DIRAC CONTINUUM RADIAL WAVE FUNCTIONS

By H. S. Perlman* and B. A. Robson*

[*Manuscript received August* 12, 1958]

Summary

Tables of Dirac continuum radial wave functions for an electron in the Coulomb field of a mercury nucleus ($Z=80$) are presented for several energies and l's. Approximate methods of interpolation and extrapolation to regions not covered explicitly are indicated.

I. Introduction

The Dirac continuum wave functions for an electron in the Coulomb field of a heavy nucleus are required for the theoretical investigation of many atomic and nuclear problems. However, analytically, these functions are notoriously intractable. Firstly, they are non-separable in the many coordinate systems which induce separation in the Schrödinger wave functions and this imposes a severe restriction on the facility with which they can be used in certain problems. Allied difficulties such as non-integral parameters can also be cited. Furthermore, to the authors' knowledge, the only available tables of Dirac continuum radial wave functions are those of Glassgold and Mack (1954), which are valid only in the extreme relativistic energy region. For these reasons, the following tables, which were calculated during an investigation of specific ionization at relativistic energies, are presented separately.

II. Dirac Wave Functions for a Coulomb Field

The following units are used throughout: energy is measured in m_0c^2, length in $\hbar/m_0c$, and momentum in m_0c. Following Rose (1937), the four components, ψ_1, ψ_2, ψ_3, ψ_4, of the Dirac wave function for an electron, which has momentum p and energy $W=(p^2+1)^{\frac{1}{2}}$ in the continuum and which is in a Coulomb field due to Z times the electronic charge, may be expressed in spherical polar coordinates r, θ, φ for the two spin states $j=l\pm\frac{1}{2}$.

For $j=l+\frac{1}{2}$:

$$\left.\begin{aligned}\psi_1&=\mathrm{i}\left(\frac{l-m+\frac{3}{2}}{2l+3}\right)^{\frac{1}{2}}Y_{l+1,\,m-\frac{1}{2}}f_\chi,\\ \psi_2&=\mathrm{i}\left(\frac{l+m+\frac{3}{2}}{2l+3}\right)^{\frac{1}{2}}Y_{l+1,\,m+\frac{1}{2}}f_\chi,\\ \psi_3&=\left(\frac{l+m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m-\frac{1}{2}}g_\chi,\\ \psi_4&=-\left(\frac{l-m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m+\frac{1}{2}}g_\chi,\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \textbf{(1a)}$$

$$l\geqslant 0, \quad -(l+1)\leqslant m-\tfrac{1}{2}\leqslant l.$$

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* Physics Department, University of Melbourne.

For $j=l-\frac{1}{2}$:

$$\left.\begin{aligned}
\psi_1&=\mathrm{i}\left(\frac{l+m-\frac{1}{2}}{2l-1}\right)^{\frac{1}{2}}Y_{l-1,\,m-\frac{1}{2}}f_\varkappa,\\
\psi_2&=-\mathrm{i}\left(\frac{l-m-\frac{1}{2}}{2l-1}\right)^{\frac{1}{2}}Y_{l-1,\,m+\frac{1}{2}}f_\varkappa,\\
\psi_3&=\left(\frac{l-m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m-\frac{1}{2}}g_\varkappa,\\
\psi_4&=\left(\frac{l+m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m+\frac{1}{2}}g_\varkappa,
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(\mathbf{1b})$$

$$l\geqslant 1,\quad -l\leqslant m-\tfrac{1}{2}\leqslant l-1.$$

$Y_{l,m}(\theta,\varphi)$ are normalized spherical harmonics, and the $f_\varkappa(p,\rho)$, $g_\varkappa(p,\rho)$ are the radial wave functions, with $\rho=2pr$. j represents the total angular momentum quantum number, l the quantum number which in the non-relativistic limit is identified with the orbital momentum quantum number, and m the magnetic quantum number.

The parameter $\varkappa$ is defined so that

$$\left.\begin{aligned}
\varkappa&=-(j+\tfrac{1}{2})=-(l+1), &&\text{when } j=l+\tfrac{1}{2},\\
\varkappa&=(j+\tfrac{1}{2})=l, &&\text{when } j=l-\tfrac{1}{2}.
\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(\mathbf{2})$$

The regular continuum radial wave functions f, g, normalized in the energy scale, can be written

$$\left\{\begin{matrix}f_\varkappa(p,\rho)\\ g_\varkappa(p,\rho)\end{matrix}\right\}=N_\mp(\varkappa,\rho)[R_\varkappa(p,\rho)\mp R_\varkappa^*(p,\rho)],\quad\ldots\ldots\ldots\ldots\ldots\quad(\mathbf{3a})$$

where

$$R_\varkappa(p,\rho)=\rho^{-3/2}K(\varkappa,p)M(\varkappa,p,\rho),\quad\ldots\ldots\ldots\ldots\ldots\quad(\mathbf{3b})$$

$$N_\mp(\varkappa,p)=p^{\frac{1}{2}}(1\mp W)^{\frac{1}{2}}\mathrm{e}^{\pi\eta/2}\mid\Gamma(\gamma+\mathrm{i}\eta)\mid[\pi^{\frac{1}{2}}\Gamma(2\gamma+1)]^{-1},\quad\ldots\quad(\mathbf{3c})$$

$$M(\varkappa,p,\rho)=\mathrm{e}^{-\mathrm{i}\rho/2}(\mathrm{i}\rho)^{\gamma+\frac{1}{2}}{}_1F_1(\gamma+1+\mathrm{i}\eta,\,2\gamma+1,\,\mathrm{i}\rho),\quad\ldots\ldots\quad(\mathbf{3d})$$

$$K(\varkappa,p)=\mathrm{i}^{-\gamma-\frac{1}{2}}[(\gamma+\mathrm{i}\eta)(-\varkappa+\mathrm{i}\alpha Z/p)]^{\frac{1}{2}},\quad\ldots\ldots\ldots\ldots\quad(\mathbf{3e})$$

$$\gamma=(\varkappa^2-\alpha^2Z^2)^{\frac{1}{2}},\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(\mathbf{3f})$$

$$\eta=\alpha ZW/p,\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(\mathbf{3g})$$

and α, the fine structure constant, is taken as 1/137.

The functions $M(\varkappa,p,\rho)$ are the Whittaker functions $M_{-\mathrm{i}\eta-\frac{1}{2},\,\gamma}(\mathrm{i}\rho)$ and are solutions of the differential equation (Erdelyi *et al.* 1953)

$$\mathrm{d}^2M(\mathrm{i}\rho)/\mathrm{d}\rho^2+\{\tfrac{1}{4}+\mathrm{i}(-\mathrm{i}\eta-\tfrac{1}{2})/\rho+(\tfrac{1}{4}-\gamma^2)/\rho^2\}M(\mathrm{i}\rho)=0.\quad\ldots\quad(\mathbf{4})$$

Substituting

$$M=X+\mathrm{i}Y\quad\ldots\ldots\ldots\quad(\mathbf{5a})$$

and

$$\tfrac{1}{4}+\mathrm{i}(-\mathrm{i}\eta-\tfrac{1}{2})/\rho+(\tfrac{1}{4}-\gamma^2)/\rho^2=F+\mathrm{i}G\quad\ldots\ldots\ldots\quad(\mathbf{5b})$$

into (4) gives two coupled real equations :

$$X''+FX-GY=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(6a)}$$

$$Y''+FY+GX=0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(6b)}$$

III. Method of Computation of the Radial Functions

The equations (**6a**), (**6b**) were solved by numerical integration from near the origin, using the recurrence formulae

$$(z_{n+1})_{\text{approx}}=2z_{n-1}-z_{n-3}+4h^2(z''_{n-1}+\tfrac{1}{3}\Delta^2 z''_{n-1}) \quad \ldots\ldots \quad \textbf{(7a)}$$

to integrate ahead, and

$$z_{n+1}=2z_n-z_{n-1}+h^2\left(z''_n+\frac{1}{12}\Delta^2 z''_n\right) \quad \ldots\ldots\ldots\ldots \quad \textbf{(7b)}$$

to provide the correction. It was assumed that the interval in ρ,h, was taken sufficiently small so that only one application of the above was required. For $0<\rho\leqslant 8$, h was taken to be $0\cdot 02$ while for $8<\rho\leqslant 16$, h was taken to be $0\cdot 2$. The initial values near the origin were computed using the series expansion for the Whittaker confluent hypergeometric functions.

These calculations were carried out on the CSIRAC (Physics Department, University of Melbourne) for $Z=80$ and for a range of $\varkappa$ and p.

The reason for solving for M rather than R is that M depends only upon $|\varkappa|$ and not $\varkappa$ and thus the values were obtained simultaneously for the two spin states. The complex gamma functions required for the coefficients N were obtained by interpolation from the National Bureau of Standards table (1954).

IV. Description of Tables

The radial wave functions were calculated for $\varkappa=\pm 1$, ± 2, ± 3, ± 4, and $p=0\cdot 2$, $0\cdot 4$, $0\cdot 6$, $0\cdot 8$. In terms of l, the above values of $\varkappa$ correspond, for the case $j=l+\frac{1}{2}$, to $l=0$, 1, 2, 3, and for the case $j=l-\frac{1}{2}$, to $l=1$, 2, 3, 4. The integrations were effected from near the origin to a distance $\rho=16$, which corresponds to a radial distance, for $p=0\cdot 8$, of five times the K shell radius of mercury.

The $M=X+\mathrm{i}Y$ are tabulated in Table 1 in terms of mantissa and power of ten. In the range $0<\rho\leqslant 8$, the values are accurate to better than $0\cdot 1$ per cent. In the range $8<\rho\leqslant 16$, the error in the values X and Y is considerably larger (generally $\sim$2 per cent.) on account of the larger interval employed in the numerical integration.

The $K=x+\mathrm{i}y$ are tabulated in Table 2 correct to $0\cdot 1$ per cent. The normalizing factors N are given in Table 3 to an accuracy of $0\cdot 5$ per cent.

The normalized radial wave functions can be written explicitly in terms of the tabulated functions

$$f_\varkappa(p,\rho)=2\mathrm{i}\rho^{-3/2}N_-(\varkappa,p)[y(\varkappa,p)X(\varkappa,p)+x(\varkappa,p)Y(\varkappa,p)], \quad \ldots \quad \textbf{(8a)}$$

$$g_\varkappa(p,\rho)=2\rho^{-3/2}N_+(\varkappa,p)[x(\varkappa,p)X(\varkappa,p)-y(\varkappa,p)Y(\varkappa,p)]. \quad \ldots \quad \textbf{(8b)}$$

TABLE 1

$M(\kappa,p,\rho)=X(\kappa,p,\rho)+iY(\kappa,p,\rho)$

ρ	$\lvert\kappa\rvert=1$															
	$p=0·2$				$p=0·4$				$p=0·6$				$p=0·8$			
	X		Y		X		Y		X		Y		X		Y	
0·2	−4·840	−2	8·250	−2	−5·405	−2	9·241	−2	−5·590	−2	9·566	−2	−5·677	−2	9·717	−2
0·4	−1·004	−1	1·539	−1	−1·263	−1	1·962	−1	−1·352	−1	2·109	−1	−1·394	−1	2·178	−1
0·6	−1·402	−1	1·900	−1	−2·006	−1	2·821	−1	−2·225	−1	3·159	−1	−2·332	−1	3·322	−1
0·8	−1·638	−1	1·909	−1	−2·706	−1	3·422	−1	−3·116	−1	4·013	−1	−3·319	−1	4·304	−1
1·0	−1·705	−1	1·624	−1	−3·324	−1	3·743	−1	−3·981	−1	4·627	−1	−4·312	−1	5·071	−1
1·2	−1·615	−1	1·122	−1	−3·832	−1	3·787	−1	−4·785	−1	4·982	−1	−5·275	−1	5·598	−1
1·4	−1·387	−1	4·840	−2	−4·209	−1	3·577	−1	−5·500	−1	5·079	−1	−6·179	−1	5·873	−1
1·6	−1·047	−1	−2·170	−2	−4·445	−1	3·141	−1	−6·105	−1	4·926	−1	−6·995	−1	5·895	−1
1·8	−6·232	−2	−9·152	−2	−4·531	−1	2·516	−1	−6·580	−1	4·541	−1	−7·702	−1	5·674	−1
2·0	−1·450	−2	−1·557	−1	−4·466	−1	1·742	−1	−6·911	−1	3·947	−1	−8·277	−1	5·223	−1
2·2	3·583	−2	−2·100	−1	−4·254	−1	8·576	−2	−7·088	−1	3·173	−1	−8·705	−1	4·563	−1
2·4	8·592	−2	−2·515	−1	−3·902	−1	−9·512	−3	−7·105	−1	2·248	−1	−8·973	−1	3·718	−1
2·6	1·333	−1	−2·781	−1	−3·422	−1	−1·077	−1	−6·961	−1	1·206	−1	−9·072	−1	2·714	−1
2·8	1·757	−1	−2·892	−1	−2·827	−1	−2·053	−1	−6·657	−1	8·093	−3	−8·995	−1	1·581	−1
3·0	2·114	−1	−2·848	−1	−2·133	−1	−2·987	−1	−6·198	−1	−1·094	−1	−8·743	−1	3·510	−2
3·2	2·390	−1	−2·655	−1	−1·358	−1	−3·851	−1	−5·590	−1	−2·285	−1	−8·317	−1	−9·446	−2
3·4	2·574	−1	−2·328	−1	−5·214	−2	−4·617	−1	−4·846	−1	−3·461	−1	−7·722	−1	−2·273	−1
3·6	2·661	−1	−1·886	−1	3·559	−2	−5·264	−1	−3·978	−1	−4·591	−1	−6·966	−1	−3·604	−1
3·8	2·649	−1	−1·351	−1	1·253	−1	−5·775	−1	−3·001	−1	−5·648	−1	−6·061	−1	−4·907	−1
4·0	2·538	−1	−7·493	−2	2·149	−1	−6·138	−1	−1·934	−1	−6·608	−1	−5·021	−1	−6·153	−1
4·2	2·334	−1	−1·058	−2	3·023	−1	−6·342	−1	−7·955	−2	−7·446	−1	−3·861	−1	−7·316	−1
4·4	2·044	−1	5·527	−2	3·855	−1	−6·386	−1	3·939	−2	−8·145	−1	−2·599	−1	−8·369	−1
4·6	1·678	−1	1·201	−1	4·627	−1	−6·270	−1	1·612	−1	−8·686	−1	−1·254	−1	−9·290	−1
4·8	1·248	−1	1·815	−1	5·321	−1	−5·994	−1	2·838	−1	−9·059	−1	1·524	−2	−1·006	
5·0	7·684	−2	2·372	−1	5·923	−1	−5·567	−1	4·049	−1	−9·255	−1	1·598	−1	−1·066	
5·2	2·540	−2	2·855	−1	6·420	−1	−4·998	−1	5·223	−1	−9·267	−1	3·059	−1	−1·109	
5·4	−2·795	−2	3·249	−1	6·800	−1	−4·299	−1	6·340	−1	−9·096	−1	4·512	−1	−1·133	
5·6	−8·164	−2	3·540	−1	7·054	−1	−3·486	−1	7·380	−1	−8·743	−1	5·935	−1	−1·137	
5·8	−1·341	−1	3·720	−1	7·178	−1	−2·577	−1	8·324	−1	−8·215	−1	7·305	−1	−1·101	
6·0	−1·840	−1	3·785	−1	7·167	−1	−1·588	−1	9·153	−1	−7·519	−1	8·598	−1	−1·086	
6·2	−2·298	−1	3·734	−1	7·024	−1	−5·386	−2	9·854	−1	−6·667	−1	9·795	−1	−1·031	
6·4	−2·703	−1	3·570	−1	6·747	−1	5·504	−2	1·041		−5·675	−1	1·087		−9·576	−1
6·6	−3·045	−1	3·297	−1	6·340	−1	1·658	−1	1·082		−4·561	−1	1·182		−8·666	−1
6·8	−3·314	−1	2·925	−1	5·809	−1	2·765	−1	1·106		−3·340	−1	1·262		−7·592	−1
7·0	−3·505	−1	2·465	−1	5·161	−1	3·850	−1	1·114		−2·035	−1	1·325		−6·370	−1
7·2	−3·610	−1	1·930	−1	4·406	−1	4·894	−1	1·105		−6·660	−2	1·370		−5·018	−1
7·4	−3·629	−1	1·335	−1	3·441	−1	5·638	−1	1·079		7·439	−2	1·397		−3·557	−1
7·6	−3·559	−1	6·958	−2	2·501	−1	6·457	−1	1·037		2·171	−1	1·405		−2·008	−1
7·8	−3·403	−1	3·003	−3	1·500	−1	7·165	−1	9·778	−1	3·592	−1	1·393		−3·927	−2
8·0	−3·164	−1	−6·453	−2	4·539	−2	7·751	−1	9·029	−1	4·983	−1	1·361		1·264	−1
8·2	−2·84	−1	−1·31	−1	−6·20	−2	8·20	−1	8·12	−1	6·32	−1	1·31		2·74	−1
8·4	−2·45	−1	−1·96	−1	−1·70	−1	8·51	−1	7·08	−1	7·58	−1	1·24		4·60	−1
8·6	−1·99	−1	−2·56	−1	−2·78	−1	8·66	−1	5·92	−1	8·74	−1	1·15		6·22	−1
8·8	−1·49	−1	−3·10	−1	−3·82	−1	8·66	−1	4·64	−1	9·78	−1	1·04		7·79	−1
9·0	−9·36	−2	−3·58	−1	−4·82	−1	8·51	−1	3·27	−1	1·07		9·15	−1	9·27	−1

TABLE 1 (*Continued*)

ρ	$\vert\varkappa\vert=1$															
	$p=0\cdot2$				$p=0\cdot4$				$p=0\cdot6$				$p=0\cdot8$			
	X		*Y*		*X*		*Y*		*X*		*Y*		*X*		*Y*	
9·2	−3·57	−2	−3·98	−1	−5·76	−1	8·20	−1	1·83	−1	1·15		7·76	−1	1·06	
9·4	2·40	−2	−4·28	−1	−6·63	−1	7·75	−1	3·39	−2	1·20		6·24	−1	1·19	
9·6	8·40	−2	−4·50	−1	−7·39	−1	7·15	−1	−1·18	−1	1·25		4·61	−1	1·30	
9·8	1·32	−1	−4·60	−1	−8·06	−1	6·42	−1	−2·71	−1	1·27		2·89	−1	1·39	
10·0	2·00	−1	−4·61	−1	−8·60	−1	5·57	−1	−4·23	−1	1·27		1·10	−1	1·46	
10·2	2·53	−1	−4·51	−1	−9·02	−1	4·61	−1	−5·71	−1	1·26		−7·31	−2	1·52	
10·4	3·02	−1	−4·31	−1	−9·30	−1	3·56	−1	−7·14	−1	1·23		−2·58	−1	1·55	
10·6	3·46	−1	−4·01	−1	−9·45	−1	2·44	−1	−8·49	−1	1·17		−4·43	−1	1·56	
10·8	3·82	−1	−3·62	−1	−9·44	−1	1·26	−1	−9·74	−1	1·10		−6·25	−1	1·55	
11·0	4·12	−1	−3·15	−1	−9·29	−1	4·22	−3	−1·09		1·01		−8·01	−1	1·52	
11·2	4·33	−1	−2·61	−1	−9·00	−1	−1·19	−1	−1·19		9·10	−1	−9·70	−1	1·47	
11·4	4·47	−1	−2·00	−1	−8·57	−1	−2·43	−1	−1·27		7·91	−1	−1·13		1·40	
11·6	4·51	−1	−1·35	−1	−8·00	−1	−3·63	−1	−1·34		6·59	−1	−1·27		1·30	
11·8	4·47	−1	−6·61	−2	−7·30	−1	−4·80	−1	−1·39		5·16	−1	−1·41		1·19	
12·0	4·33	−1	4·64	−3	−6·48	−1	−5·91	−1	−1·43		3·63	−1	−1·52		1·07	
12·2	4·12	−1	7·63	−2	−5·56	−1	−6·94	−1	−1·44		2·03	−1	−1·62		9·09	−1
12·4	3·82	−1	1·47	−1	−4·54	−1	−7·87	−1	−1·44		3·76	−2	−1·70		7·47	−1
12·6	3·44	−1	2·16	−1	−3·44	−1	−8·69	−1	−1·41		−1·31	−1	−1·75		5·72	−1
12·8	2·99	−1	2·80	−1	−2·27	−1	−9·39	−1	−1·37		−2·99	−1	−1·79		3·88	−1
13·0	2·48	−1	3·40	−1	−1·06	−1	−9·96	−1	−1·31		−4·66	−1	−1·80		1·96	−1
13·2	1·92	−1	3·94	−1	1·85	−2	−1·04		−1·23		−6·28	−1	−1·79		−2·03	−3
13·4	1·32	−1	4·41	−1	1·44	−1	−1·06		−1·13		−7·84	−1	−1·76		−2·02	−1
13·6	6·87	−2	4·71	−1	2·69	−1	−1·08		−1·01		−9·31	−1	−1·71		−4·03	−1
13·8	3·26	−3	5·09	−1	3·92	−1	−1·07		−8·86	−1	−1·07		−1·63		−6·00	−1
14·0	−6·32	−2	5·30	−1	5·11	−1	−1·05		−7·44	−1	−1·19		−1·53		−7·93	−1
14·2	−1·29	−1	5·40	−1	6·24	−1	−1·01		−5·91	−1	−1·30		−1·41		−9·77	−1
14·4	−1·93	−1	5·41	−1	7·29	−1	−9·60	−1	−4·28	−1	−1·39		−1·27		−1·15	
14·6	−2·54	−1	5·31	−1	8·26	−1	−8·93	−1	−2·57	−1	−1·47		−1·12		−1·31	
14·8	−3·12	−1	5·12	−1	9·12	−1	−8·12	−1	−8·16	−2	−1·53		−9·47	−1	−1·46	
15·0	−3·65	−1	4·82	−1	9·86	−1	−7·19	−1	9·73	−2	−1·56		−7·62	−1	−1·59	
15·2	−4·12	−1	4·44	−1	1·05		−6·14	−1	2·77	−1	−1·58		−5·65	−1	−1·70	
15·4	−4·52	−1	3·98	−1	1·09		−4·99	−1	4·55	−1	−1·57		−3·60	−1	−1·79	
15·6	−4·84	−1	3·43	−1	1·14		−3·75	−1	6·30	−1	−1·55		−1·47	−1	−1·86	
15·8	−5·09	−1	2·82	−1	1·14		−2·45	−1	7·98	−1	−1·50		6·92	−2	−1·91	
16·0	−5·25	−1	2·16	−1	1·15		−1·15	−1	9·58	−1	−1·44		2·87	−1	−1·93	
	$\vert\varkappa\vert=2$															
0·2	−1·423	−2	−1·127	−2	−1·510	−2	−1·195	−2	−1·538	−2	−1·217	−2	−1·551	−2	−1·227	−2
0·4	−6·536	−2	−5·415	−2	−7·379	−2	−6·102	−2	−7·659	−2	−6·330	−2	−7·790	−2	−6·436	−2
0·6	−1·489	−1	−1·293	−1	−1·794	−1	−1·551	−1	−1·898	−1	−1·640	−1	−1·948	−1	−1·681	−1
0·8	−2·534	−1	−2·314	−1	−3·269	−1	−2·959	−1	−3·529	−1	−3·188	−1	−3·655	−1	−3·297	−1
1·0	−3·660	−1	−3·522	−1	−5·075	−1	−4·815	−1	−5·595	−1	−5·288	−1	−5·848	−1	−5·517	−1
1·2	−4·744	−1	−4·829	−1	−7·107	−1	−7·076	−1	−8·006	−1	−7·925	−1	−8·451	−1	−8·344	−1
1·4	−5·682	−1	−6·147	−1	−9·251	−1	−9·683	−1	−1·066		−1·107		−1·137		−1·176	
1·6	−6·394	−1	−7·396	−1	−1·140		−1·256		−1·344		−1·466		−1·449		−1·572	
1·8	−6·820	−1	−8·501	−1	−1·344		−1·563		−1·625		−1·864		−1·770		−2·018	
2·0	−6·925	−1	−9·397	−1	−1·528		−1·881		−1·897		−2·293		−2·090		−2·505	

Table 1 (*Continued*)

ρ	$\lvert\kappa\rvert=2$															
	$p=0\cdot2$				$p=0\cdot4$				$p=0\cdot6$				$p=0\cdot8$			
	X		*Y*		*X*		*Y*		*X*		*Y*		*X*		*Y*	
2·2	−6·695	−1	−1·003		−1·682		−2·200		−2·149		−2·744		−2·397		−3·028	
2·4	−6·138	−1	−1·038		−1·800		−2·513		−2·374		−3·208		−2·681		−3·575	
2·6	−5·279	−1	−1·041		−1·876		−2·811		−2·561		−3·675		−2·932		−4·139	
2·8	−4·156	−1	−1·012		−1·904		−3·086		−2·702		−4·137		−3·140		−4·707	
3·0	−2·817	−1	−9·509	−1	−1·882		−3·330		−2·792		−4·585		−3·297		−5·272	
3·2	−1·319	−1	−8·594	−1	−1·810		−3·535		−2·825		−5·010		−3·397		−5·825	
3·4	2·762	−2	−7·401	−1	−1·686		−3·695		−2·797		−5·400		−3·434		−6·352	
3·6	1·906	−1	−5·966	−1	−1·513		−3·805		−2·707		−5·748		−3·403		−6·845	
3·8	3·507	−1	−4·330	−1	−1·295		−3·861		−2·553		−6·045		−3·302		−7·292	
4·0	5·021	−1	−2·540	−1	−1·035		−3·859		−2·336		−6·282		−3·131		−7·684	
4·2	6·392	−1	−6·462	−2	−7·380	−1	−3·797		−2·060		−6·454		−2·890		−8·012	
4·4	7·574	−1	1·297	−1	−4·111	−1	−3·674		−1·728		−6·555		−2·580		−8·267	
4·6	8·524	−1	3·236	−1	−6·054	−2	−3·491		−1·343		−6·579		−2·205		−8·447	
4·8	9·211	−1	5·118	−1	3·061	−1	−3·250		−9 129	−1	−6·524		−1·768		−8·543	
5·0	9·614	−1	6·892	−1	6·810	−1	−2·953		−4·438	−1	−6·389		−1·276		−8·552	
5·2	9·719	−1	8·512	−1	1·056		−2·604		5·654	−2	−6·176		−7·353	−1	−8·470	
5·4	9·524	−1	9·934	−1	1·423		−2·208		5·798	−1	−5·882		−1·536	−1	−8·291	
5·6	9·033	−1	1·112		1·775		−1·771		1·117		−5·510		4·605	−1	−8·013	
5·8	8·261	−1	1·205		2·104		−1·299		1·658		−5·061		1·097		−7·639	
6·0	7·232	−1	1·269		2·403		−7·984	−1	2·194		−4·539		1·747		−7·168	
6·2	5·974	−1	1·302		2·665		−2·767	−1	2·715		−3·950		2·398		−6·605	
6·4	4·522	−1	1·305		2·884		2·581	−1	3·213		−3·301		3·042		−5·956	
6·6	2·917	−1	1·276		3·056		7·975	−1	3·678		−2·597		3·667		−5·225	
6·8	1·200	−1	1·217		3·177		1·333		4·100		−1·849		4·262		−4·422	
7·0	−5·810	−2	1·129		3·242		1·857		4·474		−1·064		4·819		−3·552	
7·2	−2·381	−1	1·013		3·251		2·361		4·790		−2·524	−1	5·327		−2·627	
7·4	−4·153	−1	8·734	−1	3·201		2·836		5·043		5·757	−1	5·777		−1·654	
7·6	−5·854	−1	7·119	−1	3·094		3·275		5·228		1·410		6·160		−6·467	−1
7·8	−7·441	−1	5·325	−1	2·929		3·671		5·339		2·238		6·469		3·850	−1
8·0	−8·874	−1	3·390	−1	2·709		4·017		5·374		3·051		6·698		1·428	
8·2	−1·01		1·35	−1	2·44		4·31		5·33		3·84		6·84		2·47	
8·4	−1·11		−7·36	−2	2·11		4·53		5·21		4·58		6·89		3·50	
8·6	−1·19		−2·84	−1	1·75		4·70		5·00		5·28		6·85		4·50	
8·8	−1·24		−4·90	−1	1·35		4·79		4·72		5·93		6·72		5·46	
9·0	−1·26		−6·89	−1	9·12	−1	4·81		4·37		6·50		6·49		6·38	
9·2	−1·25		−8·75	−1	4·53	−1	4·77		3·94		7·00		6·16		7·23	
9·4	−1·22		−1·05		−2·30	−2	4·65		3·44		7·42		5·75		8·00	
9·6	−1·15		−1·20		−5·09	−1	4·46		2·89		7·74		5·25		8·69	
9·8	−1·06		−1·33		−9·96	−1	4·20		2·28		7·97		4·66		9·29	
10·0	−9·50	−1	−1·43		−1·48		3·88		1·63		8·11		4·00		9·78	
10·2	−8·13	−1	−1·51		−1·94		3·49		8·50	−1	8·06		3·28		1·02	1
10·4	−6·56	−1	−1·55		−2·39		3·05		1·23	−1	7·97		2·49		1·04	1
10·6	−4·84	−1	−1·57		−2·80		2·56		−6·21	−1	7·78		1·66		1·06	1
10·8	−2·98	−1	−1·56		−3·18		2·03		−1·37		7·49		7·79	−1	1·06	1
11·0	−1·04	−1	−1·51		−3·52		1·47		−2·12		7·09		−1·26	−1	1·05	1
11·2	9·46	−2	−1·44		−3·81		8·70	−1	−2·85		6·61		−1·05		1·03	1
11·4	2·94	−1	−1·34		−4·04		4·60	−1	−3·55		6·03		−1·98		9·91	
11·6	4·91	−1	−1·21		−4·22		−3·70	−1	−4·23		5·36		−2·90		9·43	
11·8	6·79	−1	−1·06		−4·33		−9·98	−1	−4·85		4·62		−3·80		8·83	
12·0	8·57	−1	−8·84	−1	−4·38		−1·62		−5·42		3·82		−4·68		8·13	

TABLE 1 (*Continued*)

ρ	$\lvert\varkappa\rvert=2$															
	$p=0\cdot2$				$p=0\cdot4$				$p=0\cdot6$				$p=0\cdot8$			
	X		Y		X		Y		X		Y		X		Y	
12·2	1·02		−6·92	−1	−4·37		−2·22		−5·93		2·95		−5·63		7·00	
12·4	1·17		−4·86	−1	−4·29		−2·80		−6·37		2·04		−6·39		6·07	
12·6	1·29		−2·69	−1	−4·14		−3·35		−6·73		1·09		−7·09		5·06	
12·8	1·39		−4·44	−2	−3·94		−3·86		−7·00		1·41	−1	−7·71		3·98	
13·0	1·46		1·83	−1	−3·66		−4·32		−7·19		−8·72	−1	−8·25		2·83	
13·2	1·51		4·09	−1	−3·34		−4·72		−7·28		−1·86		−8·69		1·64	
13·4	1·53		6·30	−1	−2·95		−5·06		−7·28		−2·83		−9·02		4·17	−1
13·6	1·52		8·42	−1	−2·52		−5·34		−7·19		−3·78		−9·25		−8·26	−1
13·8	1·49		1·04		−2·05		−5·55		−6·99		−4·69		−9·37		−2·07	
14·0	1·42		1·22		−1·54		−5·68		−6·71		−5·55		−9·37		−3·31	
14·2	1·33		1·39		−1·00		−5·73		−6·33		−6·36		−9·25		−4·52	
14·4	1·21		1·53		−4·41	−1	−5·71		−5·86		−7·09		−9·02		−5·68	
14·6	1·07		1·64		1·34	−1	−5·62		−5·31		−7·74		−8·67		−6·80	
14·8	9·05	−1	1·73		7·15	−1	−5·44		−4·68		−8·30		−8·21		−7·84	
15·0	7·24	−1	1·79		1·29		−5·19		−3·99		−8·77		−7·65		−8·80	
15·2	5·29	−1	1·82		1·86		−4·87		−3·23		−9·13		−6·97		−9·67	
15·4	3·22	−1	1·82		2·41		−4·48		−2·42		−9·39		−6·21		−1·04	1
15·6	1·07	−1	1·78		2·93		−4·03		−1·57		−9·53		−5·36		−1·11	1
15·8	−1·12	−1	1·72		3·42		−3·52		−6·86	−1	−9·55		−4·43		−1·16	1
16·0	−3·31	−1	1·63		3·87		−2·95		2·18	−1	−9·46		−3·43		−1·20	1
	$\lvert\varkappa\rvert=3$															
0·2	2·343	−3	−1·091	−2	2·441	−3	−2·841	−3	2·473	−3	−2·878	−3	2·487	−3	−2·894	−3
0·4	2·369	−2	−2·674	−2	2·573	−2	−2·907	−2	2·640	−2	−2·982	−2	2·671	−2	−3·018	−2
0·6	8·874	−2	−9·712	−2	1·006	−1	−1·102	−1	1·045	−1	−1·146	−1	1·064	−1	−1·167	−1
0·8	2·210	−1	−2·344	−1	2·615	−1	−2·780	−1	2·753	−1	−2·929	−1	2·818	−1	−3·000	−1
1·0	4·397	−1	−4·513	−1	5·434	−1	−5·602	−1	5·796	−1	−5·985	−1	5·970	−1	−6·168	−1
1·2	7·579	−1	−7·524	−1	9·793	−1	−9·786	−1	1·059		−1·060		1·097		−1·099	
1·4	1·182		−1·134		1·599		−1·547		1·751		−1·706		1·826		−1·775	
1·6	1·712		−1·585		2·426		−2·273		2·693		−2·535		2·825		−2·663	
1·8	2·341		−2·088		3·479		−3·155		3·917		−3·571		4·134		−3·776	
2·0	3·057		−2·623		4·771		−4·184		5·321		−4·788		5·785		−5·119	
2·2	3·843		−3·166		6·307		−5·343		7·303		−6·235		7·806		−6·686	
2·4	4·677		−3·692		8·084		−6·611		9·498		−7·838		1·022	1	−8·466	
2·6	5·534		−4·177		1·009	1	−7·957		1·203	1	−9·593		1·303	1	−1·044	1
2·8	6·391		−4·597		1·232	1	−9·347		1·491	1	−1·147	1	1·625	1	−1·257	1
3·0	7·220		−4·930		1·473	1	−1·075	1	1·811	1	−1·343	1	1·988	1	−1·484	1
3·2	7·994		−5·157		1·731	1	−1·213	1	2·162	1	−1·543	1	2·390	1	−1·720	1
3·4	8·689		−5·264		2·002	1	−1·344	1	2·541	1	−1·744	1	2·828	1	−1·960	1
3·6	9·279		−5·240		2·280	1	−1·465	1	2·943	1	−1·941	1	3·300	1	−2·200	1
3·8	9·744		−5·077		2·563	1	−1·572	1	3·366	1	−2·129	1	3·802	1	−2·435	1
4·0	1·006	1	−4·774		2·846	1	−1·661	1	3·804	1	−2·303	1	4·329	1	−2·660	1
4·2	1·022	1	−4·335		3·125	1	−1·729	1	4·251	1	−2·459	1	4·875	1	−2·868	1
4·4	1·021	1	−3·766		3·394	1	−1·773	1	4·701	1	−2·591	1	5·434	1	−3·056	1
4·6	1·001	1	−3·079		3·649	1	−1·791	1	5·149	1	−2·696	1	6·000	1	−3·216	1
4·8	9·630		−2·289		3·883	1	−1·780	1	5·590	1	−2·769	1	6·566	1	−3·346	1
5·0	9·068		−1·414		4·094	1	−1·739	1	6·017	1	−2·806	1	7·128	1	−3·440	1

TABLE 1 (*Continued*)

ρ	|κ|=3															
	p=0·2				p=0·4				p=0·6				p=0·8			
	X		*Y*		*X*		*Y*		*X*		*Y*		*X*		*Y*	
5·2	8·331		−4·749	−1	4·276	1	−1·667	1	6·422	1	−2·801	1	7·676	1	−3·494	1
5·4	7·428		5·059	−1	4·424	1	−1·563	1	6·802	1	−2·763	1	8·205	1	−3·504	1
5·6	6·372		1·504		4·534	1	−1·428	1	7·144	1	−2·677	1	8·704	1	−3·468	1
5·8	5·182		2·494		4·603	1	−1·261	1	7·447	1	−2·548	1	9·166	1	−3·383	1
6·0	3·879		3·451		4·628	1	−1·066	1	7·703	1	−2·374	1	9·583	1	−3·247	1
6·2	2·486		4·352		4·607	1	−8·448		7·906	1	−2·157	1	9·948	1	−3·059	1
6·4	1·029		5·173		4·537	1	−5·991		8·051	1	−1·897	1	1·025	2	−2·816	1
6·6	−4·652	−1	5·894		4·417	1	−3·327		8·133	1	−1·597	1	1·049	2	−2·521	1
6·8	−1·968		6·495		4·249	1	−4·916	−1	8·147	1	−1·257	1	1·066	2	−2·176	1
7·0	−3·451		6·960		4·033	1	2·474		8·091	1	−8·830		1·074	2	−1·783	1
7·2	−4·886		7·279		3·768	1	5·525		7·962	1	−4·777		1·074	2	−1·345	1
7·4	−6·247		7·439		3·456	1	8·611		7·760	1	−4·582	−1	1·066	2	−8·663	
7·6	−7·507		7·435		3·101	1	1·169	1	7·484	1	4·074		1·048	2	−3·515	
7·8	−8·642		7·267		2·704	1	1·470	1	7·136	1	8·761		1·022	2	1·938	
8·0	−9·631		6·934		2·271	1	1·760	1	6·714	1	1·354	1	9·857	1	7·635	
8·2	−1·04	1	6·44		1·81	1	2·03	1	6·22	1	1·83	1	9·40	1	1·35	1
8·4	−1·11	1	5·80		1·31	1	2·29	1	5·66	1	2·31	1	8·86	1	1·95	1
8·6	−1·15	1	5·01		7·98		2·52	1	5·03	1	2·78	1	8·23	1	2·55	1
8·8	−1·17	1	4·11		2·67		2·72	1	4·35	1	3·23	1	7·51	1	3·14	1
9·0	−1·17	1	3·10		−2·82		2·87	1	3·61	1	3·63	1	6·67	1	3·72	1
9·2	−1·15	1	2·00		−8·27		3·00	1	2·83	1	4·04	1	5·79	1	4·28	1
9·4	−1·11	1	8·40	−1	−1·36	1	3·08	1	2·00	1	4·40	1	4·84	1	4·80	1
9·6	−1·04	1	−3·59	−1	−1·90	1	3·12	1	1·15	1	4·72	1	3·83	1	5·29	1
9·8	−9·58		−1·57		−2·41	1	3·12	1	2·71		4·98	1	2·77	1	5·73	1
10·0	−8·57		−2·78		−2·90	1	3·07	1	−7·01		5·19	1	1·67	1	6·12	1
10·2	−7·39		−3·94		−3·35	1	2·98	1	−1·60	1	5·30	1	5·43		6·45	1
10·4	−6·07		−5·06		−3·77	1	2·84	1	−2·49	1	5·39	1	−6·04		6·71	1
10·6	−4·63		−6·09		−4·15	1	2·66	1	−3·37	1	5·40	1	−1·76	1	6·90	1
10·8	−3·09		−7·01		−4·47	1	2·43	1	−4·22	1	5·35	1	−2·91	1	7·02	1
11·0	−2·97		−7·82		−4·74	1	2·17	1	−5·03	1	5·23	1	−4·04	1	7·05	1
11·2	1·65	−1	−8·49		−4·96	1	1·87	1	−5·79	1	5·05	1	−5·14	1	7·00	1
11·4	1·82		−9·01		−5·12	1	1·54	1	−6·49	1	4·79	1	−6·20	1	6·87	1
11·6	3·47		−9·36		−5·21	1	1·18	1	−7·13	1	4·47	1	−7·20	1	6·66	1
11·8	5·07		−9·54		−5·23	1	7·89		−7·70	1	4·09	1	−8·14	1	6·36	1
12·0	6·59		−9·54		−5·19	1	3·83		−8·18	1	3·64	1	−9·00	1	5·98	1
12·2	8·01		−9·37		−5·09	1	−3·57	−1	−8·58	1	3·14	1	−9·78	1	5·52	1
12·4	9·32		−9·02		−4·92	1	−4·63		−8·89	1	2·60	1	−1·05	2	4·99	1
12·6	1·05	1	−8·49		−4·68	1	−8·91		−9·09	1	2·00	1	−1·10	2	4·38	1
12·8	1·15	1	−7·81		−4·39	1	−1·32	1	−9·20	1	1·37	1	−1·15	2	3·71	1
13·0	1·23	1	−6·97		−4·03	1	−1·73	1	−9·20	1	7·12		−1·18	2	2·98	1
13·2	1·29	1	−5·99		−3·62	1	−2·13	1	−9·10	1	2·89	−1	−1·21	2	2·20	1
13·4	1·32	1	−4·89		−3·16	1	−2·51	1	−8·89	1	−6·68		−1·21	2	1·38	1
13·6	1·34	1	−3·69		−2·66	1	−2·86	1	−8·58	1	−1·37	1	−1·21	2	5·20	
13·8	1·34	1	−2·41		−2·12	1	−3·17	1	−8·17	1	−2·07	1	−1·19	2	−3·58	
14·0	1·31	1	−1·06		−1·54	1	−3·45	1	−7·67	1	−2·76	1	−1·16	2	−1·25	1
14·2	1·26	1	3·18	−1	−9·44		−3·69	1	−7·07	1	−3·42	1	−1·12	2	−2·15	1
14·4	1·19	1	1·71		−3·29		−3·88	1	−6·38	1	−4·06	1	−1·06	2	−3·03	1
14·6	1·10	1	3·09		2·96		−4·02	1	−5·62	1	−4·66	1	−9·96	1	−3·90	1
14·8	9·92		4·44		9·22		−4·12	1	−4·78	1	−5·21	1	−9·17	1	−4·74	1
15·0	8·67		5·72		1·54	1	−4·15	1	−3·88	1	−5·71	1	−8·27	1	−5·54	1

Table 1 (*Continued*)

	$\lvert\varkappa\rvert=3$															
	$p=0\cdot2$				$p=0\cdot4$				$p=0\cdot6$				$p=0\cdot8$			
ρ	*X*		*Y*		*X*		*Y*		*X*		*Y*		*X*		*Y*	
15·2	7·26		6·92		2·15	1	−4·14	1	−2·93	1	−6·15	1	−7·27	1	−6·29	1
15·4	5·73		8·02		2·73	1	−4·07	1	−1·94	1	−6·52	1	−6·19	1	−6·98	1
15·6	4·09		9·00		3·29	1	−3·94	1	−9·15		−6·82	1	−5·02	1	−7·60	1
15·8	2·37		9·83		3·81	1	−3·76	1	1·28		−7·04	1	−3·80	1	−8·15	1
16·0	6·07	−1	1·05	1	4·28	1	−3·52	1	1·18	1	−7·19	1	−2·52	1	−8·61	1
	$\lvert\varkappa\rvert=4$															
0·2	5·343	−4	4·756	−4	5·515	−4	4·928	−4	5·570	−4	4·976	−4	5·595	−4	4·998	−4
0·4	1·084	−2	9·912	−3	1·156	−2	1·056	−2	1·179	−2	1·077	−2	1·189	−2	1·087	−2
0·6	6·090	−2	5·700	−2	6·707	−2	6·273	−2	6·910	−2	6·461	−2	7·005	−2	6·549	−2
0·8	2·020	−1	1·936	−1	2·299	−1	2·201	−1	2·393	−1	2·289	−1	2·437	−1	2·331	−1
1·0	5·014	−1	4·922	−1	5·903	−1	5·782	−1	6·206	−1	6·076	−1	6·350	−1	6·215	−1
1·2	1·035		1·041		1·261		1·265		1·340		1·343		1·377		1·380	
1·4	1·881		1·940		2·373		2·437		2·547		2·613		2·631		2·697	
1·6	3·111		3·291		4·065		4·276		4·410		4·633		4·577		4·804	
1·8	4·783		5·193		6·480		6·985		7·107		7·647		7·412		7·966	
2·0	6·940		7·738		9·757		1·078	1	1·082	1	1·192	1	1·134	1	1·248	1
2·2	9·603		1·100	1	1·402	1	1·589	1	1·572	1	1·776	1	1·656	1	1·868	1
2·4	1·277	1	1·505	1	1·938	1	2·253	1	2·198	1	2·544	1	2·327	1	2·689	1
2·6	1·641	1	1·990	1	2·592	1	3·091	1	2·974	1	3·529	1	3·164	1	3·748	1
2·8	2·047	1	2·558	1	3·369	1	4·126	1	3·911	1	4·761	1	4·182	1	5·082	1
3·0	2·487	1	3·206	1	4·270	1	5·373	1	5·017	1	6·269	1	5·394	1	6·725	1
3·2	2·952	1	3·930	1	5·295	1	6·849	1	6·298	1	8·081	1	6·807	1	8·711	1
3·4	3·431	1	4·723	1	6·437	1	8·565	1	7·754	1	1·022	2	8·426	1	1·107	2
3·6	3·910	1	5·573	1	7·687	1	1·053	2	9·379	1	1·271	2	1·025	2	1·384	2
3·8	4·377	1	6·468	1	9·030	1	1·275	2	1·117	2	1·557	2	1·227	2	1·704	2
4·0	4·817	1	7·393	1	1·045	2	1·521	2	1·310	2	1·880	2	1·448	2	2·068	2
4·2	5·215	1	8·331	1	1·192	2	1·792	2	1·516	2	2·241	2	1·686	2	2·479	2
4·4	5·557	1	9·265	1	1·343	2	2·085	2	1·732	2	2·640	2	1·939	2	2·937	2
4·6	5·830	1	1·018	2	1·494	2	2·400	2	1·956	2	3·077	2	2·204	2	3·442	2
4·8	6·020	1	1·105	2	1·643	2	2·734	2	2·184	2	3·550	2	2·477	2	3·994	2
5·0	6·117	1	1·186	2	1·787	2	3·084	2	2·413	2	4·057	2	2·756	2	4·590	2
5·2	6·112	1	1·259	2	1·923	2	3·446	2	2·639	2	4·595	2	3·035	2	5·230	2
5·4	5·993	1	1·323	2	2·047	2	3·818	2	2·860	2	5·161	2	3·312	2	5·909	2
5·6	5·759	1	1·374	2	2·157	2	4·194	2	3·070	2	5·750	2	3·582	2	6·624	2
5·8	5·411	1	1·412	2	2·249	2	4·570	2	3·265	2	6·357	2	3·840	2	7·369	2
6·0	4·949	1	1·435	2	2·320	2	4·944	2	3·442	2	6·976	2	4·081	2	8·139	2
6·2	4·378	1	1·442	2	2·368	2	5·311	2	3·594	2	7·602	2	4·299	2	8·928	2
6·4	3·704	1	1·432	2	2·389	2	5·665	2	3·719	2	8·228	2	4·489	2	9·729	2
6·6	2·937	1	1·404	2	2·381	2	6·001	2	3·811	2	8·850	2	4·647	2	1·054	3
6·8	2·086	1	1·358	2	2·343	2	6·314	2	3·867	2	9·462	2	4·767	2	1·134	3
7·0	1·168	1	1·294	2	2·274	2	6·599	2	3·883	2	1·006	3	4·844	2	1·214	3
7·2	1·951		1·211	2	2·171	2	6·850	2	3·856	2	1·062	3	4·874	2	1·292	3
7·4	−8·136		1·112	2	2·035	2	7·062	2	3·783	2	1·116	3	4·853	2	1·368	3
7·6	−1·840	1	9·957	1	1·866	2	7·231	2	3·664	2	1·166	3	4·777	2	1·440	3
7·8	−2·866	1	8·645	1	1·664	2	7·351	2	3·495	2	1·210	3	4·645	2	1·508	3
8·0	−3·872	1	7·197	1	1·430	2	7·419	2	3·276	2	1·249	3	4·453	2	1·571	3

TABLE 1 (*Continued*)

ρ	$\lvert\kappa\rvert=4$															
	$p=0\cdot2$				$p=0\cdot4$				$p=0\cdot6$				$p=0\cdot8$			
	X		Y		X		Y		X		Y		X		Y	
8·2	−4·84	1	5·63	1	1·17	2	7·43	2	3·01	2	1·28	3	4·20	2	1·63	3
8·4	−5·74	1	3·96	1	8·73	1	7·38	2	2·69	2	1·31	3	3·89	2	1·68	3
8·6	−6·56	1	2·22	1	5·56	1	7·26	2	2·32	2	1·33	3	3·51	2	1·72	3
8·8	−7·29	1	4·25		2·17	1	7·08	2	1·91	2	1·34	3	3·08	2	1·75	3
9·0	−7·91	1	−1·39	1	−1·40	1	6·84	2	1·45	2	1·34	3	2·59	2	1·77	3
9·2	−8·39	1	−3·20	1	−5·11	1	6·54	2	9·52	1	1·33	3	2·04	2	1·78	3
9·4	−8·75	1	−4·98	1	−8·92	1	6·17	2	4·19	1	1·31	3	1·44	2	1·78	3
9·6	−8·95	1	−6·70	1	−1·28	2	5·73	2	−1·45	1	1·28	3	7·97	1	1·77	3
9·8	−9·00	1	−8·33	1	−1·66	2	5·24	2	−7·35	1	1·25	3	1·08	1	1·74	3
10·0	−8·89	1	−9·85	1	−2·04	2	4·70	2	−1·34	2	1·20	3	−6·17	1	1·71	3
10·2	−8·62	1	−1·12	2	−2·41	2	4·10	2	−1·99	2	1·11	3	−1·37	2	1·66	3
10·4	−8·19	1	−1·25	2	−2·76	2	3·45	2	−2·62	2	1·04	3	−2·15	2	1·60	3
10·6	−7·61	1	−1·35	2	−3·09	2	2·77	2	−3·25	2	9·61	2	−2·94	2	1·52	3
10·8	−6·89	1	−1·43	2	−3·39	2	2·05	2	−3·86	2	8·72	2	−3·73	2	1·43	3
11·0	−6·03	1	−1·50	2	−3·66	2	1·30	2	−4·46	2	7·74	2	−4·52	2	1·34	3
11·2	−5·05	1	−1·54	2	−3·89	2	5·32	1	−5·03	2	6·68	2	−5·31	2	1·23	3
11·4	−3·96	1	−1·56	2	−4·08	2	−2·49	1	−5·56	2	5·55	2	−6·07	2	1·10	3
11·6	−2·79	1	−1·55	2	−4·23	2	−1·04	2	−6·06	2	4·36	2	−6·80	2	9·73	2
11·8	−1·54	1	−1·52	2	−4·33	2	−1·82	2	−6·50	2	3·12	2	−7·50	2	8·32	2
12·0	−2·41		−1·47	2	−4·38	2	−2·59	2	−6·89	2	1·83	2	−8·15	2	6·84	2
12·2	1·08	1	−1·39	2	−4·38	2	−3·33	2	−7·22	2	5·23	1	−8·70	2	4·54	2
12·4	2·42	1	−1·29	2	−4·33	2	−4·05	2	−7·48	2	−8·04	1	−9·19	2	2·89	2
12·6	3·73	1	−1·17	2	−4·22	2	−4·73	2	−7·66	2	−2·14	2	−9·60	2	1·20	2
12·8	5·00	1	−1·04	2	−4·05	2	−5·37	2	−7·77	2	−3·46	2	−9·93	2	−5·20	1
13·0	6·22	1	−8·82	1	−3·83	2	−5·95	2	−7·80	2	−4·76	2	−1·02	3	−2·25	2
13·2	7·35	1	−7·13	1	−3·57	2	−6·47	2	−7·74	2	−6·03	2	−1·03	3	−3·97	2
13·4	8·37	1	−5·32	1	−3·25	2	−6·93	2	−7·60	2	−7·25	2	−1·03	3	−5·67	2
13·6	9·28	1	−3·41	1	−2·88	2	−7·31	2	−7·37	2	−8·40	2	−1·03	3	−7·34	2
13·8	1·01	2	−1·44	1	−2·47	2	−7·61	2	−7·05	2	−9·49	2	−1·01	3	−8·95	2
14·0	1·07	2	5·70		−2·02	2	−7·83	2	−6·65	2	−1·05	3	−9·81	2	−1·05	3
14·2	1·11	2	2·58	1	−1·54	2	−7·96	2	−6·17	2	−1·14	3	−9·42	2	−1·19	3
14·4	1·14	2	4·58	1	−1·03	2	−8·01	2	−5·61	2	−1·22	3	−8·92	2	−1·33	3
14·6	1·15	2	6·51	1	−4·96	1	−7·97	2	−4·98	2	−1·29	3	−8·32	2	−1·45	3
14·8	1·14	2	8·37	1	5·43		−7·83	2	−4·28	2	−1·34	3	−7·61	2	−1·57	3
15·0	1·11	2	1·01	2	6·15	1	−7·61	2	−3·51	2	−1·39	3	−6·82	2	−1·66	3
15·2	1·07	2	1·17	2	1·18	2	−7·30	2	−2·69	2	−1·41	3	−5·93	2	−1·74	3
15·4	1·01	2	1·32	2	1·74	2	−6·90	2	−1·83	2	−1·43	3	−4·96	2	−1·81	3
15·6	9·25	1	1·44	2	2·29	2	−6·43	2	−9·24	1	−1·43	3	−3·97	2	−1·86	3
15·8	8·29	1	1·55	2	2·82	2	−5·87	2	6·85	−1	−1·41	3	−2·82	2	−1·89	3
16·0	7·19	1	1·63	2	3·32	2	−5·25	2	9·55	1	−1·38	3	−1·67	2	−1·90	3

TABLE 2

$K(\varkappa,p)=x(\varkappa,p)+iy(\varkappa,p)$

$\varkappa$	$p=0{\cdot}2$		$p=0{\cdot}4$		$p=0{\cdot}6$		$p=0{\cdot}8$	
	x	y	x	y	x	y	x	y
1	2·775	—1·346	1·611	—0·7315	1·285	—0·5444	1·155	—0·4538
2	—1·700	—3·105	—1·438	—2·016	—1·269	—1·826	—1·195	—1·762
3	—3·240	2·651	—2·600	2·087	—2·477	1·953	—2·439	1·896
4	3·231	3·753	2·755	3·245	2·645	3·154	2·596	3·130
—1	2·176	—2·188	0·913	—1·516	0·5065	—1·301	0·3113	—1·201
—2	—3·336	—1·184	—2·476	—0·0267	—2·198	0·342	—2·064	0·520
—3	—0·367	4·176	0·736	3·257	1·092	2·963	1·264	2·823
—4	4·936	—0·398	3·993	—1·475	3·691	—1·824	3·545	—1·992

TABLE 3

$N_{\pm}(\varkappa,p)$

$\lvert\varkappa\rvert$	$p=0{\cdot}2$		$p=0{\cdot}4$	
	N_+	N_-	N_+	N_-
1	$8{\cdot}727\times10^{-1}$	$i8{\cdot}641\times10^{-2}$	$1{\cdot}017$	$i1{\cdot}958\times10^{-1}$
2	$2{\cdot}369\times10^{-1}$	$i2{\cdot}346\times10^{-2}$	$1{\cdot}522\times10^{-1}$	$i2{\cdot}932\times10^{-2}$
3	$2{\cdot}810\times10^{-2}$	$i2{\cdot}782\times10^{-3}$	$1{\cdot}251\times10^{-2}$	$i2{\cdot}410\times10^{-3}$
4	$2{\cdot}064\times10^{-3}$	$i2{\cdot}044\times10^{-4}$	$7{\cdot}302\times10^{-4}$	$i1{\cdot}406\times10^{-4}$

$\lvert\varkappa\rvert$	$p=0{\cdot}6$		$p=0{\cdot}8$	
	N_+	N_-	N_+	N_-
1	$1{\cdot}145$	$i3{\cdot}171\times10^{-1}$	$1{\cdot}271$	$i4{\cdot}460\times10^{-1}$
2	$1{\cdot}331\times10^{-1}$	$i3{\cdot}687\times10^{-2}$	$1{\cdot}303\times10^{-1}$	$i4{\cdot}572\times10^{-2}$
3	$9{\cdot}816\times10^{-3}$	$i2{\cdot}719\times10^{-3}$	$9{\cdot}185\times10^{-3}$	$i3{\cdot}223\times10^{-3}$
4	$5{\cdot}412\times10^{-4}$	$i1{\cdot}499\times10^{-4}$	$4{\cdot}963\times10^{-4}$	$i1{\cdot}642\times10^{-4}$

V. EXTENSION OF THE TABLES

For values of $Z\neq80$, it is evident that the use of these tables for $75\leqslant Z\leqslant85$ would involve errors of 5 per cent. or less.

In the region $|\varkappa|>4$, the Sommerfeld-Maue type of approximation (Bethe and Maximon 1954) is generally satisfactory.

For values of p not tabulated, the following possibilities arise

(i) extrapolation to $p<0{\cdot}2$;
(ii) interpolation for p in the region $0{\cdot}2<p<0{\cdot}8$;
(iii) extrapolation to $p>0{\cdot}8$.

For case (i), it usually suffices to approximate to the Dirac wave functions by means of the Darwin wave functions (Darwin 1928).

For case (ii), graphical interpolation of f,g will generally be accurate to a few per cent.

For case (iii), the following procedure is suggested. Plot the Whittaker functions for a given $\varkappa,\rho$, against $\eta=Z(1+p^2)^{\frac{1}{2}}/p$, using the tables. Then, determine the appropriate η corresponding to the required p ($>0\cdot8$) and read off the unknown Whittaker function from the extrapolated curve. This procedure is justified because the Whittaker functions are fairly smooth functions of η in the region from $\eta=1\cdot6$ ($p=0\cdot4$) through $\eta=1\cdot1$ ($p=0\cdot6$) and $\eta=0\cdot9$ ($p=0\cdot8$) to $\eta=0\cdot58$ ($p=\infty$). The method was used for a number of cases and found to yield radial wave functions near extrema to within a few per cent. Increased accuracy may possibly be secured if, in addition, the values of the Whittaker functions at $\eta=0$ are used. These are readily obtainable since, at $\eta=0$, the Whittaker functions can be expressed, with the help of contiguity relations, in terms of real Bessel functions.

VI. ACKNOWLEDGMENTS

The authors wish to thank Dr. F. Hirst and his computer staff for their willing assistance in the operation of CSIRAC, and Associate Professor C. B. O. Mohr for his encouragement and interest in this work.

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