ON THE EQUIVALENCE OF CURRENT LOOPS AND MAGNETIC SHELLS

By C. A. HURST*

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Summary

A paradox in the theory of the magnetic effects of stationary currents is discussed and is shown to arise from the neglect of a singular magnetic field which is required to complete the equivalence of magnetic shells and current loops.

I. INTRODUCTION

It appears paradoxical that whilst the force between two permanent magnets decreases when they are immersed in a medium with relative permeability greater than unity, the force between two current loops under the same circumstances increases. For the standard theory states that the magnetic effects due to a current loop are the same as those produced by a magnetic shell spanning the region bounded by the current loop. So the naive view would be that both magnets and current loops would behave similarly in a material medium. The usual explanation of this difference is to point out that the magnetic shell is only a fictitious shell, and the equivalence is only partial. For the field of a magnetic shell is conservative, and the potential single valued, whereas the field of a current loop is non-conservative and the potential multiple valued. But the precise way in which this lack of complete equivalence may be expressed mathematically, and the way in which it may lead to different consequences in the two cases, such as mentioned above, has not, to my knowledge, been shown. It appears that the proper way to distinguish between the two cases is given by an application of the mathematical theory of distributions, i.e. of the δ-function and its derivatives. For a magnetic shell correctly represents the field of the current loop *everywhere* except at the shell itself, and so, to distinguish between the two fields, a function which is zero everywhere except on a surface will be required.

As is well known, the potential at two points, close together but on opposite sides of the shell, differs by $4\pi M$, where M is the magnitude of the magnetic moment per unit area, and as the field of the shell is conservative, the potential must suffer a jump by $4\pi M$ on passing through the shell, in order to cancel this difference. This means that the magnetic field *on* the shell is highly singular, and it is just to cancel this singularity in the case of a current loop that an additional field is needed. But as a consequence the potential is continuous but not single valued.

When a current loop is fully immersed in a material medium, the additional magnetic field which distinguishes it from a magnetic shell, will produce an

* Department of Mathematical Physics, University of Adelaide.

additional polarization *over the surface of the shell only*, and this additional polarization will augment the equivalent magnetic moment from $\mu_0 IA$ to μIA where I is the current in the loop, and A is the area of the loop (assumed for the moment to be plane).

II. Mathematical Theory of the Current Loop in Vacuo

From Maxwell's equations, we have

$$\mathbf{B}=\text{curl } \mathbf{A}, \tag{1}$$

$$\nabla^2\mathbf{A}=-\mu_0\mathbf{I}, \tag{2}$$

where $\mathbf{I}$ is the current in a loop.

If the equation of the loop is parametrically given by

$$x=x_0(s),\ y=y_0(s),\ z=z_0(s)\ (0\leqslant s\leqslant 1)$$

then

$$\mathbf{I}(x,y,z)=\mathbf{I}\int_0^1 \mathrm{d}s\,\delta(x-x_0(s))\delta(y-y_0(s))\delta(z-z_0(s))\left(\mathbf{i}\frac{\mathrm{d}x_0}{\mathrm{d}s}+\mathbf{j}\frac{\mathrm{d}y_0}{\mathrm{d}s}+\mathbf{k}\frac{\mathrm{d}z_0}{\mathrm{d}s}\right), \tag{3}$$

and the solution of (2) is

$$\mathbf{A}(x,y,z)=\mu_0\int\int\int\frac{\mathbf{I}(x',y',z')}{4\pi r}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \text{ with } r=\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}$$

$$=\frac{I\mu_0}{4\pi}\int\frac{\mathbf{t}}{r}\mathrm{d}s \text{ with } \mathbf{t}=\mathbf{i}\frac{\mathrm{d}x_0}{\mathrm{d}s}+\mathbf{j}\frac{\mathrm{d}y_0}{\mathrm{d}s}+\mathbf{k}\frac{\mathrm{d}z_0}{\mathrm{d}s}, \tag{4}$$

when (3) is substituted.

From (1), we get

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}=\frac{I}{4\pi}\int_c(\mathbf{t}\times\nabla')\frac{1}{r}\mathrm{d}s. \tag{5}$$

A suitable generalization of Stokes' theorem gives, using tensor notation,

$$H_i=\frac{I}{4\pi}\int\int\mathrm{d}S\left(n_k\frac{\partial^2}{\partial x_i'\partial x_k'}-n_i\frac{\partial^2}{\partial x_k'^2}\right)\frac{1}{r},$$

or, in vector notation,

$$\mathbf{H}=\frac{I}{4\pi}\int\int_S\mathrm{d}S(\mathbf{n}\cdot\nabla'\nabla'-\mathbf{n}\nabla'^2)\frac{1}{r} \tag{5'}$$

$$=\mathbf{H}_m+\mathbf{H}_c.$$

The first term $\mathbf{H}_m$ is the field due to the equivalent magnetic shell, whereas the second term is a correction required to make $\mathbf{H}$ continuous through S. As $\nabla^2(1/r)=-\delta(r)/r^2$, where $\delta(r)$ is the Dirac δ-function, H_c describes a field which is zero everywhere except on S.

This is the term which is non-conservative, because

$$\int_{c'}\mathbf{H}\cdot\mathbf{t}'\mathrm{d}s=\int_{c'}\mathbf{H}_m\cdot\mathbf{t}'\mathrm{d}s+\int_{c'}\mathbf{H}_c\cdot\mathbf{t}'\mathrm{d}s'=\int_{c'}\mathbf{H}_c\cdot\mathbf{t}'\mathrm{d}s$$

for any closed circuit. If c' loops the circuit c, then

$$\int_{c'} \mathbf{H}_c \cdot \mathbf{t}' \mathrm{d}s = -\frac{I}{4\pi}\iiint_{Sc'} \mathrm{d}s\mathrm{d}S\mathbf{t}' \cdot \mathbf{n}\nabla^2\frac{1}{r} = \pm\frac{I}{4\pi}\iiint \mathrm{d}V\frac{\delta(r)}{r^2} \qquad (6)$$
$$= \pm I,$$

with the sign determined by the relative sense of $\mathbf{t}'$ and $\mathbf{n}$. The " source " of such a field will be that which produces a " potential "

$$\Omega_{ik} = \frac{I}{4\pi}\iint_s \mathrm{d}Sn_i\frac{\partial}{\partial x_k'}\frac{1}{r}, \qquad (7)$$

which is a tensor, and not a scalar, quantity. So the magnetic field of a current is the sum of a true magnetostatic field together with a pseudo-magnetostatic field produced by a distribution of " sources " over the surface of the shell. The combined effect of these sources will be to produce a field which is everywhere regular and non-conservative.

III. Extension to a Permeable Medium

When a current loop is placed in a medium, of permeability μ, a distributed magnetization is produced with a magnetic moment density given by $\mathbf{M}=\chi\mathbf{H}^R$ (in non-ferromagnetic materials), with $\mathbf{H}^R$ the resultant field in the medium. So $\mathbf{H}^R$ is determined by the integral equation

$$\mathbf{H}^R = \mathbf{H}_m + \mathbf{H}_c + \nabla\int_V \chi\mathbf{H}^R \cdot \nabla\left(\frac{1}{r}\right)\mathrm{d}V. \qquad (8)$$

$\mathbf{H}^R$ is non-conservative, but the quantity $\mathbf{H}^R - \mathbf{H}_c$ satisfies the equation

$$\mathbf{H}^R - \mathbf{H}_c = \mathbf{H}_m + \nabla\int_V \chi\mathbf{H}_c \cdot \nabla\left(\frac{1}{r}\right)\mathrm{d}V + \nabla\int_V \chi(\mathbf{H}^R - \mathbf{H}_c)\cdot\nabla\left(\frac{1}{r}\right)\mathrm{d}V, \qquad (8')$$

in which no non-conservative terms appear. In fact the non-homogeneous term

$$\mathbf{H}_m + \nabla\int_V \chi\mathbf{H}_c \cdot \nabla\left(\frac{1}{r}\right)\mathrm{d}V$$

is the field due a magnetic shell of moment $\mu_0 I + \chi I = \mu I$ per unit area, i.e. the effective magnetic moment has been increased. Hence the mutual potential energy of two current loops, which *in vacuo* is proportional to $\mu_0 I_1 I_2$, becomes, in a medium of permeability μ, proportional to $\mu I_1 I_2$, which may be written as

$$(\mu I_1)(\mu I_2)/\mu = M_1 M_2/\mu, \qquad (9)$$

where M_1, M_2 are the equivalent moments of the magnetic shells. So, unlike two magnetic sources, the force between the current sources in a medium increases (if $\mu > \mu_0$) because the strength of both sources increases and more than compensates the decrease in field produced by the shielding effect of the induced dipoles.