THE USE OF COMPLETE TEMPERATURE–TIME CURVES FOR DETERMINATION OF THERMAL CONDUCTIVITY WITH PARTICULAR REFERENCE TO ROCKS

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Summary

Most of the transient methods at present in use for the determination of thermal conductivity involve the study of the asymptote of a temperature-time curve. This implies that they require relatively long times of experiment and make no use of the information contained in measurements of temperature at smaller times.

A simple method of reducing observations is described which uses measurements of temperature at equally spaced intervals of time, together with a curve calculated from the theory. Such curves are given for a number of experimental situations likely to be useful for the measurement of the thermal conductivities of rocks. Among these are the line source : (i) in the semi-infinite solid, (ii) along the axis of a cylinder, (iii) along the surface of a cylinder. These methods yield values for both the thermal conductivity and diffusivity, and the desirability of such measurements is stressed. The application of the method to the thermal conductivity probe and to the measurement of the diffusivity of solids in the form of slabs and cylinders is also discussed. An alternative method of the same type, using the times at which specified temperatures are attained, is also described.

I. Introduction

In the early literature of conduction of heat many transient methods were given for determining the thermal diffusivity of specimens of simple geometrical shapes. These usually involve cooling the body from a constant initial temperature and use only the late stages of the cooling curve in which the theoretical solution reduces to a single exponential term (for references see Carslaw and Jaeger (1959), Sections 6.5, 8.5, 9.5). More recently, " time-lag " methods which involve an experimental situation in which the temperature-time curve has a linear asymptote (cf. Barrer 1939), and " probe " methods in which there is an asymptote linear in the logarithm of the time (cf. Blackwell 1954 ; de Vries and Peck 1958) have enjoyed a considerable vogue.

These methods all have the disadvantage that they completely waste the information available in the early parts of the temperature-time curves. Also, since they involve the approach to an asymptote they require relatively long experimental times when applied to specimens of poor conductors and of moderate size.

The object of the present paper is to describe a simple numerical method of discussing complete temperature-time curves which is applicable to a wide variety of cases.

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In many simple experimental situations, the temperature $v(t)$ measured at some point at time t after the beginning of the experiment has the theoretical form

$$v(t)=A(K,\varkappa)f(\varkappa t/a^2), \qquad (1)$$

where A is a constant depending on the thermal conductivity K and the diffusivity $\varkappa$ of the material, and $f(\varkappa t/a^2)$ is a function of the dimensionless quantity $\varkappa t/a^2$, where a is a length which specifies the size of the specimen. The diffusivity $\varkappa=K/\rho c$, where ρ is the density and c the specific heat of the material.

It follows from (1) that

$$\frac{v(2t)}{v(t)}=\frac{f(2\varkappa t/a^2)}{f(\varkappa t/a^2)}, \qquad (2)$$

so that the value of $\varkappa t/a^2$ can be found from an experimental value of $v(2t)/v(t)$ with the aid of a theoretical curve of the function $f(2T)/f(T)$.

In practice, a convenient unit of time t_0 is chosen and the values $v(nt_0)$ of the temperature at times nt_0, $n=1, 2, \ldots$, are measured. Each ratio $v(2nt_0)/v(nt_0)$ gives a value of $\varkappa t_0/a^2$ and these should agree within experimental error. If they show any systematic drift it is a valuable indication that the theory does not adequately represent the experimental situation. When $\varkappa t_0/a^2$ has been found, the value of $A(K,\varkappa)$ follows from (1) by comparing the observed values of $v(t)$ with the theoretical value $A(K,\varkappa)f(\varkappa t/a^2)$. In this way both K and $\varkappa$ can be found. Even if both are not needed the fact that $K/\varkappa=\rho c$, the heat capacity of the material per unit volume, which is usually known approximately, provides a useful check of the method. There is, of course, no need except practical convenience for the choice of the particular number 2 in (2), any other number can be used in the same way if desired.

The above method of reduction is most useful when temperatures are recorded continuously. If readings are made manually with a galvanometer or potentiometer it is much more convenient to record the times at which the temperature attains a definite value. Suppose that t and t_N are, respectively, the times at which temperatures v and Nv are attained, then by (1)

$$f(\varkappa t_N/a^2)=N f(\varkappa t/a^2), \qquad (3)$$

and $\varkappa t/a^2$ can be determined from a theoretical curve of the root T of $f(Tt_N/t)=N f(T)$ as a function of t_N/t.

Either of these methods of reduction can be applied to a great many methods for the determination of thermal conductivity. The necessary theoretical results for a number of systems likely to arise in connexion with measurements of the conductivities of rocks are given below and illustrated by reducing results obtained with actual experiments on the systems. The reason for the choice of these systems is given in Section II. The case of a line source in an infinite medium is discussed in Section III. It has already been described by Jaeger (1959) for the corresponding hydrological problem. The case of a line source along the axis of a circular cylinder is discussed in Section IV, and that of a line source on the surface of a cylinder in Section V. The application of the method to the thermal conductivity probe is given in Section VI. Finally, in Section VII the determination of diffusivity from slabs, cylinders, and spheres with constant initial and surface temperatures, as in many of the classical methods, is discussed.

In all cases where numerical values are given, units will be c.g.s., calorie, and degC.

II. The Determination of the Thermal Conductivity of Rocks

The measurement of the thermal conductivity of rocks is a relatively slow and difficult matter. The standard steady-state method, the divided bar (Benfield 1939), involves the use of a set of disks with accurately ground surfaces, and even when these have been made takes considerable experimental time. The " probe " methods, used for measurement of conductivities *in situ* in drill holes, are also slow and cumbersome.

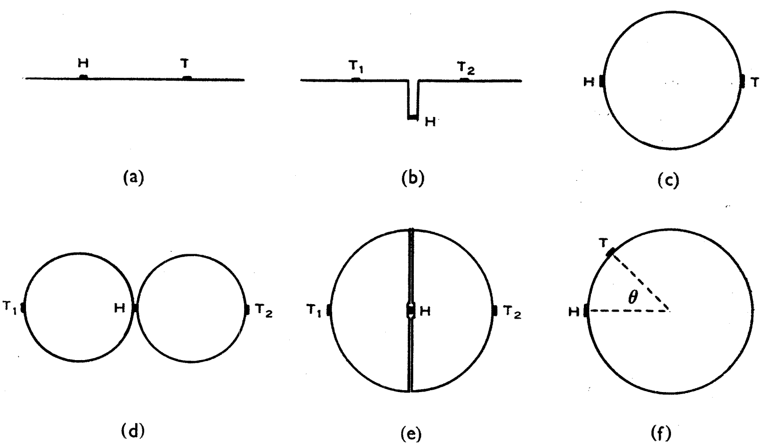

Fig. 1.—Cross sections of some possible systems for the determination of the thermal conductivity of rock specimens. H, heater wire ; T, T_1, T_2, thermocouples.

An ideal method requires (i) small experimental time, and (ii) small time of preparation of specimens. Condition (i) suggests, not merely that transient methods should be used, but also that the early part of the regime in which asymptotes have not been attained should be studied, this has the additional advantage that heat losses from insulated surfaces should be negligible. Condition (ii) implies that preparation of specimens should all be done with diamond tools and should involve no hand finishing.

The mechanical operations which can rapidly and accurately be performed on rocks with diamond tools are : (i) drilling cylindrical cores, (ii) facing off their ends to make finite cylinders, (iii) cutting flat surfaces. Surfaces cut by a saw will not be accurately flat so that, for example, a wire pressed between two of them will not be in contact over the whole of the length ; however, it is easy with an accurate wheel or saw to take off a shaving over a limited strip which will be plane to within $0 \cdot 001$ cm or better (cf. Fig. 1 (*e*)).

A number of experimental situations which may be obtained by the operations described above and are covered by the theory given below are shown in Figure 1. They are:

(i) A semi-infinite solid whose plane surface has a long straight heater wire H and a thermocouple T attached to it and covered with insulating material (Fig. 1 (a)). This simple situation, which must be viewed with some suspicion because of possible heat losses from the heater, may readily be replaced by that described in (ii) below.

(ii) A semi-infinite solid with a heater wire H in a narrow slot filled with cement to give good contact (Fig. 1 (b)). These methods can be used on nearly plane surfaces *in situ* or on large specimens of friable rock which cannot be machined readily.

(iii) If thin core only, say <2 cm in diameter is available, the heater H and thermocouple T may be placed along diametrically opposite generators (Fig. 1 (c)). A more satisfactory arrangement, Figure 1 (d), has narrow flats ground on two cylinders with the heater H pressed between. Two thermocouples T_1 and T_2 in parallel or series are used.

(iv) Core of moderate diameter, say 5 cm, may be slit along an axial plane and have an axial heater wire H between narrow, accurately ground, slots (Fig. 1 (e)).

(v) For core of greater diameter which it is not desired to cut, the heater H and thermocouple T may be on the surface along generators distant θ apart (Fig. 1 (f)).

(vi) Many other systems suggest themselves immediately, for example, slabs or finite cylinders with constant initial and surface temperatures.

III. The Continuous Line Source* in an Infinite Medium

The temperature $v(t)$ at time t at distance a from a continuous line source which emits heat at the constant rate Q per unit length per unit time for $t>0$ is given by (cf. Carslaw and Jaeger 1959, Section 10.4)

$$v(t)=-\frac{Q}{4\pi K}\mathrm{Ei}\left(-\frac{a^2}{4\varkappa t}\right), \quad \ldots\ldots\ldots\ldots \quad (4)$$

where K and $\varkappa$ are the thermal conductivity and diffusivity of the surrounding medium and

$$-\mathrm{Ei}(-x)=\int_x^\infty \frac{\mathrm{e}^{-u}}{u}\mathrm{d}u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

is the tabulated exponential integral. For the present purposes the Works Projects Administration Tables (1940) are the most convenient, but a useful shorter table is given by Jahnke and Emde (1933).

* The justification for using the simple line-source formula for the case of a thin heating wire at the distances $a>1$ cm contemplated here is essentially contained in the theoretical and experimental results on thin probes. Calculation shows that the effect of small finite diameter of the probe is negligible at the distances considered here, and experimental agreement of conductivities measured by thin probes with those determined by steady-state methods implies that for quite small times the heat flow outside the probe is determined by (4).

The result corresponding to (**2**) is

$$\frac{v(2t)}{v(t)}=\frac{\mathrm{Ei}(-a^2/8\varkappa t)}{\mathrm{Ei}(-a^2/4\varkappa t)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{6})$$

Values of $\mathrm{Ei}(-x)/\mathrm{Ei}(-2x)$ are shown in curve I of Figure 2, and values of $-\mathrm{Ei}(-x)$ in curve II.

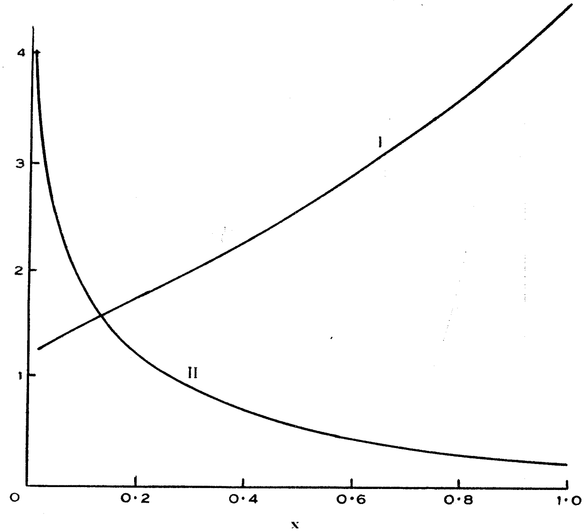

Fig. 2.—Curve I: values of $\mathrm{Ei}(-x)/\mathrm{Ei}(-2x)$. Curve II: $-\mathrm{Ei}(-x)$.

The method is, of course, ideally adapted for use with powders, but to illustrate its application to rocks the results of an experiment on a block of granite using the system of Figure 1 (*b*) are shown in Table 1. The surface of the solid was covered with lightweight insulating powder to minimize heat loss. The thickness of the block was greater than 10 cm, so that for the purposes of the present calculation it may be regarded as semi-infinite. A heating wire of

TABLE 1

REDUCTION OF OBSERVATIONS ON A LINE SOURCE IN AN INFINITE MEDIUM

n	$v(nt_0)$	$v(2nt_0)/v(nt_0)$	$a^2/8\varkappa nt_0$	$a^2/8\varkappa t_0$	$-0\cdot6\,\mathrm{Ei}(-4\cdot66/n)$
0	0·000				0·000
1	0·000				0·000
2	0·014				0·019
3	0·054	3·67	0·81	2·43	0·055
4	0·100	2·84	0·59	2·36	0·100
5	0·146	2·48	0·47	2·35	0·148
6	0·198	2·20	0·38	2·28	0·194
7	0·241	2·06	0·33	2·31	0·239
8	0·284	1·94	0·28	2·24	0·283
10	0·362				0·362
12	0·436				0·434
14	0·497				0·498
16	0·550				0·557

resistance $0 \cdot 278\ \Omega/\mathrm{cm}$ was placed at the base of a narrow slot $0 \cdot 5$ cm deep, which was filled with cement to ensure good contact. The heater current was $0 \cdot 57$ A. Temperatures in the surface were measured by thermocouples T_1, T_2, in parallel, each distant $1 \cdot 23$ cm from the heater.

Actual temperatures $v(nt_0)$ measured from the recorder at times nt_0, $n=1, 2, \ldots$, where $t_0=7 \cdot 5$ sec, are shown in the second column. The third column shows the values of $v(2nt_0)/v(nt_0)$ and the fourth the values of $a^2/8\varkappa nt_0$ which are the abscissae of curve I of Figure 2 corresponding to the ordinates

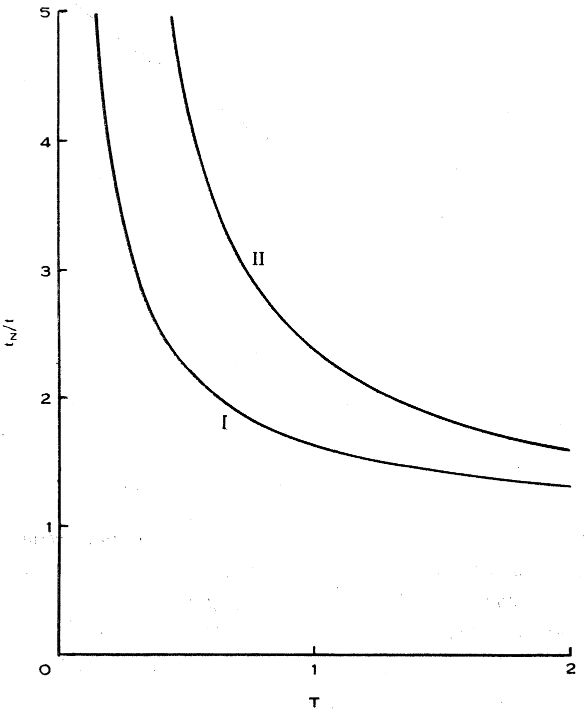

Fig. 3.—Values of t_N/t for which $\mathrm{Ei}(-tT/t_N)=N\mathrm{Ei}(-T)$. Curve I: $N=2$; curve II: $N=3$.

$v(2nt_0)/v(nt_0)$. The values of $a^2/8\varkappa t_0$ in the fifth column are seen to be approximately constant, as they should be, with a mean value of $2 \cdot 33$. Using this mean value, the observed temperatures should be proportional to $\mathrm{Ei}(-4 \cdot 66/n)$.

The mean value of the constant of proportionality calculated from the values $n=3, \ldots, 8$ is $0 \cdot 60$ and the last column shows values of $-0 \cdot 6\ \mathrm{Ei}(-4 \cdot 66/n)$, which are seen to agree to within experimental error with the observed temperatures. By (**1**), allowing for an image source, the value of the constant of proportionality is $Q/2\pi K$ and this leads to the result $K=0 \cdot 0057$. It should be noticed that this result does not require a knowledge of the distance a, which might be uncertain.

If the value of a is known, $\varkappa$ follows from $a^2/8\varkappa t_0=2 \cdot 33$, and using the measured value $a=1 \cdot 23$ cm gives $\varkappa=0 \cdot 0108$. This value of $\varkappa$ does not involve a knowledge either of the heat input or of the calibration of the thermocouples.

From these values of K and $\varkappa$, the heat capacity ρc per unit volume of the rock is found to be $0 \cdot 53$, which is consistent with the density of the rock and the specific heat of granite. This provides a valuable check on the overall reliability of the method.

If the second method of reduction described in the Introduction is applied to this problem, (**3**) becomes, using (**4**),

$$\mathrm{Ei}(-tT/t_N) = N\,\mathrm{Ei}(-T), \quad \ldots\ldots\ldots\ldots \quad (\mathbf{7})$$

where $T = a^2/4\varkappa t$. In Figure 3 values of t_N/t satisfying (**7**) are shown as a function of T for the values 2 and 3 of N.

TABLE 2

ALTERNATIVE REDUCTION OF THE EXPERIMENT OF TABLE 1

n	$v(x_n t_0)$	x_n	x_{2n}/x_n	$a^2/4\varkappa x_n t_0$	$a^2/4\varkappa t_0$
0	0				
1	$0 \cdot 062$	$3 \cdot 18$	$1 \cdot 43$	$1 \cdot 45$	$4 \cdot 61$
2	$0 \cdot 124$	$4 \cdot 54$	$1 \cdot 57$	$1 \cdot 07$	$4 \cdot 86$
3	$0 \cdot 186$	$5 \cdot 80$	$1 \cdot 76$	$0 \cdot 82$	$4 \cdot 76$
4	$0 \cdot 248$	$7 \cdot 14$	$1 \cdot 94$	$0 \cdot 65$	$4 \cdot 64$
6	$0 \cdot 372$	$10 \cdot 22$			
8	$0 \cdot 496$	$13 \cdot 86$			

The treatment by this method of the curve reduced in Table 1 is shown in Table 2. Choosing the same unit of time $t_0 = 7 \cdot 5$ sec, the times $x_n t_0$ at which the temperatures attain the values $n \times 0 \cdot 062$ degC, $n = 1, 2, \ldots$, are shown in the third column. The fourth column shows the ratios x_{2n}/x_n and the fifth the value of $a^2/4\varkappa x_n t_0$ read off from curve I of Figure 3. The last column shows $a^2/4\varkappa t_0$, which should be the same for all rows. The mean value of $a^2/4\varkappa t_0$ is $4 \cdot 72$, which may be compared with $4 \cdot 64$ found earlier. The determination of K and $\varkappa$ now proceeds as before.

IV. A Circular Cylinder with Axial Heating

The cylinder is supposed to be initially at zero temperature and to have a continuous line source of heat along its axis which emits Q heat units per unit time per unit length for $t > 0$. The surface of the cylinder is assumed to be insulated, so that its length does not enter the problem.

With the previous notation, the temperature $v(t)$ at time t at a point of the surface of the cylinder is given by (Carslaw and Jaeger 1959, Section 14.18 (**11**), (**12**))

$$v(t) = \frac{Q}{\pi K} f_1(\varkappa t/a^2), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{8})$$

where

$$f_1(T) = T - \tfrac{1}{8} - \sum_{s=1}^{\infty} \frac{1}{\alpha_s^2 J_0(\alpha_s)} \exp(-\alpha_s^2 T), \quad \ldots\ldots\ldots \quad (\mathbf{9})$$

and the α_s, $s = 1, 2, \ldots$, are the positive roots of $J_1(\alpha) = 0$.

For large values of t, v tends to a straight line of slope $Q/\pi a^2 \rho c$ and intercept $a^2/8\varkappa$ on the t-axis. By measuring both of these, both K and $\varkappa$ can be determined. This is an example of the " time-lag " methods referred to earlier.

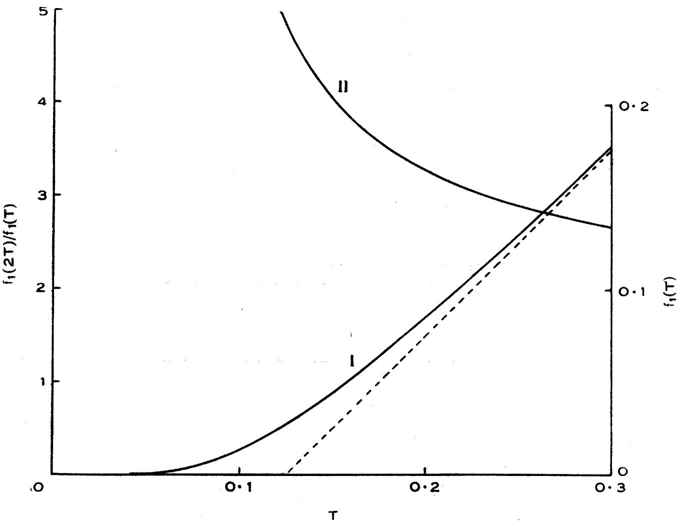

Fig. 4.—Curve I : $f_1(T)$, right-hand scale ; the asymptote is shown dotted. Curve II : $f_1(2T)/f_1(T)$, left-hand scale.

To enable the present method to be used for smaller values of time along the lines described in Section I, values of $f_1(T)$ and $f_1(2T)/f_1(T)$ calculated from (9) are shown in Figure 4.

TABLE 3

REDUCTION OF OBSERVATIONS FOR AXIAL HEATING OF A CIRCULAR CYLINDER

n	$v(nt_0)$	$v(2nt_0)/v(nt_0)$	$\varkappa nt_0/a^2$	$\varkappa t_0/a^2$	$1\cdot34f_1\ (0\cdot036n)$
0	0				
1	0·000				
2	0·001				0·003
3	0·020				0·023
4	0·051	4·31	0·140	0·0350	0·052
5	0·090	3·49	0·182	0·0364	0·090
6	0·131	3·14	0·215	0·0358	0·131
7	0·177	2·86	0·257	0·0367	0·176
8	0·220				0·221
10	0·314				0·316
12	0·411				0·411
14	0·506				0·508

As an example of a practical case, Table 3 shows the surface temperature of a cylinder of quartz-porphyry of diameter 4·76 cm heated by an axial wire of resistance 0·278 Ω/cm carrying a current of 0·64 A for $t > 0$. The cylinder was immersed in lightweight insulating powder.

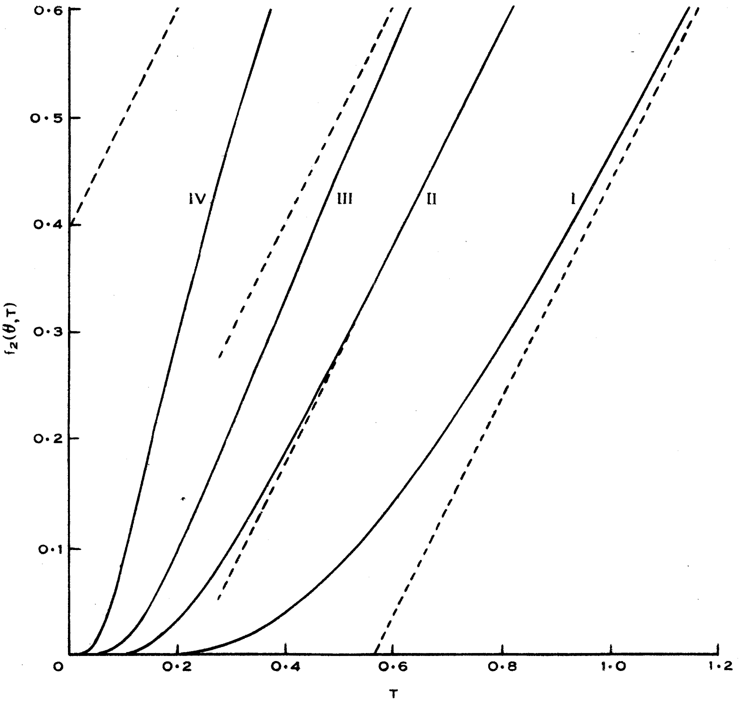

Fig. 5.—Values of $f_2(\theta,T)$. Curves I, II, III, IV are for $\theta=180$, 90, 69, 45° respectively. The asymptotes are shown by dotted lines.

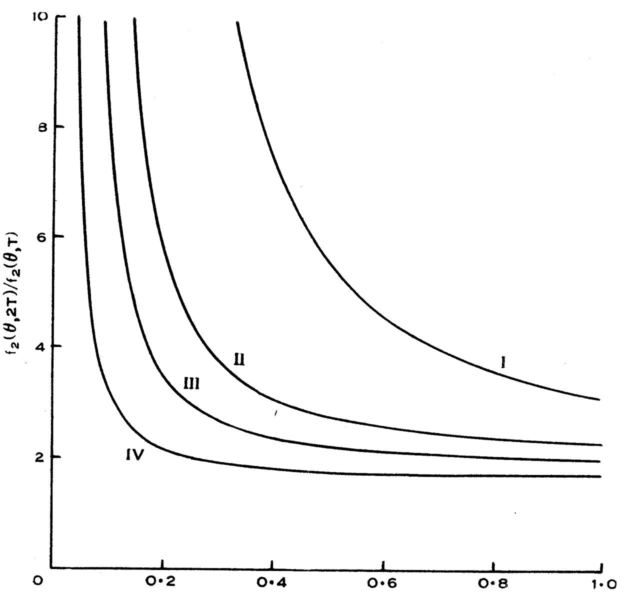

Fig. 6.—Values of $f_2(\theta,2T)/f_2(\theta,T)$. Curves I, II, III, IV are for $\theta=180$, 90, 69, 45° respectively.

The unit of time chosen is $t_0=15$ sec ; the second column gives the temperature $v(nt_0)$ at time nt_0, and the third the ratio $v(2nt_0)/v(nt_0)$. The fourth column gives values of $\varkappa nt_0/a^2$ read off from curve II of Figure 4. The mean of the values of $\varkappa t_0/a^2$ in the fifth column is 0·036 and, choosing a constant of proportionality of 1·34, the calculated results in the last column agree well with the experimental values.

From $Q/\pi K=1{\cdot}34$, we get $K=0{\cdot}0064$, and $\varkappa t_0/a^2=0{\cdot}036$ gives $\varkappa=0{\cdot}0136$. These give $\rho c=0{\cdot}47$, leading, with $\rho=2{\cdot}7$, to the reasonable value $c=0{\cdot}17$.

V. A Circular Cylinder Heated along a Generator

The cylinder is supposed to be initially at zero temperature and to have a continuous line source along a generator which emits Q heat units per unit length per unit time for $t>0$. There is supposed to be no loss of heat from the surface of the cylinder, so its length does not matter.

Table 4

Reduction of observations on a circular cylinder heated along a generator

n	$v(nt_0)$	$v(2nt_0)/v(nt_0)$	$\varkappa nt_0/a^2$	$\varkappa t_0/a^2$	$1{\cdot}53 f_2(\pi,\ 0{\cdot}055n)$
0	0				0
4	0·002				0·004
5	0·010				0·012
6	0·027	10·3	0·323	0·054	0·027
7	0·050	8·04	0·38	0·054	0·050
8	0·082	6·5	0·443	0·056	0·081
10	0·171				0·168
12	0·278				0·275
14	0·402				0·401
16	0·533				0·540

The temperature $v(t)$ at time t at a point on the surface at an angular distance θ from the source is (Carslaw and Jaeger 1959, Section 14.18 (**11**), (**12**))

$$v(t)=\frac{Q}{\pi K}f_2(\theta,\varkappa t/a^2), \quad \ldots\ldots\ldots\ldots \quad (\mathbf{10})$$

where

$$f_2(\theta,T)=T+\tfrac{1}{8}-\ln\,(2\sin\tfrac{1}{2}\theta)-\sum_{n=0}^{\infty}\varepsilon_n\cos n\theta\sum_{m=1}^{\infty}\frac{\exp\,(-\alpha_{n,m}^2T)}{(\alpha_{n,m}^2-n^2)}, \quad \ldots \quad (\mathbf{11})$$

where $\varepsilon_1=1$, $\varepsilon_n=2$ if $n\geqslant 2$, and the $\alpha_{n,m}$, $m=1, 2, \ldots$, are the roots of $J_n'(\alpha)=0$. These are tabulated by Smith, Rodgers, and Traub (1944).

For large values of the time, $v(t)$ is asymptotically a straight line of slope $Q/\pi a^2\rho c$ and intercept $a^2\{8\ln\,(2\sin\tfrac{1}{2}\theta)-1\}/8\varkappa$ on the t-axis. If $0<\theta<69°$, approximately, this intercept is negative and if $69°<\theta<180°$ it is positive. Values of $f_2(\theta,T)$ for the values 45, 69, 90, and 180° of θ are shown in Figure 5. These results lead to a "time-lag" method for determining both K and $\varkappa$;

it may be remarked that the asymptote is approached most rapidly when θ is about 90°.

For use with the present method the values of the ratios $f_2(\theta,2T)/f_2(\theta,T)$ for the values 45, 69, 90, 180° of θ are shown in Figure 6.

As an example, Table 4 shows some observations for the case $\theta=\pi$ made with a heating wire of resistance $0 \cdot 278\ \Omega$/cm carrying a current of $0 \cdot 8$ A and running between two dolerite cylinders of diameter $2 \cdot 22$ cm in the arrangement of Figure 1 (d). The unit of time t_0 is $7 \cdot 5$ sec, and temperatures at times nt_0 are shown in the second column. The third column gives the ratios $v(2nt_0)/v(nt_0)$, and the fourth shows values of $\kappa nt_0/a^2$ read off from curve I of Figure 6. The mean of the values of $\kappa t_0/a^2$ in the fifth column is $0 \cdot 055$, leading to $\kappa=0 \cdot 0090$. Choosing a constant of proportionality of $1 \cdot 53$, the theoretical values in the last column are obtained which agree reasonably well with the measured temperatures. The result $Q/\pi K=1 \cdot 53$ now gives $K=0 \cdot 0044$, $\rho c=0 \cdot 49$.

VI. The Thermal Conductivity Probe

This is the extension of the line source theory of Section III to cover the case in which the source has a finite diameter and thermal capacity. The simplest idealization of such a system is as follows : an infinitely long circular cylinder of radius a is surrounded by an infinite medium of thermal properties K, ρ, c, κ and contains perfect conductor of thermal capacity S per unit length to which heat is supplied at the constant rate Q per unit length per unit time for times $t>0$. There is a thermal contact resistance $1/H$ per unit area at the surface $r=a$. Then, if the whole system is at zero temperature at time $t=0$, the temperature v of the perfect conductor at time t is given by

$$Kv/Q=G(h,\alpha,\tau), \qquad \ldots\ldots\ldots\ldots \quad (\mathbf{12})$$

where

$$\tau=\kappa t/a^2, \quad \alpha=2\pi a^2\rho c/S, \quad h=K/aH, \quad \ldots\ldots \quad (\mathbf{13})$$

and $G(h,\alpha,\tau)$ is a complicated integral expression for which some numerical values are given by Jaeger (1956).

In principle, the use of a probe containing a heater and a temperature-measuring element in a hole drilled in rock provides a simple method of measuring rock conductivities *in situ*. In fact the experiments are difficult to interpret since all three parameters (**13**) occurring in (**12**) have to be determined, and, while (**12**) has a logarithmic asymptote from which K may be determined directly, to do so in a hole of diameter 4 cm requires an experiment of 2 hr or more in duration, which is most undesirable.

In order to be able to use the earlier parts of the temperature-time curve, Blackwell (1954) has proposed to use several terms of the asymptotic expansion of $G(h,\alpha,\tau)$, while Beck, Jaeger, and Newstead (1956) have suggested a method of curve fitting. The present method is very suitable for the reduction of these results, though more numerical results than those given by Jaeger (1956) are needed for its complete application.

Firstly, it should be said that in most cases a reasonably accurate estimate of the value of α can be made from a knowledge of the dimensions and material of the probe and the nature of the surrounding rock. Secondly, for the case of holes containing water it may be assumed that h is zero or small, whereas, if there is an air-space between the probe and the rock, h will be relatively large. Only the former case will be discussed here, and in Figure 7 values of $G(0,\alpha,\tau)$ and $G(0,\alpha,2\tau)/G(0,\alpha,\tau)$ for various values of α are shown. These have been taken from Jaeger (1956), where some information about the case $h>0$ is also given.

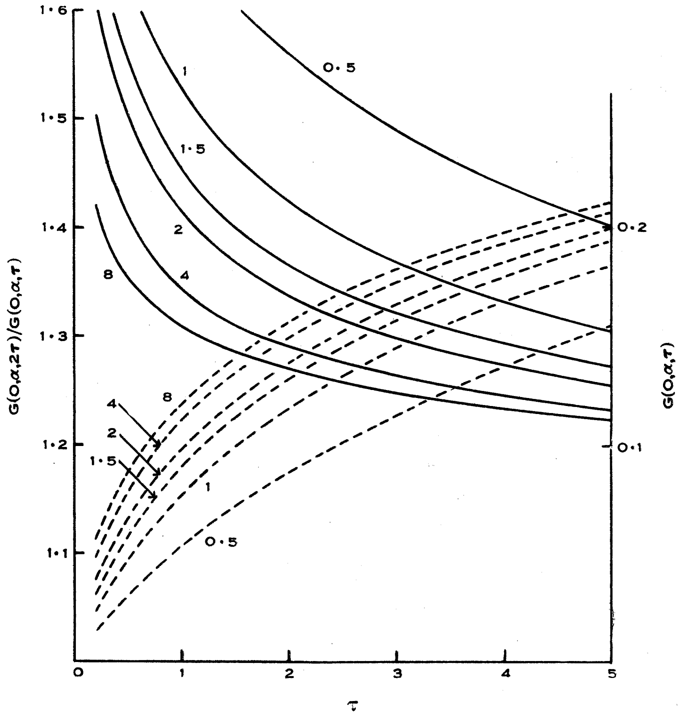

Fig. 7.—Values of $G(0,\alpha,\tau)$: dotted lines, right-hand scale. Values of $G(0,\alpha,2\tau)/G(0,\alpha,\tau)$: full lines, left-hand scale. The numbers on the curves are the values of α.

As an illustration, some observations made in a water-filled hole of diameter $3\cdot5$ cm in vesicular basalt are shown in Table 5. They are reduced on the assumption (derived from the dimensions of the probe) that $\alpha=2$. The unit of time t_0 is 300 sec.

As usual, the second column shows values of the observed temperature at time nt_0, the third gives the ratio $v(2nt_0)/v(nt_0)$, the fourth gives the values of $\varkappa nt_0/a^2$ read off from Figure 7 for the case $\alpha=2$. The mean of the values of $\varkappa t_0/a^2$ is $0\cdot62$, and the last column shows calculated temperatures for this value with a constant of proportionality $Q/K=52$.

The value of Q was $0\cdot22$ cal cm^{-1} sec^{-1}, so that from (**12**) it follows that $K=0\cdot0042$. Also since $\varkappa/a^2=0\cdot0021$ and $a=1\cdot75$ cm we find $\varkappa=0\cdot0063$. These results lead to the rather high value $\rho c=0\cdot67$, which may be attributed to water filling the vesicles in the rock.

The above reduction is incomplete since it merely assumed that $h=0$ and $\alpha=2$. An actual experimental curve can usually be fitted equally well for a range of values of both h and α. There are, however, two additional criteria which are usually available: (i) if the assumed values of h or α are flagrantly incorrect the values of $\varkappa t_0/a^2$ in the fifth column of the table will cease to be approximately constant, (ii) the value of ρc obtained must be consistent with that assumed in α. This last criterion is not available for the example given in Table 5 because of the peculiar nature of the material, so that the reduction cannot be carried further.

TABLE 5

REDUCTION OF OBSERVATIONS MADE WITH A THERMAL CONDUCTIVITY PROBE

n	$v(nt_0)$	$v(2nt_0)/v(nt_0)$	$\varkappa nt_0/a^2$	$\varkappa t_0/a^2$	$52G(0,2,0\cdot 62n)$
0	0				0
2	5·69	1·381	1·32	0·66	5·72
3	6·92	1·344	1·86	0·62	6·92
4	7·86	1·318	2·42	0·60	7·90
5	8·68	1·297	3·00	0·60	8·68
6	9·30				9·36
8	10·36				10·40
10	11·26				11·23

VII. Measurement of Diffusivity from Slabs or Cylinders with Constant Initial and Surface Temperature

These represent some of the oldest and simplest of the classical methods for measuring diffusivity. The formulae are all well known.

(i) For the slab of thickness $2a$ with zero initial temperature and surface temperature V, the temperature $v(t)$ on the mid-plane of the slab at time t is given by (Carslaw and Jaeger 1959, Section 3.4)

$$v=V-\frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\{-\tfrac{1}{4}(2n+1)^2\pi^2T\}, \quad \ldots\ldots \quad \mathbf{(14)}$$

where

$$T=\varkappa t/a^2. \quad \ldots\ldots \quad \mathbf{(15)}$$

Numerical values of v at close intervals of T are given by Olson and Schultz (1942). Values of the ratio $v(2t)/v(t)$ for this case are shown in Figure 8, curve I.

(ii) For an infinite circular cylinder of radius a with zero initial temperature and surface temperature V, the temperature on the axis is given by (Carslaw and Jaeger 1959, Section 7.6)

$$v=V-2V\sum_{n=1}^{\infty}\frac{\exp(-\alpha_n^2T)}{\alpha_nJ_1(\alpha_n)}, \quad \ldots\ldots \quad \mathbf{(16)}$$

where T is given by (**15**) and α_n, $n=1, 2, \ldots$, are the positive roots of $J_0(\alpha)=0$. Numerical values of (**16**) are given by Olson and Schultz (1942). Values of the ratio $v(2t)/v(t)$ are shown in Figure 8, curve II.

(iii) Numerical values of the temperatures in finite circular cylinders with zero initial and constant surface temperatures can readily be obtained (Carslaw and Jaeger 1959, Section 8.4; or Olson and Schultz 1942). In curve III of Figure 8 values of the ratio $v(2t)/v(t)$ for the centre temperature of a finite cylinder of length $2a$ and diameter $2a$ are shown.

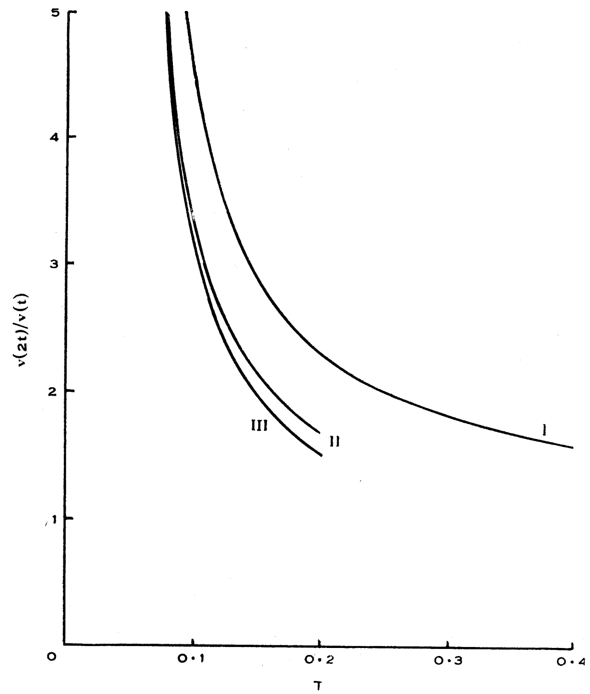

Fig. 8.—Values of $v(2t)/v(t)$ at the centre of a solid with zero initial temperature and constant surface temperature as a function of $T=\varkappa t/a^2$. Curve I: a slab; curve II: a cylinder; curve III: a finite cylinder whose length is equal to its diameter.

These methods have the well-known disadvantage that it is difficult to be certain that the prescribed temperature is in fact maintained at the surface. This disadvantage is less important for poor conductors, and it may be noted that such methods are particularly well suited to rocks whose diffusivities have to be measured when saturated with water.

As an example, Table 6 shows the recorded temperature changes (on an arbitrary scale) measured by a thermocouple at the centre of a finite circular cylinder whose length and diameter were both $5 \cdot 08$ cm and which was initially at a constant temperature and was immersed at $t=0$ in a well-stirred mixture of ice and water. The mean of the values of $\varkappa t_0/a^2$ in the fourth column is $0 \cdot 0079$ and the unit of time t_0 was $3 \cdot 75$ sec, leading to $\varkappa=0 \cdot 0136$. The material of the cylinder was the quartz porphyry previously studied in Section IV, where the same value of $\varkappa$ was found. While the complete agreement is, of course, accidental, it suggests that the boundary condition of zero surface temperature may in fact have been fairly well realized in the present experiment. It will be noticed that the absolute values of the temperature are not needed in the present method.

TABLE 6

REDUCTION OF OBSERVATIONS ON THE DIFFUSIVITY OF A FINITE CYLINDER

n	$v(nt_0)$	$v(2nt_0)/v(nt_0)$	$\varkappa nt_0/a^2$	$\varkappa t_0/a^2$
0	0			
8	0·085	6·73	0·064	0·0080
9	0·128	5·53	0·071	0·0079
10	0·182	4·61	0·079	0·0079
11	0·236	4·08	0·085	0·0077
16	0·572			
18	0·708			
20	0·839			
22	0·962			

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