# RADIATIVE TRANSFER IN NON-UNIFORM MEDIA

By R. G. GIOVANELLI*

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*Summary*

The equation of radiative transfer incorporating Eddington's approximation is expressed in a form applicable to an inhomogeneous medium, as

$$\tfrac{1}{3}\nabla^2 J=\frac{1}{3\varkappa}\nabla J \,.\, \nabla\varkappa+\varkappa^2(\lambda J-4\pi S).$$

Here $J$ is the total intensity, $\varkappa$ the attenuation coefficient, $\lambda$ the ratio of absorption and attenuation coefficients, and $S$ the source function.

A study is made of the emission from a simple model inhomogeneous semi-infinite medium, including the relative effects of variations in $\varkappa$, $\lambda$, and $S$ across the surface. Particular attention is drawn to the significance of variations in attenuation coefficient in connexion with the appearance of chromospheric granulation.

## I. INTRODUCTION

Solutions of the equation of radiative transfer used in astrophysics are customarily those for a plane parallel or spherically symmetrical medium. It is by no means clear, however, to what extent such solutions are applicable to atmospheres exhibiting granulation, and particularly whether they may be used in deducing from observation the spatial variation of physical conditions.

To study this question, the differential equation of radiative transfer, embodying Eddington's approximation, is generalized here for non-uniform media. A solution is obtained for a semi-infinite medium in which the attenuation coefficient, scattering parameter, and source function are independent of depth but may have small sinusoidal variations as functions of a coordinate parallel to the surface. This investigation yields some insight as to the effect of structure size on the appearance of granules, one of the important deductions being that, with structures that are not too coarse or too fine, variations in attenuation coefficient alone are sufficient to result in marked variations in brightness.

## II. THE EQUATION OF RADIATIVE TRANSFER

Consider a coherently scattering medium in which $\varkappa$, $\lambda$, and $S$ are functions of position, $\varkappa$ being the attenuation coefficient and $\lambda$ the scattering parameter ($=1-\tilde{\omega}_0$, where $\tilde{\omega}_0$ is the albedo for single scattering), while $S$, the source function, is the ratio of the emission per unit volume and solid angle to the attenuation coefficient.

The intensity of radiation, $I$, is a function of direction and position. To establish an equation for the intensity we note that the change of intensity of a beam of radiation in traversing a distance d$s$ is due to an attenuation loss $\varkappa I$d$s$

* Division of Physics, C.S.I.R.O., University Grounds, Chippendale, N.S.W.

and a gain due to scattering and emission. The gain due to isotropic coherent scattering is $(1/4\pi)\tilde{\omega}_0\varkappa J\mathrm{d}s$, where

$$J=\int_{4\pi} I\mathrm{d}\Omega$$

is the total intensity, and the contribution due to emission is $\varkappa S\mathrm{d}s$. So

$$\mathrm{d}I/\mathrm{d}s=-\varkappa I+(1/4\pi)\tilde{\omega}_0\varkappa J+\varkappa S.$$

In rectangular coordinates,

$$\sin\theta\cos\varphi\frac{\partial I}{\partial x}+\sin\theta\sin\varphi\frac{\partial I}{\partial y}+\cos\theta\frac{\partial I}{\partial z}=-\varkappa I+\frac{1}{4\pi}\tilde{\omega}_0\varkappa J+\varkappa S, \quad \ldots (1)$$

$\theta$ being the angle between the beam and the $z$-axis and $\varphi$ the azimuth referred to the $x$-axis.

In its most general form, $I$ may be expanded in a series of spherical harmonics

$$I=\sum_{n=0}^{\infty}\left[I_nP_n(\mu)+\sum_{m=1}^{n}(a_n^m\cos m\varphi+b_n^m\sin m\varphi)P_n^m(\mu)\right], \quad \ldots (2)$$

where $\mu=\cos\theta$, $P_n(\mu)$ and $P_n^m(\mu)$ being Legendre polynomials and associated Legendre functions respectively, and $I_n$, $a_n^m$, and $b_n^m$ the corresponding amplitudes. Substituting (2) in (1) and integrating around $\varphi$, it follows that

$$\mu\frac{\partial}{\partial z}\sum_{n=0}^{\infty}I_nP_n(\mu)+\tfrac{1}{2}\sin\theta\sum_{n=1}^{\infty}\left(\frac{\partial a_n^1}{\partial x}+\frac{\partial b_n^1}{\partial y}\right)P_n^1(\mu)=-\varkappa\sum_{n=0}^{\infty}I_nP_n(\mu)+\frac{1}{4\pi}\tilde{\omega}_0\varkappa J+\varkappa S. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Multiplying by $P_m(\mu)\mathrm{d}\mu$ and integrating from $-1$ to $+1$, using the recurrence and integral relations for Legendre functions, and noting that $J=4\pi I_0$, it follows that

$$\left.\begin{aligned}\frac{1}{3}\frac{\partial I_1}{\partial z}+\frac{1}{3}\frac{\partial a_1^1}{\partial x}+\frac{1}{3}\frac{\partial b_1^1}{\partial y}&=-\varkappa\lambda I_0+\varkappa S \qquad (m=0),\\ \frac{\partial I_0}{\partial z}+\frac{2}{5}\frac{\partial I_2}{\partial z}+\frac{3}{5}\left(\frac{\partial a_2^1}{\partial x}+\frac{\partial b_2^1}{\partial y}\right)&=-\varkappa I_1 \qquad (m=1).\end{aligned}\right\} \quad \ldots (4)$$

Subject to the approximation that $I_2$, $a_2^1$, and $b_2^1$ are zero, the latter equation becomes

$$\partial I_0/\partial z=-\varkappa I_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

In the plane parallel case when $a_1^1$ and $b_1^1$ are zero and $\varkappa$ a function only of $z$, equations (4) and (5) lead to the well-known Eddington equation of radiative transfer.

When $\varkappa$ is a general function of position, we may note that, by symmetry, two other equations similar to (5) can be obtained immediately:

$$\partial I_0/\partial x=-\varkappa a_1^1, \qquad \partial I_0/\partial y=-\varkappa b_1^1.$$

From these and (5),

$$\nabla^2 I_0 = -\left(a_1^1 \frac{\partial\varkappa}{\partial x} + b_1^1 \frac{\partial\varkappa}{\partial y} + I_1 \frac{\partial\varkappa}{\partial z}\right) - \varkappa\left(\frac{\partial a_1^1}{\partial x} + \frac{\partial b_1^1}{\partial y} + \frac{\partial I_1}{\partial z}\right).$$

Using the same equations and (4), this leads to

$$\nabla^2 I_0 = \frac{1}{\varkappa}\left(\frac{\partial I_0}{\partial x}\frac{\partial\varkappa}{\partial x} + \frac{\partial I_0}{\partial y}\frac{\partial\varkappa}{\partial y} + \frac{\partial I_0}{\partial z}\frac{\partial\varkappa}{\partial z}\right) + 3\varkappa^2(\lambda I_0 - \varkappa S).$$

In terms of the total intensity, the generalized equation of radiative transfer may be written in the form

$$\frac{1}{3}\nabla^2 J = \frac{1}{3\varkappa}\nabla J \,.\, \nabla\varkappa + \varkappa^2(\lambda J - 4\pi S). \quad \ldots\ldots\ldots\ldots \quad (6)$$

## III. Radiative Transfer in a Model Sinusoidal Medium

In general, the solution of (6) is rather lengthy, though we can gain worthwhile insight as to the effect of structure in stellar atmospheres from a restricted study of a model semi-infinite medium in which the $x$ and $y$ axes lie in the surface, the $z$ axis being directed outwards, and the quantities $\varkappa$, $\lambda$, and $S$ are independent of $y$ and $z$ but have small sinusoidal variations, all of the same phase, in the $x$ direction. Then

$$\left.\begin{aligned} \varkappa &= \varkappa_0 + \varkappa_1 \cos lx, \\ \lambda &= \lambda_0 + \lambda_1 \cos lx, \\ S &= S_0 + S_1 \cos lx. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7)$$

Here $l$ is a measure of the structure size.

To solve (6), $J$ may be expressed in the form

$$J = \sum_0^\infty \mathscr{J}_n(z) \cos nlx,$$

and, together with (7), substituted in the equation of radiative transfer. Now $\mathscr{J}_0(z)$ is the average value of $J$ at depth $z$, while $\mathscr{J}_1(z)$ describes the variations of $J$ with the same periodicity as the medium. Higher terms in $\mathscr{J}_n(z)$, $n \geqslant 2$, represent distortion of the intensity distribution and may be disregarded for small enough values of $\varkappa_1$, $\lambda_1$, and $S_1$. Then (6) becomes

$$\begin{aligned} \tfrac{1}{3}\mathrm{D}^2\mathscr{J}_0 + \tfrac{1}{3}(\mathrm{D}^2 - l^2)\mathscr{J}_1 \cos lx &= \tfrac{1}{3}(\varkappa_0 + \varkappa_1 \cos lx)^{-1}(l^2 \mathscr{J}_1 \varkappa_1 \sin^2 lx) \\ &\quad + (\varkappa_0 + \varkappa_1 \cos lx)^2\{(\lambda_0 + \lambda_1 \cos lx)(\mathscr{J}_0 + \mathscr{J}_1 \cos lx) - 4\pi(S_0 + S_1 \cos lx)\}, \end{aligned}$$

where $\mathrm{D} \equiv \partial/\partial z$ and $\mathscr{J}_n \equiv \mathscr{J}_n(z)$.

Terms of the type $\cos^n lx$ are now replaced by a sum of terms in $\cos mlx$, in order to permit separation of equations for the various harmonic components. After some straightforward analysis in which terms involving second orders of small quantities are neglected, it is found that terms independent of $x$ yield

$$\mathrm{D}^2\mathscr{J}_0 = 3\varkappa_0^2\lambda_0\mathscr{J}_0 + \left[\frac{3}{2}(2\varkappa_0\varkappa_1\lambda_0 + \varkappa_0^2\lambda_1) + \frac{1}{2}\, l^2 \frac{\varkappa_1}{\varkappa_0}\right]\mathscr{J}_1 - 12\pi\varkappa_0^2 S_0.$$

Terms involving $\cos lx$ yield

$$(\mathrm{D}^2-l^2)\mathscr{J}_1=3\varkappa_0^2\lambda_0\mathscr{J}_1+3(2\varkappa_0\varkappa_1\lambda_0+\varkappa_0^2\lambda_1)\mathscr{J}_0-12\pi(2\varkappa_0\varkappa_1 S_0+\varkappa_0^2 S_1).$$

The solutions of these simultaneous differential equations, subject to the boundary condition that the intensity remain finite deep within the medium, are

$$\left.\begin{aligned}\mathscr{J}_0&=A\mathrm{e}^{p_1 z}+B\mathrm{e}^{p_2 z}+\frac{c(l^2+a)-(b+f)d}{a^2+al^2-2b(b+f)},\\ \mathscr{J}_1&=(p_1^2-a)(b+f)^{-1}A\mathrm{e}^{p_1 z}+(p_2^2-a)(b+f)^{-1}B\mathrm{e}^{p_2 z}+\frac{ad-2bc}{a^2+al^2-2b(b+f)},\end{aligned}\right\} \qquad (8)$$

where $a=3\varkappa_0^2\lambda_0$,

$$b=\frac{3}{2}(2\varkappa_0\varkappa_1\lambda_0+\varkappa_0^2\lambda_1),$$
$$c=12\pi\varkappa_0^2 S_0,$$
$$d=12\pi(2\varkappa_0\varkappa_1 S_0+\varkappa_0^2 S_1),$$
$$f=\frac{1}{2}\,l^2\,\frac{\varkappa_1}{\varkappa_0},$$
$$p_1=\left[\frac{1}{2}l^2+a+\frac{1}{2}(l^4+8b^2+8bf)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$
$$p_2=\left[\frac{1}{2}l^2+a-\frac{1}{2}(l^4+8b^2+8bf)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

The constants $A$ and $B$ are chosen to ensure zero inward flux at the boundary, $z=0$. In particular, this applies at points where $\partial J/\partial x=0$, or $\sin lx=0$; i.e. $\cos lx=\pm 1$. As in the usual treatment of the plane parallel equation of transfer, this leads to the condition

$$\varkappa J=-\tfrac{2}{3}\mathrm{D}J,$$

when $z=0$, $\cos lx=\pm 1$. Then, after some straightforward analysis, it is found that

$$A=(\gamma\varepsilon-\beta\eta)/(\varepsilon\alpha-\beta\delta), \qquad B=(\alpha\eta-\gamma\delta)/(\varepsilon\alpha-\beta\delta),$$

where $\alpha=\varkappa_0+\frac{1}{2}\varkappa_1(b+f)^{-1}[l^2+(l^4+8b^2+8bf)^{\frac{1}{2}}]+\frac{2}{3}p_1$,

$$\beta=\varkappa_0+\tfrac{1}{2}\varkappa_1(b+f)^{-1}[l^2-(l^4+8b^2+8bf)^{\frac{1}{2}}]+\tfrac{2}{3}p_2,$$
$$\gamma=-\{\varkappa_0[c(l^2+a)-(b+f)d]+\varkappa_1(ad-2bc)\}\{a^2+al^2-2b(b+f)\}^{-1},$$
$$\delta=\varkappa_1+(\tfrac{1}{2}\varkappa_0+\tfrac{1}{3}p_1)(b+f)^{-1}[l^2+(l^4+8b^2+8bf)^{\frac{1}{2}}],$$
$$\varepsilon=\varkappa_1+(\tfrac{1}{2}\varkappa_0+\tfrac{1}{3}p_2)(b+f)^{-1}[l^2-(l^4+8b^2+8bf)^{\frac{1}{2}}],$$
$$\eta=-\{\varkappa_1[c(l^2+a)-(b+f)d]+\varkappa_0(ad-2bc)\}\{a^2+al^2-2b(b+f)\}^{-1}.$$

To simplify these results, we note from physical considerations that, when $l$ is small enough (very coarse structures), $J$ at any point is the same as in a uniform medium having the same values of $\varkappa$, $\lambda$, and $S$. When $l$ is large enough (very fine structures) the intensity variations are negligible, $J$ being the same as in a uniform medium where the constants are $\varkappa_0$, $\lambda_0$, and $S_0$. The condition $l^4 \gg 8b(b+f)$ is satisfied almost throughout the entire transition between these extremes, in which case the average value of the total intensity at the surface is

$$\mathscr{J}_0 = \frac{2}{3}\,\frac{c}{\sqrt{a}\left(\varkappa_0+\frac{2}{3}\sqrt{a}\right)} = \frac{8\pi S_0}{\sqrt{(3\lambda_0)}(1+2\sqrt{(\lambda_0/3)})} \quad (z=0). \quad \ldots (9)$$

Thus the first important result emerges, that the mean value of $J$ at the surface is unaffected by small sinusoidal variations in $\varkappa$, $\lambda$, or $S$ in a direction parallel to the surface; it can also be shown that this conclusion applies to the mean value of $J$ at all depths.

Under the same conditions, $l^4 \gg 8b(b+f)$, the fractional variation in $J$ across the surface is

$$\frac{\mathscr{J}_1}{\mathscr{J}_0} = \left\{c(a+l^2)\left[\varkappa_0+\frac{2}{3}\sqrt{(a+l^2)}\right]\right\}^{-1}$$
$$\times\left\{-2\varkappa_0\frac{a}{l^2}bc[1-\sqrt{(1+l^2/a)}]-2\varkappa_0 bc+\varkappa_0 ad\sqrt{(1+l^2/a)}-\varkappa_1 ac(1+l^2/a)\right.$$
$$\left.+\frac{2}{3}\sqrt{(a+l^2)}\,.\,(ad-2bc)\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

This relation is easiest discussed in some special cases.

(i) As $l\to\infty$, $\mathscr{J}_1/\mathscr{J}_0\to 0$.

(ii) When $l^2 \gg a$, i.e., for fine enough structures, greater changes are produced in the total intensity at the surface by a given fractional change in *attenuation* than by equal fractional changes in $S$ or $\lambda$. This can be appreciated by noting that the dominant term in the numerator of (**10**) is then $-\varkappa_1 ac(1+l^2/a)$.

(iii) When $S_1=0$ and $\lambda_1=0$ (variations in $\varkappa$ alone),

$$\frac{\mathscr{J}_1}{\mathscr{J}_0} = \frac{\varkappa_1}{(1+l^2/a)\left[\varkappa_0+\frac{2}{3}\sqrt{(a+l^2)}\right]}\left\{-\frac{2a}{l^2}\left[1-\sqrt{\left(1+\frac{l^2}{a}\right)}\right]+2\sqrt{\left(1+\frac{l^2}{a}\right)}-3-\frac{l^2}{a}\right\} \quad (z=0).$$

Thus, when the variations in medium are of attenuation alone,

$$\frac{\mathscr{J}_1}{\mathscr{J}_0}\to 0, \qquad l^2 \ll a,$$
$$\frac{\mathscr{J}_1}{\mathscr{J}_0} = -\frac{\varkappa_1(3-2\sqrt{2})}{\varkappa_0+\frac{2}{3}\sqrt{2}}, \qquad l^2 = a,$$
$$\frac{\mathscr{J}_1}{\mathscr{J}_0} \approx -\frac{\varkappa_1}{\varkappa_0+\frac{2}{3}l}, \qquad l^2 \gg a,$$
$$\frac{\mathscr{J}_1}{\mathscr{J}_0}\to 0, \qquad l^2\to\infty.$$

This sequence shows that the variation in the total intensity across the surface, or the contrast, rises from zero to a maximum and decreases to zero again as the size of the structures diminishes. The maximum value of $\mathscr{J}_1/\mathscr{J}_0$ is of the order of but less than the fractional variation in $\varkappa$, and occurs for a value of $l$ somewhere in the range $\sqrt{a}\lesssim l\lesssim 3\varkappa_0/2$. Maximum values of $J$ across the surface correspond to minima of attenuation coefficient ; this is because radiation can escape from greater depths at such positions.

(iv) When $\varkappa_1=0$ and $\lambda_1=0$ (variations in $S$ alone),

$$\frac{\mathscr{J}_1}{\mathscr{J}_0}=\frac{S_1}{S_0\sqrt{(1+l^2/a)}}\cdot\frac{\varkappa_0+\frac{2}{3}\sqrt{a}}{\varkappa_0+\frac{2}{3}\sqrt{(a+l^2)}}\qquad (z=0).$$

Thus the contrast remains high from the coarsest structures down to those for which $l^2\approx a$, and diminishes for smaller structures. The maximum value of $\mathscr{J}_1/\mathscr{J}_0$ is $S_1/S_0$, maxima of $J$ coinciding with maxima of $S$.

(v) When $\varkappa_1=0$ and $S_1=0$ (variations in $\lambda$ alone),

$$\frac{\mathscr{J}_1}{\mathscr{J}_0}=-\frac{\lambda_1}{\lambda_0}\cdot\frac{\varkappa_0 a/l^2\{1-\sqrt{(1+l^2/a)}\}+\varkappa_0+2\sqrt{a}\sqrt{(1+l^2/a)}}{(1+l^2/a)\{\varkappa_0+\frac{2}{3}\sqrt{(a+l^2)}\}}\qquad (z=0).$$

Thus

$$\frac{\mathscr{J}_1}{\mathscr{J}_0}\to-\frac{\lambda_1}{2\lambda_0}\cdot\frac{4\lambda_0+\sqrt{(3\lambda_0)}}{2\lambda_0+\sqrt{(3\lambda_0)}},\qquad l\to 0,$$

which is of the order of $-\frac{1}{2}\lambda_1/\lambda_0$ for resonance radiation ($\lambda_0\ll 1$).

$$\frac{\mathscr{J}_1}{\mathscr{J}_0}=-\frac{\lambda_1}{2\lambda_0}\cdot\frac{2-\sqrt{2}+2\sqrt{(6\lambda_0)}}{1+\frac{2}{3}\sqrt{(6\lambda_0)}},\qquad l^2=a,$$

$$\frac{\mathscr{J}_1}{\mathscr{J}_0}=-\frac{\lambda_1}{\lambda_0}\cdot\frac{a(\varkappa_0+2l)}{l^2(\varkappa_0+\frac{2}{3}l)},\qquad l^2\gg a.$$

As the size of the structure diminishes, so does the contrast, being substantially reduced when $l^2=a$.

(vi) When the emission per unit volume and solid angle is proportional to the absorption coefficient, then $S/\lambda$ is uniform throughout, and $ad=2bc$. Thus $\mathscr{J}_1$ vanishes deep in the medium, as is to be expected from very general considerations.

## IV. Discussion

In showing that emission from a non-uniform medium is dependent on structure, the present work complements an earlier investigation by the author where, using a different method, the reflectance of a semi-infinite diffuse medium was also found to depend on structure (Giovanelli 1957*).

The most significant result obtained here is that, in a certain size range, variations in brightness of the medium are most sensitive to variations in attenuation coefficient, a quantity which in the case of a semi-infinite plane parallel medium does not appear explicitly in the expression for the brightness. For very coarse structures, variations in attenuation coefficient have no influence,

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but variations in scattering parameter and source function then have their greatest effects. As the structure size decreases, the effects of variations in $\lambda$ and $S$ become less significant and variations in $\varkappa$ become more important. After an optimum size, further decrease in the structure dimensions results in a reduction of brightness variations towards zero.

The solutions obtained here are for a problem too simplified to have direct application to the Sun's atmosphere. However, it is instructive to calculate the orders of magnitude of the important structure sizes using data appropriate to the solar chromosphere. For example, if $\varkappa_0 \lesssim 2 \times 10^{-8}$ cm$^{-1}$ in H$\alpha$ and if $\lambda_0 \approx 10^{-2}$, then the structural dimensions corresponding to $l = 3\varkappa_0/2$ and $\sqrt{a}$ are $x = 2\pi/l \lesssim 2 \times 10^8$ cm and $2 \times 10^9$ cm respectively. This range includes such a large fraction of actual chromospheric structures that it is clearly essential to use appropriate solutions of the equation of radiative transfer, taking into account non-uniformities of the chromosphere, in deriving spatial variations of physical conditions there. Again, it would appear that even with perfect telescopic resolution there is a natural limit to the size of structure which can be detected with given contrast.

A further point requires stressing: the average intensity $\mathscr{I}_0$ is certainly influenced by structure when the variations in physical conditions are large.

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