ON THE RELATION BETWEEN LUMINOSITY DISTANCE AND DOPPLER SHIFT IN RELATIVISTIC COSMOLOGY*

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Introduction

An approximate relation between luminosity distance D and Doppler shift δ in cosmology is usually obtained by a succession of complicated expansions in series.

In the present paper it is shown that an expression for D, even to a higher approximation in δ, can be obtained in a much simpler way.

The relation between luminosity distance D and red-shift δ commonly used in relativistic cosmology is (McVittie 1956)‡

$$D=\frac{c\delta}{h_1}\left(1+\frac{h_1^2+h_2}{2h_1^2}\delta\right), \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where

$$h_1=\dot{R}(t_0)/R(t_0)=\dot{R}_0/R_0, \text{ i.e. the Hubble constant,} \qquad \ldots\ldots\ldots\ldots \quad (2)$$

and

$$h_2=\ddot{R}_0/R_0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

R being a function of t.

The exact expression for D is

$$D=\frac{R_0^2}{R}\,\frac{r}{1+\alpha r^2/4}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where r is a function of R given by the null-geodesic equation

$$c\int_t^{t_0}\frac{\mathrm{d}t}{R(t)}=\int_0^r\frac{\mathrm{d}r}{1+\alpha r^2/4}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

The constant $\alpha=+1$ for an elliptic space,
$=0$ for a flat space,
$=-1$ for a hyperbolic space.

The left-hand side of (5) could be written

$$c\int_t^{t_0}\frac{\mathrm{d}t}{R(t)}=c\int_R^{R_0}\frac{\mathrm{d}R}{R\dot{R}},$$

and therefore

$$\int_0^r\frac{\mathrm{d}r}{1+\alpha r^2/4}=\psi, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

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‡ McVITTIE, G. C. (1956).—"General Relativity and Cosmology." (Chapman & Hall: London.)

where

$$\psi = c\int_R^{R_0} \frac{dR}{R\dot{R}}. \qquad (7)$$

If $\alpha=+1$ it follows from (6) that

$$r = 2\tan\tfrac{1}{2}\psi,$$

and substituting this in (4) we have the following expression for the luminosity distance

$$D = (R_0^2/R)\sin\psi. \qquad (8)$$

It is also easily seen that:

if $\alpha=0$,
$$D = (R_0^2/R)\psi, \qquad (9)$$

if $\alpha=-1$,
$$D = (R_0^2/R)\sinh\psi. \qquad (10)$$

Elliptic and Hyperbolic Spaces

For these spaces

$$D = (R_0^2/R)F(R), \qquad (11)$$

where

$$F(R) = \begin{matrix}\sin\\ \sinh\end{matrix}\left(c\int_R^{R_0}\frac{dR}{R\dot{R}}\right), \qquad (12)$$

the *sin* corresponding to the elliptic space and *sinh* to the hyperbolic one.

Using a Taylor expansion for F, we have

$$F(R) = F(R_0) + \left(\frac{dF}{dR}\right)_{R_0}(R-R_0) + \left(\frac{d^2F}{dR^2}\right)_{R_0}\frac{(R-R_0)^2}{2} + \left(\frac{d^3F}{dR^3}\right)_{R_0}\frac{(R-R_0)^3}{6} + \ldots, \qquad (13)$$

where, firstly, $F(R_0)=0$,

secondly,

$$\frac{dF}{dR} = -c\begin{matrix}\cos\\ \cosh\end{matrix}\left(c\int_R^{R_0}\frac{dR}{R\dot{R}}\right)\frac{1}{R\dot{R}},$$

that is,

$$\left(\frac{dF}{dR}\right)_{R_0} = \frac{-c}{R_0\dot{R}_0}, \qquad (14)$$

thirdly,

$$\frac{d^2F}{dR^2} = \pm c^2\begin{matrix}\sin\\ \sinh\end{matrix}\left(c\int_R^{R_0}\frac{dR}{R\dot{R}}\right)\frac{-1}{R^2\dot{R}^2} - c\begin{matrix}\cos\\ \cosh\end{matrix}\left(c\int_R^{R_0}\frac{dR}{R\dot{R}}\right)\frac{d}{dR}\left(\frac{1}{R\dot{R}}\right). \qquad (15)$$

But

$$\frac{d}{dR}\left(\frac{1}{R\dot{R}}\right) = \frac{d}{dt}\left(\frac{1}{R\dot{R}}\right)\frac{dt}{dR} = -\frac{\dot{R}^2 + R\ddot{R}}{R^2\dot{R}^3}. \qquad (16)$$

Substituting this in (15),

$$\frac{d^2F}{dR^2} = \mp c^2\begin{matrix}\sin\\ \sinh\end{matrix}\left(c\int_R^{R_0}\frac{dR}{R\dot{R}}\right)\frac{1}{R^2\dot{R}^2} + c\begin{matrix}\cos\\ \cosh\end{matrix}\left(c\int_R^{R_0}\frac{dR}{R\dot{R}}\right)\frac{\dot{R}^2 + R\ddot{R}}{R^2\dot{R}^3}, \qquad (17)$$

and

$$\left(\frac{\mathrm{d}^2F}{\mathrm{d}R^2}\right)_{R_0}=c\frac{\dot{R}_0^2+R_0\ddot{R}_0}{R_0^2\dot{R}_0^3}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{18})$$

Fourthly,

$$\frac{\mathrm{d}^3F}{\mathrm{d}R^3}=\mp c^3\,\begin{matrix}\cos\\ \cosh\end{matrix}\left(c\int_R^{R_0}\frac{\mathrm{d}R}{R\dot{R}}\right)\frac{-1}{R^3\dot{R}^3}$$
$$\mp c^2\,\begin{matrix}\sin\\ \sinh\end{matrix}\left(c\int_R^{R_0}\frac{\mathrm{d}R}{R\dot{R}}\right)(-2R^{-3}\dot{R}^{-1}-2R^{-2}\dot{R}^{-3}\ddot{R})\frac{1}{\dot{R}}$$
$$\pm c^2\,\begin{matrix}\sin\\ \sinh\end{matrix}\left(c\int_R^{R_0}\frac{\mathrm{d}R}{R\dot{R}}\right)\frac{\dot{R}^2+R\ddot{R}}{R^3\dot{R}^4}$$
$$+c\,\begin{matrix}\cos\\ \cosh\end{matrix}\left(c\int_R^{R_0}\frac{\mathrm{d}R}{R\dot{R}}\right)\left(-\frac{2}{R^3}-\frac{2\ddot{R}}{R^2\dot{R}^2}+\frac{\dddot{R}}{\dot{R}^3R}-\frac{3\ddot{R}^2}{R\dot{R}^4}\right)\frac{1}{\dot{R}},$$

and

$$\left(\frac{\mathrm{d}^3F}{\mathrm{d}R^3}\right)_{R_0}=\pm\frac{c^3}{R_0^3\dot{R}_0^3}+\frac{c}{\dot{R}_0}\left\{-\frac{2}{R_0^3}-\frac{2\ddot{R}_0}{R_0^2\dot{R}_0^2}+\frac{\dddot{R}_0}{\dot{R}_0^3R_0}-\frac{3\ddot{R}_0^2}{R_0\dot{R}_0^4}\right\}. \quad \ldots (\mathbf{19})$$

Substituting (**14**), (**18**), and (**19**) in (**13**) we have

$$F(R)=-\frac{c}{R_0\dot{R}_0}(R-R_0)+c\frac{\dot{R}_0^2+R_0\ddot{R}_0}{R_0^2\dot{R}_0^3}\,\frac{(R-R_0)^2}{2}$$
$$+\left\{\pm\frac{c^3}{R_0^3\dot{R}_0^3}+\frac{c}{\dot{R}_0}\left[-\frac{2}{R_0^3}-\frac{2\ddot{R}_0}{R_0^2\dot{R}_0^2}-\frac{3\ddot{R}_0^2}{R_0\dot{R}_0^4}+\frac{\dddot{R}_0}{\dot{R}_0^3R_0}\right]\right\}\frac{(R-R_0)^3}{6}. \quad \ldots\ldots\ldots\ldots (\mathbf{20})$$

Substituting this in (**11**) and remembering that

$$1-R_0/R=-\delta,$$

and

$$R=R_0/(1+\delta)=R_0(1-\delta+\delta^2-\delta^3\ldots),$$

we have

$$D=-\frac{cR_0}{R}(-\delta)+c\frac{\dot{R}_0^2+R_0\ddot{R}_0}{\dot{R}_0^3}R_0(1-\delta+\delta^2-\delta^3\ldots)\frac{(-\delta)^2}{2}$$
$$+\left\{\pm\frac{c^3}{R_0\dot{R}_0^3}+\frac{c}{\dot{R}_0}\left[-\frac{2}{R_0}-\frac{2\ddot{R}_0}{\dot{R}_0^2}+\frac{\dddot{R}_0R_0}{\dot{R}_0^3}-\frac{3\ddot{R}_0^2R_0}{\dot{R}_0^4}\right]\right\}\frac{(-\delta)^3}{6}R_0^2(1-\delta+\delta^2\ldots)^2,$$

or, after a few reductions,

$$D=\frac{c}{h_1}\delta+c\frac{h_1^2+h_2}{2h_1^3}\delta^2$$
$$-\left\{\frac{c}{6}\,\frac{h_1^2+h_2}{h_1^3}+\frac{c}{6h_1}\left[\frac{h_3}{h_1^3}-3\,\frac{h_2^2}{h_1^4}\right]\pm\frac{c^3}{6R_0^2h_1^3}\right\}\delta^3. \quad \ldots\ldots (\mathbf{21})$$

This is the luminosity distance–red-shift relation, correct to the third order in δ. The plus sign in the last term is used for elliptic spaces and the minus sign for hyperbolic ones.

Flat Spaces

In this case the luminosity distance is given by

$$D=\frac{R_0^2}{R}c\int_R^{R_0}\frac{\mathrm{d}R}{R\dot{R}},\quad \text{i.e.}\quad F=C\int_R^{R_0}\frac{\mathrm{d}R}{R\dot{R}}. \qquad \text{(22)}$$

Then, firstly,

$$\left(\frac{\mathrm{d}F}{\mathrm{d}R}\right)_{R_0}=-\frac{c}{R_0\dot{R}_0}, \qquad \text{(23)}$$

secondly,

$$\left(\frac{\mathrm{d}^2F}{\mathrm{d}R^2}\right)_{R_0}=c\frac{\dot{R}_0^2+R_0\ddot{R}_0}{R_0^2\dot{R}_0^3}, \qquad \text{(24)}$$

thirdly,

$$\left(\frac{\mathrm{d}^3F}{\mathrm{d}R^3}\right)_{R_0}=\frac{c}{\dot{R}_0}\left(-\frac{2}{R_0^3}-\frac{2\ddot{R}_0}{R_0^2\dot{R}_0^2}-\frac{3\ddot{R}_0^2}{R_0\dot{R}_0^4}+\frac{\dddot{R}_0}{\dot{R}_0^3R_0}\right). \qquad \text{(25)}$$

It is evident, by comparison of (**14**), (**18**), (**19**) and (**23**), (**24**), (**25**) that the luminosity distance–red-shift relation is

$$D=\frac{c}{h_1}\delta\left\{1+\frac{h_1^2+h_2}{2h_1^2}\delta-\frac{1}{6}\left[\frac{h_1^2+h_2}{h_1^2}+\left(\frac{h_3}{h_1^3}-3\frac{h_2^2}{h_1^4}\right)+\frac{\alpha c^2}{R_0^2h_1^2}\right]\delta^2\right\}, \qquad \text{(26)}$$

where $\alpha=+1$ for an elliptic space,
$\alpha=0$ for a flat space,
$\alpha=-1$ for a hyperbolic space,

and

$$h_3=\dddot{R}_0/R_0.$$

The first two terms in (**26**) give

$$D=\frac{c}{h_1}\delta+c\frac{h_1^2+h_2}{2h_1^3}\delta^2, \qquad \text{(27)}$$

which is the formula commonly used in relativistic cosmology and which does not distinguish between the three types of spaces.