to an analysis correctly made in terms of the restricted theory.

II. The Predictions of the Restricted Theory of Relativity

To apply the restricted theory of relativity to our hypothetical experiment, we consider three inertial reference systems S, S', and S'' whose corresponding axes are all parallel and whose origins coincide at the instant $t=t'=t''=0$. The system S' moves with uniform speed v, relative to S, in the direction of the positive x-axis of S ; the system S'' moves with the same speed v relative to S, but in the opposite direction. These reference systems must be regarded as having continuous existence before, during, and after the experiment, quite independent of the motions of the moving observer M. The restricted theory then gives rigorously the relations between measurements made in the three systems. The Lorentz transformations give, for the relevant relations between measurements made in S and S',

$$x'=\gamma(x-vt), \qquad t'=\gamma(t-vx/c^2), \qquad \text{(1a)}$$

$$x=\gamma(x'+vt'), \qquad t=\gamma(t'+vx'/c^2), \qquad \text{(1b)}$$

$$\gamma=(1-v^2/c^2)^{-\frac{1}{2}}>1, \qquad \text{(1c)}$$

and corresponding relations between measurements made in S and S''. These equations relate the S-coordinates *of an event* to the S' or S''-coordinates of *the same event.*

In our hypothetical experiment there are four identifiable events.

We suppose that the observers R and M are, initially, at rest at the origin of S, that their clocks are set to agree with the clocks of this system, and that M's journey commences at the instant $t_1=t_1'=t_1''=0$. The *first event* E_1 is the beginning of M's journey; it is marked by M's departure from the origin of S and his coming to rest at the origin of S'; its S-coordinates are $x_1=0$, $t_1=0$ and its S'-coordinates $x_1'=0$, $t_1'=0$. The *second event* E_2 is the termination of M's outward journey; it is marked by M's departure from the origin of S' and by his coming to rest in S; its S-coordinates are $x_2=X$, $t_2=T$ and its S'-coordinates $x_2'=0$, $t_2'=T'$.

If it happens that we know the time interval $t_2-t_1=T$ between these events, as determined by R, we can predict the time interval $t_2'-t_1'=T'$ between the events as read by M on his clock. Noting that $vT=X$, equations (**1a**) give

$$T'=t_2'-t_1'=\gamma(T-vX/c^2)=T/\gamma. \qquad \text{(2a)}$$

Conversely, if T' were known, the value of T could be predicted. Noting that $x_1'=x_2'=0$, equations (**1b**) give

$$T=(t_2-t_1)=\gamma T'. \qquad \text{(2b)}$$

Equations (**2a**) and (**2b**) are identical. They specify unambiguously the relation between the measurements made by R and M, each using his own clock, of the time occupied by M's outward journey. They show that M's measure of this interval is less than R's.

The *third event* E_3 is the beginning of M's return journey; it is marked by M's departure from the point $x=X$ in S at the time $t=T+\tau$, and by his coming to rest in S''. The *fourth event* E_4 is the termination of M's return journey; it is marked by his coming to rest at $x=0$ in S at the time $t=2T+\tau$ and by his ceasing to be at rest in S''

The interval read by M on his clock between the events E_3 and E_4 may be denoted by T''. Using the relations corresponding to equations (**1**) it may be shown that this is related to T by

$$T''=T/\gamma=T'. \qquad \text{(3)}$$

The interval read by M on his clock between the events E_2 and E_3, when he is at rest in S, will obviously be equal to τ.

Thus when M reaches R at the end of the experiment the total elapsed time as read by him on his clock will be

$$T''+\tau+T'=2T'+\tau=2T/\gamma+\tau.$$

Remembering that γ is greater than unity, it follows that his clock will be retarded relative to R's by the amount

$$2T-2T'=2T-2T/\gamma, \qquad \text{(4)}$$

when he rejoins R.

This result must be regarded as the unique and unambiguous prediction of the restricted theory. It is not possible to obtain any different result by direct application of the Lorentz transformations; it is immaterial whether M's readings of his clock are calculated from given readings by R of his clock, or vice versa.

How then can any paradox, or appearance of paradox, arise in the application of the restricted theory to the hypothetical experiment considered?

It can and does arise only through an incorrect application of the Lorentz transformations, in which a calculation is first made of the "rate" of a moving clock and this "rate" is then applied without due regard to its exact meaning.

III. The "Rate" of a Moving Clock

If the observers in the inertial reference system S wish to determine the rate of the moving clock M, any method they can devise is, of necessity, equivalent to one basic procedure. They can, in effect, only compare this clock with two synchronized clocks C_1 and C_2, at rest at x and $x+\delta x$ in their own system, with which it is coincident at times t and $t+\delta t$.

To predict the result of this procedure, using the restricted theory, we consider the inertial reference system S' in which the clock M is at rest at its origin. Then the *two events*, which are the coincidences of the clock M with the clocks C_1 and C_2, have respectively the S-coordinates x,t and $x+\delta x$, $t+\delta t$ and the S'-coordinates $0,t'$ and $0,t'+\delta t'$, where t' and $t'+\delta t'$ are the readings of the clock M at the two coincidences. Using the equations (**1**), noting that $x'=0$ for both events, we find

$$\delta t'=\delta t/\gamma \quad (x' \text{ constant}). \qquad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{5a})$$

Thus the clock M records a shorter interval between these events, *which occur at the same place in* S$'$, than do the clocks of the system S. No other method is available to the S-observers to measure the rate of the moving clock M, so we are justified in describing the ratio $\delta t'/\delta t=1/\gamma$ *as the "rate" of the clock* M *according to the* S-*observers*.

If we suppose that observers also exist in the system S', and if they wish to determine the rate of the clock R, which is stationary in the system S, they must adopt a similar procedure, and we can predict that they will find

$$\delta t=\delta t'/\gamma \quad (x \text{ constant}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{5b})$$

for the relation between the interval δt, read on the clock R, and the interval $\delta t'$ read on the clocks in their own system, between two clock coincidences *at the same place in* S. Again we are justified in describing the ratio $\delta t/\delta t'=1/\gamma$ as the "*rate*" *of the clock* R *according to the* S$'$ *observers*.

Although there is good reason to refer to equations (**5a**) and (**5b**) as giving the "rate" of a moving clock according to the S-observers, and according to the S'-observers, respectively, *the exact significance of the equations must never be forgotten in applying them in particular cases*: they are, strictly speaking, relations between measurements, made in S and S', of time intervals between events which occur, on the one hand, at the same place in S' and, on the other hand, at the same place in S: this is expressed by the restrictive conditions "x' constant" and "x constant" in the two equations.

If then the Lorentz transformations are to be utilized by first calculating clock " rates ", the relevance of each such " rate " to the particular case must be carefully considered.

IV. The Origin and Resolution of the " Paradox "

The events E_1 and E_2, which are the beginning and end of M's outward journey, occur at the same point in the system S'. Equation (**5a**) therefore gives correctly the relation between the measurements by R and M of the time interval between these events; this relation is the same as that, given in equations (**2**), obtained by direct application of the Lorentz transformation. Similarly, the corresponding measurement by R of the rate of M's clock during his return journey will give correctly the relation between the measurements made by M and R of the interval between the events E_3 and E_4, as given in equation (**3**). The measurements by R and M of the interval between the events E_2 and E_3 will, of course, both give the time τ. The retardation of M's clock, relative to R's, during the experiment will therefore be given by equation (**4**), just as was found by direct application of the Lorentz transformations.

On the other hand, since the events E_1 and E_2 do not occur at the same place in S, equation (**5b**) cannot be used to find a relation between the measurements made by R and M of the interval between these events. This applies also to the interval between the events E_3 and E_4.

The " paradox " has arisen through ignoring this distinction. It is completely resolved once the exact significance of equations (**5a**) *and* (**5b**) *is taken into account.*

There is no need to reconcile the symmetry of the equations (**5a**) and (**5b**) with the asymmetry of the predicted retardation because equation (**5b**) has no direct relevance in this particular case.

The asymmetry of the predicted retardation is obviously consistent with the dynamical asymmetry involved in the original specification of the experiment, which required that M should be subjected to acceleration and that R should not. On the other hand, the Lorentz transformations do not in themselves take into account any such dynamical asymmetry, for they are solely relations between measurements made in inertial systems. How then does the application of these transformations lead to an asymmetrical clock retardation corresponding to the prescribed dynamical asymmetry of the motions of R and M?

The answer is not far to seek. Because M is the accelerated observer, i.e. the one to whom something happens, the identifiable events E_1, E_2, E_3, E_4 are all coincident with M. The events E_1 and E_2 which mark the beginning and end of M's outward journey therefore occur at the same place in S', but at different places in S. The dynamical asymmetry is thus displayed as an essential asymmetry in the relations of the events E_1 and E_2 to the systems S and S' and, similarly, in the relations of the events E_3 and E_4 to the systems S and S''.

We may therefore fairly, if somewhat picturesquely, claim that the restricted theory is not deceived by the apparent kinematical symmetry of the motions, of M relative to R and R relative to M.

V. The Invariance of the Relative Retardation

The retardation of M's clock, relative to R's, given by equation (**4**), is an invariant for *all* observers and for *all* reference systems. That this must be so is obvious on general physical grounds ; for the reading on any clock of the time of an event coincident with it must be the same for *all* observers. Thus the interval read on a clock between two events coincident with it is an invariant, i.e. the *proper* time of the clock. In the present case the events E_1 and E_4, which mark the beginning and end of the experiment, are coincident with both clocks, so that $2T+\tau$ and $2T'+\tau$ are their proper times. The relative retardation, given by the difference between these proper times, is therefore also an invariant and may be calculated in any way, or in any reference system whatsoever, that may be convenient.

The proper time θ' of any clock, in arbitrary motion with instantaneous speed u in an inertial reference system S, between two events coincident with it at the S-times θ_1 and θ_4, is given by

$$\theta' = \int_{\theta_1}^{\theta_4} (1-u^2/c^2)^{\frac{1}{2}} d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{5c})$$

on the accepted assumption that equation (**5a**) gives the " rate " of the clock at each instant when the intervals δt and $\delta t'$ become infinitesimal. The invariance of this integral to Lorentz transformations is readily verified.

The invariance of the relative retardation could be illustrated by calculating the proper times of the two clocks in the system S (as we have done in effect in the foregoing) and by repeating the calculation for the system S' by using the " rates " of both clocks, according to the S' observers, given by equation (**5b**).

VI. Measurements and Perceptions of the Moving Observer

The observer M, while at rest in S', would have found the " rate " of R's clock to be less than that of his own by the factor $1/\gamma$, and would have found that R was moving away from him with speed v. It may therefore be *inferred* that, had he remained at rest in S' until the corresponding light data from R had reached him, his measurements, like those of hypothetical permanent S' observers, would have shown the reading on R's clock to be T'/γ, and the distance of R to be vT', at the instant immediately preceding the event E_2. Yet, having come to rest in S, if he remained so for long enough for the corresponding light from R to reach him, his own measurements would show that the reading of R's clock was $\gamma T'$, and that the distance of R was $\gamma vT'$, at the instant immediately after event E_2.

Such inferences and possible measurements are summarized in Table 1. The sudden changes in the readings of R's clock and in R's distance are fully accounted for by the corresponding sudden changes in the systems of reference. Appearance of paradox can arise only if, as has often happened, the reading T'/γ of R's clock at the instant before event E_2 is torn from the context of Table 1 and is contrasted with the predicted retardation of M's clock relative to R's at the end of the experiment.

The data of Table 1 indicate a sudden change in the reading of R's clock during the event E_2 by the amount $\gamma(v^2/c^2)T'$. If the duration of this event is denoted by τ_2' it may be argued that, in some sense, the average " rate " of R's clock is $\gamma(v^2/c^2)T'/\tau_2'$ during the event. If, as we have assumed, τ_2' is very small compared with T', this " rate " will be enormous unless v^2 is correspondingly small compared with c^2. It may also be argued that, in a similar sense, R's speed reaches the enormous average value $(\gamma-1)vT'/\tau_2'$ during this event.

It must be clearly recognized that the systems of reference indicated in Table 1 are those in which M happens to be at rest at the instants indicated; they are not, in general, the systems of reference that could be used by him in making measurements relating to distant events occurring at these instants.

TABLE 1

INFERENCES AND POSSIBLE MEASUREMENTS RELATING TO THE INSTANTS IMMEDIATELY BEFORE AND IMMEDIATELY AFTER THE EVENTS E_2 AND E_3, AND RELATED DATA

	Event E_2		Event E_3	
	Instant Before	Instant After	Instant Before	Instant After
M's own clock reading ..	T'	T'	$T'+\tau$	$T'+\tau$
System of reference.. ..	S'	S	S	S''
R's speed	v	0	0	v
" Rate " of R's clock ..	$1/\gamma$	1	1	$1/\gamma$
Reading of R's clock ..	$\gamma T'\left(1-\frac{v^2}{c^2}\right)$	$\gamma T'$	$\gamma T'+\tau$	$\gamma T'\left(1+\frac{v^2}{c^2}\right)+\tau$
R's distance	vT'	$\gamma vT'$	$\gamma vT'$	vT'

Thus Table 1 cannot properly be regarded as an account of the experiment in terms of M's measurements and observations, and it cannot be inferred that M's observations would show that the " rate " of R's clock and the speed of R had reached enormous values during the short intervals occupied by the events E_2 and E_3. It is, in any case, obvious that he would not have any immediate perception of such effects.

To describe M'*s immediate perceptions* we must specifically exclude any allowance for light transmission time. Let us suppose that R's clock emits, and is illuminated by, a flash of light at the end of each second of its own time. The reading of R's clock which M will see, in the literal sense, at any time can then be stated in terms of his reception of these flashes. The arrival of each flash at M is a definite event coincident with M; the reading of M's clock at the instant of its arrival is therefore the same for *all* observers and may be calculated in any convenient way, e.g. in the system S. We find, of course, that what M sees, in this literal sense, can be expressed in terms of the relativistic Doppler effect.

During his outward journey M would receive the flashes of light from R's clock at the rate of $\gamma(c-v)/c$ per second so that he would see, literally, R's clock running more slowly than his own; the reading he would see on R's clock at the time T' at which his outward journey ended is $[\gamma(c-v)/c]T'$.

During the event E_2 the rate of arrival of the flashes would increase to one per second and would remain at this value while M remained at rest in S; the reading he would see on R's clock at the time $T'+\tau$, at which his return journey started, is $[\gamma(c-v)/c]T'+\tau$.

During the event E_3 the rate of arrival of the flashes would increase further to $\gamma(c+v)/c$, and would remain at this rate during M's return journey, so that he would see, literally, R's clock running faster than his own; the reading he would see on R's clock at the time $2T'+\tau$ at which he reached R is

$$\gamma\{(c-v)/c\}T'+\tau+\gamma\{(c+v)/c\}T'=2\gamma T'+\tau. \quad \ldots\ldots \quad (6)$$

Thus M's immediate perceptions would be perfectly consistent with the predicted retardation of his own clock, relative to R's, at the end of the experiment; they would not disclose, or even suggest, any "racing" of R's clock during the events E_2 and E_3.

VII. Description of the Experiment in Terms of the Coordinates of the Accelerated Reference System S_m

A complete and consistent account of the experiment can be given by describing it in terms of the coordinates x_m, t_m of the accelerated system of reference S_m in which M remains at rest, at the origin, throughout the experiment. This system S_m is identical with the inertial system S' during M's outward journey; it becomes identical with a system S_0, at rest relative to S, as a result of the event E_2; it becomes identical with a system S_0'', at rest relative to S'', as a result of event E_3, and remains so during M's return journey.

The transition of S_m from identity with S' to identity with S_0 at event E_2, or from S_0 to S_0'' at event E_3, may be specified in terms of a continuous succession of inertial reference systems in each of which M is successively at rest at its origin. The coordinates of the successive inertial systems are related by non-homogeneous infinitesimal Lorentz transformations (see, for example, Møller 1952, Ch. IV). Since we are concerned only with measurements which, in each successive inertial system, are simultaneous with the instant at which M is at rest at its origin, we may define the time t_m of M's clock as the time of the accelerated system S_m which comprises this continuous succession of inertial systems. The coordinates x_m of S_m may be identified at each instant with the x-coordinates of the inertial reference system in which M is at rest, at its origin, at that instant.

The transformations relating the coordinates of the system S_m to those of an inertial reference system, such as S_0, take a relatively simple form if the rest-acceleration of M has a constant value g during each of the events E_2, E_3, etc., as it would in fact have if M were, on each occasion, subject to a constant force, i.e. as measured in an inertial reference system. *These transformations enable us to describe, in terms of the coordinates of* $\mathrm{S_m}$, *phenomena and events of which a description is already available in terms of the coordinates of an inertial reference system.*

We will consider, as an example, the event E_2 which terminates M's outward journey, denoting by τ_2 and τ_2' the time occupied by this event as determined by R and by M respectively.

Before the instant T' at which event E_2 commences, the system S_m is identical with the inertial reference system S', so that

$$x_m = x' \; ; \quad t_m = t' \quad (t_m < T'). \quad \ldots\ldots\ldots\ldots\ldots \quad \textbf{(7a)}$$

At the instant $T' + \tau_2'$ at which event E_2 ends, S_m becomes identical with an inertial reference system S_0, at rest relative to S, defined by:

$$x_0 = x - X \; ; \quad t_0 = t - T + T_0 \; ; \quad T_0 = T' + \tau_2' - \tau_2, \quad \ldots\ldots \quad \textbf{(7b)}$$

so that

$$x_m = x_0 \quad ; \quad t_m = t_0 \qquad (t_m > T' + \tau_2'). \quad \ldots\ldots\ldots \quad \textbf{(7c)}$$

During the interval τ_2' M is subject to a rest-acceleration g (which has a negative value) and the S_m-coordinates are related to the S_0-coordinates by:

$$1 + g x_0/c^2 = (1 + g x_m/c^2) \cosh\,[g(\theta_2' - \tau_2')/c], \quad \ldots\ldots\ldots \quad \textbf{(7d)}$$

$$g(\theta_2 - \tau_2)/c = (1 + g x_m/c^2) \sinh\,[g(\theta_2' - \tau_2')/c], \quad \ldots\ldots\ldots \quad \textbf{(7e)}$$

where

$$\theta_2 = t_0 - T_0 \; ; \quad \theta_2' = t_m - T' \; ;$$
$$T_0 < t_0 < T_0 + \tau_2 \; ; \quad T' < t_m < T' + \tau_2'.$$

The coordinate x_m *of* R's *clock* is given by equation (**7d**) when we put $x_0 = -X$. Neglecting the distance travelled by M during his deceleration, it is easily verified that x_m has the values

$$x_m = -X/\gamma \;\text{ at }\; t_m = T',$$
$$x_m = -X \quad \text{ at }\; t_m = T' + \tau_2',$$

in agreement with the values given in Table 1.

The velocity u_m *of* R's *clock* at any instant may be found by differentiation of equation (**7d**). Noting that $dx_0/dt_m = 0$, this gives

$$u_m = -c(1 + g x_m/c^2) \tanh\,[g(\theta_2' - \tau_2')/c]. \quad \ldots\ldots \quad \textbf{(8a)}$$

It is easily verified that at the onset of the acceleration at $t_m = T'$ this velocity suddenly assumes the value $-v(1 + g x_m/c^2)$ but decreases to zero at $t_m = T' + \tau_2'$.

Readings $t = T + \theta_2$ *of* R's *clock* which, in the system S_m, are simultaneous with the readings $t_m = T' + \theta_2'$ of M's clock, are given by equation (**7e**) when x_m is put equal to the coordinate of R at each instant considered. It is easily verified that this gives the readings

$$t = T/\gamma^2 \quad \text{at }\; t_m = T',$$
$$t = T + \tau_2 \;\text{ at }\; t_m = T' + \tau_2',$$

in agreement with the values given in Table 1.

The " rate " of R*'s clock* in the system S_m is obtained by differentiation of equation (**7e**) and is given by

$$\mathrm{d}t/\mathrm{d}t_m=[(1+gx_m/c^2)^2-u_m^2/c^2]^{\frac{1}{2}}\,; \quad x_0=-X, \quad \ldots\ldots\ldots \quad (\mathbf{8b})$$

and this may be written

$$\mathrm{d}t/\mathrm{d}t_m=(1-gX/c^2)\ \mathrm{sech}^2\ [g(\theta_2'-\tau_2')/c]. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{8c})$$

Under the conditions being considered, this " rate " does not differ greatly from the value $(1-gX/c^2)$ during the event E_2; it approximates to $-gX/c^2$ and may reach enormous values if X is large. Since g is negative, this is consistent with the average rate $\gamma(v^2/c^2)T'/\tau_2'$ inferred in Section VI from the data given in Table 1; for we may write, approximately, $g=-v/\tau_2'$ and $X=\gamma vT'$. *The " rate " of a standard clock* stationary in S_m, for which $u_m=0$, can similarly be shown to be

$$\mathrm{d}t/\mathrm{d}t_m=1+gx_m/c^2\,; \quad x_m \text{ constant}, \quad \ldots\ldots\ldots\ldots \quad (\mathbf{9a})$$

and will, in general, be very large for large negative values of x_m, g being negative. *The velocity of light in* $\mathrm{S_m}$ at any point having the coordinate x_m is obtained by differentiating equation (**7d**) and inserting the value c for the velocity in the inertial system S_0. One obtains

$$c_m=c(1+gx_m/c^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{9b})$$

The factor $(1+gx_m/c^2)$ may reach very large values for large negative values of x_m, the value of g being negative.

Graphical Representation: We can thus build up a complete and coherent description of the experiment in terms of the coordinates of the accelerated system S_m. The foregoing results are consistent with the inferences, drawn in Section VI, from the data summarized in Table 1. Equation (**9b**), for the velocity c_m of light at any point in the system S_m, enables us to account also, in terms of the S_m coordinates, for M's immediate perceptions, summarized in Section VI, of the readings of R's clock.

Such a description, or a corresponding description in terms of the coordinates of the inertial reference system S, can best be presented graphically in the manner illustrated by the diagrams of Figure 1; these refer to an experiment similar to that so far considered, and differing only in that the times τ_2 and τ_3 occupied by the events E_2 and E_3 have been assigned relatively large values to facilitate the graphical representation.

In Figure 1 (a), any event that takes place on the x-axis of the inertial reference system S is represented by a point having coordinates x,t in the diagram. The arbitrary units of length and time used are such that the velocity c of light is unity. Points representing events which are simultaneous in S lie on a line parallel to the axis of x; any line parallel to the axis of t is the world line of an object at rest in S and, in particular, the axis of t is itself the world line of the observer R at rest at $x=0$. The world line of the observer M is shown; starting at $t=0$, he travels away from R with speed $v=0{\cdot}6c$ until $t=T=12{\cdot}5$; he is then decelerated by a constant force ($g=-0{\cdot}5c$ per unit time) and comes to rest at $x=8$ at time $t=T+\tau_2=14$; he remains at rest until $t=T+\tau_2+\tau=18$ and is then accelerated towards R by a constant force ($g=-0\cdot5c$ per unit time) to reach the speed $v=0\cdot6c$ at $t=T+\tau_2+\tau+\tau_3=19\cdot5$; he travels with this speed to reach R again at $t=2T+\tau_2+\tau+\tau_3=32$.

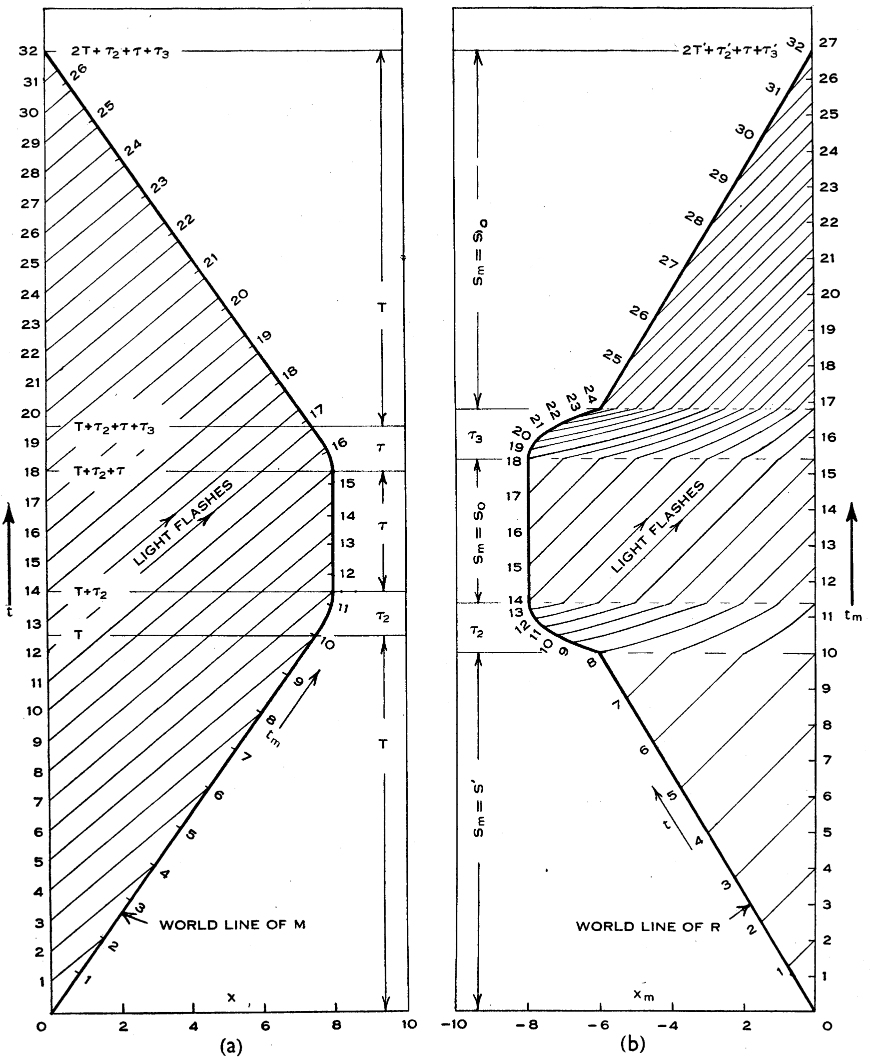

Fig. 1.—Description of clock retardation experiment in terms of the coordinates of inertial and accelerated reference systems S and S_m. World lines of light flashes emitted by R's clock are shown.

The readings t_m of M's clock which, in the system S, are simultaneous with readings t of R's clock can be calculated by means of equation (**5c**); these readings have been scaled off along M's world line. It is seen that M's clock reads only $t_m=26\cdot8$ when he rejoins R at $t=32$.

We suppose, as in Section VI, that R's clock emits, and is illuminated by, a flash of light at the end of each unit of time. The world lines of these flashes are straight lines in the diagram: the intersection of each with M's world line represents the arrival of a flash and its perception by M. The reading t_m of M's clock at the arrival of each flash can be ascertained from the time scale marked off on M's world line. It will be seen that the rates of arrival of the flashes are consistent with those given in equation (**6**), it being noted that in the present case $v=0\cdot6c$ and $1/\gamma=(1-v^2/c^2)^{\frac{1}{2}}=0\cdot8$, so that $\gamma(c-v)/c=\frac{1}{2}$ and $\gamma(c+v)/c=2$.

In Figure 1 (*b*) any event taking place on the x_m-axis of the accelerated reference system S_m is similarly represented by a point having coordinates x_m, t_m in the diagram. The axis of t_m is itself the world line of the observer M, at rest at $x_m=0$. The world line of the observer R is shown; starting at $t_m=0$ he moves away from M with speed $v=0\cdot6c$ until $t_m=T'=10$; his speed suddenly increases to the value $u_m=2\cdot4c$ (equation (**8a**)) and then decreases until he comes to rest at $x_m=-8$ at $t_m=T'+\tau_2'=11\cdot4$; he remains at rest until $t_m=T'+\tau_2'+\tau=15\cdot4$ and starts to move again, towards M, at first slowly but reaching the speed $u_m=2\cdot4c$ at $t_m=T'+\tau_2'+\tau+\tau_3'=16\cdot8$; his speed is then suddenly reduced to $v=0\cdot6c$ and he travels with this speed to reach M at $t_m=2T'+\tau_2'+\tau+\tau_3'=26\cdot8$.

The readings t of R's clock, which are simultaneous in the system S_m with readings t_m of M's clock, may be calculated by means of equation (**7e**); these readings have been scaled off along R's world line. It is seen that the reading of R's clock has only reached $t=8$ at $t_m=10$, but that its rate during the interval τ_2' is so high (equation (**8b**)) that its reading becomes $t=14$ at $t_m=11\cdot4$. Its rate is the same as that of M's clock during the interval τ but again reaches very high values during the interval τ_3', so that $t=24$ at $t_m=16\cdot8$. Its rate then becomes less than that of M's clock by the factor $1/\gamma=0\cdot8$ so that its reading is $t=32$ when R rejoins M at $t_m=26\cdot8$.

The world lines of the light flashes from R's clock are also shown. During the intervals τ_2' and τ_3' the velocity of the light is $c(1+gx_m/c^2)$, as given by equation (**9b**), and reaches the value $4c$ at $x_m=-8$. The variation of this velocity with x_m results in the world lines being, in general, non-linear and this results, in spite of the large variations in the rate of R's clock, in the perception of the flashes by M at the times predicted by Figure 1 (*a*) and by equation (**6**).

VIII. The Significance of the Accelerated Reference System

It is commonly assumed that the S_m-description, represented in Figure 1 (*b*), is the account that the observer M would give of the experiment, even if his only data were his own perceptions and measurements. This assumption is not justifiable. It presupposes that M could himself measure the S_m-coordinates of every event; but this is not, even in principle, possible.

The accelerated reference system S_m was defined by stating that its coordinates are, at any instant, identical with those of the particular inertial reference system in which M is at rest, at the origin, at that instant. The motion of M thus specifies a multiplicity, which will in general be infinite, of hypothetical inertial reference systems, each of whose coordinates is to be used at a particular instant or during a particular interval. These coordinates may be predicted, for any event, by means of the Lorentz transformation, if the coordinates of that event in any one inertial reference system, such as S, are already known. The S_m-coordinates provide a summary of all such predictions.

Thus a description of events in terms of the S_m-coordinates cannot furnish more information than is already available in terms of the coordinates of inertial reference systems.

On the other hand, measurement of the coordinates x_m, t_m of any event presupposes, by definition, actual measurement in the particular inertial reference system in which M is at rest, at its origin, at the instant t_m considered. This, in turn, presupposes the existence in that particular inertial system of observers at rest, either permanently or, at the very least, for the time required for transmission of light throughout the spatial region of interest. *Thus, in general, the observer* M *could not himself make these measurements.*

This conclusion was already foreshadowed in the discussion of the data of Table 1 in Section VI. It was there pointed out that, even though M was at rest in the system S' until his outward journey ended at the instant T', he could not verify the reading T'/γ of R's clock, or its distance vT', which we inferred would be simultaneous in the system S' with the reading T' of his own clock; for light emitted by R's clock at the instant its reading was T'/γ could not reach M until long after he had ceased to be at rest in S'. These inferences could in fact be verified experimentally only by hypothetical observers at rest in S', i.e. in the inertial reference system whose origin coincided with M at the instant $t'=T'$. Thus even the data presented in Table 1 cannot properly be regarded as M's account of his own observations during the experiment; it is no more than a summary of data, measurable in systems S, S', and S'', selected in a way defined by the motion of M.

When, as in Section VII, the accelerated reference system, from $t_m=T'$ to $t_m=T'+\tau_2'$, comprises a continuous succession of hypothetical inertial systems in each of which M is at rest only instantaneously, it is obvious that M could not, even in principle, determine the coordinates x_m, t_m of distant events because he would not remain at rest in any of these inertial systems for a finite time. Reference to Figures 1 (a) and 1 (b) also shows that, even when M is at rest in the system S from $t_m=T'+\tau_2'$ to $t_m=T'+\tau_2'+\tau$, he could not determine the coordinates of R because light signals emitted by R's clock during this interval would not reach him until after he had come to rest in S''.

It has been implied in the foregoing that M would be restricted to methods of measurement, based on the transmission of light, such as would be possible in practice using optical instruments. No account was taken of the possibility of his utilizing in any way a real physical system having continuous extension throughout the region occupied by R and himself during the experiment.

Such a physical system at rest with R is quite conceivable. If it were provided with visible markings of the spatial coordinates x and if it were equipped at appropriate points with standard clocks synchronized by the S observers, the observer M could in fact determine the S-coordinates of any events. The result would be a trivial verification of the S-description of the experiment.

On the other hand, a physical system, of sufficient extent, moving with M is difficult to imagine unless one supposes the distance travelled by M, or the accelerations to which M is subjected, to be very small. Nevertheless, it is necessary to examine whether such a system, however impracticable, could in principle enable M to measure the S_m-coordinates.

The conditions to be met can be definitely prescribed. It is necessary that a real coordinate frame, of sufficient extent, moving with M, and corresponding exactly at every instant to the coordinates x_m of the accelerated reference system S_m, should be, in principle, physically realizable and that it could be supposed to be equipped with clocks adjusted to keep the time t_m, i.e. coordinate clocks, as distinct from the standard clocks of the restricted theory of relativity (equation (**9a**)). The observer M could then determine the coordinates of any event by noting the point in the coordinate frame, and the reading of the coordinate clock, coincident with the event. These observations would not be affected by the time of transmission of light from the point of occurrence of the event to the observer; they could in fact be made by any observer who could read the coordinate clock, and identify the point in the coordinate framework, coincident with the event.

In general, such a physical system is not realizable, even in principle. The fundamental restriction is that there can be no transmission of gravitational fields, or of stresses in a physical system, with a speed greater than that of light.

The effect of this restriction is easily illustrated. The accelerated system S_m of Section VII is identical with the inertial system S' at $t_m=T'$; it becomes identical with S_0 at $t_m=T'+\tau_2'$ (Fig. 1 (*b*)). The corresponding real coordinate frame would, at every instant, have to be contracting at the same rate at every point to preserve its identity with the succession of inertial reference systems which constitute the system S_m during its transition from S' to S_0. This would require, at the onset of M's acceleration, instantaneous transmission of stress throughout the system, as judged by S' observers, and this is clearly incompatible with the postulates of the theory of relativity.

More specifically, it can be shown (e.g. Møller 1952, Ch. VIII) that the coordinates x_m, t_m defined, during the interval τ_2, by equations (**7d**) and (**7e**) correspond to a rigid coordinate frame subject to a *steady* rest-acceleration g of its origin and equipped at appropriate points with clocks keeping the time t_m, each being adjusted to a rate less, by a factor $(1+gx_m/c^2)$, than that of a standard clock at rest with it, as required by equation (**9a**). When viewed from an inertial reference system this rigid framework would be subject at each point to a Lorentz contraction determined by the speed of that point; this speed would depend on the accelerated motion of the origin of the frame and on the resultant contraction of the system as a whole. The equations (**7d**) and (**7e**) are, clearly, inapplicable unless the acceleration g of the origin has been so long maintained that a steady state has been reached throughout the system.

Thus accelerated reference systems such as S_m, in which the accelerations are transitory, as they are during the events E_2 and E_3, can have no realizable physical counterpart which would, in principle, permit an observer, such as M, to measure the system coordinates. The artificiality of the system S_m is demonstrated in Figure 1 (*b*) by the discontinuities in *R*'s velocity at $t_m = T'$ and $t_m = T' + \tau_2' + \tau + \tau_3'$ and, perhaps even more forcibly, by the discontinuities in the velocity of light, since these occur simultaneously at all points in an inertial reference system as, for example, in S' at $t_m = T'$.

It appears that the only case in which a relative retardation of clocks would occur in the interval between their successive coincidences, and in which observers at rest in a relevant accelerated reference system could measure the system coordinates of every event of interest, is that in which the accelerated reference system corresponds to a steady rotation. This case would arise if an observer R' were at rest in an inertial reference system and if an observer M' moved in a circle with constant speed so that he coincided instantaneously with R' each time he traversed the circle.

Thus, except in such a special case, the description of a clock-retardation experiment in terms of the coordinates of an accelerated reference system does not represent the account that would be given of the experiment by the accelerated observer as a result of his own perceptions and measurements.

It must not be overlooked that the observer M in our experiment would in fact be aware of effects of his own acceleration and, simultaneously, of Doppler changes in the frequency of arrival of regular light signals from *R*'s clock. The only sensible account he could give of these perceptions, taken together with such measurements as he might make by optical methods, would of necessity be similar to the account given in Section VI above.

IX. The General Theory of Relativity

The accelerated reference systems considered in Sections VII and VIII are conventionally regarded as part of the subject matter of the general theory of relativity.

The foregoing discussion shows that this part of the general theory can add nothing of physical significance to an analysis, properly made in terms of the restricted theory, as in Sections II–VI, of the relative retardations of clocks due to their arbitrary motions in a region free of gravitational fields.

Furthermore, it is easy to see that any application of the principle of equivalence of the general theory to such cases would be quite trivial.

The principle states that any accelerated system of coordinates is, physically, completely equivalent to, and indistinguishable from, a similar system at rest in a gravitational field, so far as the perceptions and measurements of observers at rest in the two systems are considered. The principle thus permits the course of events in a gravitational field to be predicted by calculating, *by means of the restricted theory of relativity*, the course of events as described in the equivalent accelerated reference system.

The converse process of calculating the course of events in an accelerated reference system from that in the equivalent gravitational field must be essentially trivial, since it would, in principle, involve first calculating the latter from the former.

It is true that the S_m-description represented in Figure 1 (*b*) could have been derived equally well by ascribing the effects experienced by M as a result of his own acceleration to the existence of a *static* gravitational field having a potential $gx_m(1+gx_m/2c^2)$ during the intervals τ_2 and τ_3, and having zero value at all other times. Expressions already known (i.e. already calculated using the principle of equivalence), for the rate of a clock and for the velocity of light in this gravitational field, would lead to the same results as those set out in equations (**8**) and (**9**) of Section VII.

This approach to the problem is based on the explicit assumption that it is the point of view of the observer M which is being considered, for he alone could suppose that the effects he experienced might be due to a gravitational field ; all inertial observers would be aware that the region was free of any such field. It has therefore been generally assumed that the result of the analysis, e.g. the S_m description, is the account that M would give of the experiment as a result of his own perceptions and measurements (e.g. Tolman 1934 ; McCrea 1951 ; Møller 1952).

It was shown in Section VIII that this assumption is not justifiable because the reference system S_m does not correspond to any physical system that is realizable even in principle. This conclusion is not affected by the introduction of the concept of the equivalent gravitational field. On the contrary, nothing could demonstrate more clearly the artificiality of the reference system S_m than the statement that its physical equivalent is a gravitational field which is everywhere zero until the instant $t_m=T'$, has the potential $gx_m(1+gx_m/2c^2)$ from $t_m=T'$ to $t_m=T'+\tau_2'$, and becomes zero everywhere again at $t_m=T'+\tau_2'$. The concept of such a field is completely incompatible with the limiting value c for all velocities measured in inertial reference systems ; for it may be seen from Figure 1 (*b*) that the time $t_m=T'$ is everywhere identical with the time $t'=T'$ of the inertial reference system S', and the time $t_m=T'+\tau_2'$ is everywhere identical with the time $t_0=T'+\tau_2'$ of the system S_0, so that the specified field would have to be created simultaneously at all points in S' and be destroyed simultaneously at all points in S_0.

Thus the principle of equivalence can contribute nothing of physical significance to the analysis ; it only accentuates the artificiality of the description of our hypothetical experiment in terms of the coordinates of the accelerated reference system S_m.

It must not be overlooked that the principle of equivalence was utilized (Tolman 1934) to resolve, by means of the general theory, the so-called paradox of the restricted theory. In effect, the " paradox " was resolved by denying the applicability of the restricted theory to the problem and then using instead conclusions that had been derived from the restricted theory by means of the principle of equivalence. This tortuous procedure succeeded in hiding the paradox rather than in resolving it ; for it scarcely need be pointed out that the procedure would be quite invalid if the restricted theory were indeed not properly applicable to the problem considered. However, the resolution of the " paradox " in Section IV above and the subsequent discussion show that the general theory can contribute nothing of physical significance to an analysis properly carried out by means of the restricted theory except when there are permanent gravitational fields to be taken into account, as in the case analysed by Mikhail (1952).

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