CHARGED PARTICLE TRAJECTORIES IN STATIC ELECTRIC AND MAGNETIC FIELDS*

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A trajectory passing through a given point in a given direction is completely determined if its curvature and torsion are known functions of its arc length. Relativistic expressions for the curvature and torsion in terms of the electric and magnetic field distributions are derived below. Besides their intrinsic interest these expressions may be useful in the analytical solution of some simple trajectory problems. In more complicated problems the trajectories may be extrapolated from their origin by means of the canonical equations, see, for example, Weatherburn (1955),

$$\left.\begin{aligned} x &= s-\varkappa_0^2 s^3/6+\ldots, \\ y &= \varkappa_0 s^2/2+\varkappa_0' s^3/6+\ldots, \\ z &= \varkappa_0\tau_0 s^3/6+\ldots, \end{aligned}\right\} \tag{1}$$

where $\varkappa_0$ and τ_0 are the curvature and torsion at the point O on the trajectory; $\varkappa_0'$ is the rate of change of curvature with regard to arc length evaluated at O; x, y, z are the rectangular Cartesian coordinates of the point at a distance s along the trajectory from O referred to the coordinate axes OX, OY, OZ which correspond respectively with the directions of the tangent, normal, and binormal to the trajectory at O.

Trajectory plotting by means of the equations (1) terminated at terms of the second degree in s has been done automatically by Gabor (1937) and Langmuir (1937) for purely electrostatic fields and low particle velocities. In this case only the curvature need be known and is $\varkappa=-E_n/2V$ where V is the potential at the point in question and E_n is the component of the electric field normal to the trajectory. The electrolytic tank analogue was used for the direct measurement of both V and E_n. If other than plane trajectories are to be plotted terms to at least the third degree in s must be used. Thus $\varkappa$, τ, and $\varkappa'$ must be known in terms of the field distributions. Relativistic formulas for these three quantities are derived below by the methods of differential geometry using vector notation. Unit vectors in the directions of the tangent, normal, and binormal to the charged particle trajectory are denoted by **t**, **n**, and **b** respectively. Derivatives with regard to arc length are denoted by a superscript dash.

Derivation of the Curvature

The trajectory of a charged particle of charge q, mass m, and speed v in an electric field **E** and magnetic field **B** is described by

$$v\mathrm{d}(mv\mathbf{t})/\mathrm{d}s=q(\mathbf{E}+v\mathbf{t}\times\mathbf{B}). \tag{2}$$

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In order to carry out the differentiation a convenient expression for m is sought. We have

$$mc^2 - m_0c^2 + qV = \text{constant} \tag{3}$$

as the energy equation. If we assume the particle, of charge q, had zero initial velocity in the region of zero potential, then the constant of (3) is zero. Note that for positive kinetic energy the product qV is negative. In what follows q and V will be treated as pure algebraic symbols so that both q and V may be of either sign but the product qV must be a negative quantity.

Equation (3) may be written as

$$m = m_0(1 - V/V_0), \tag{4}$$

where $V_0 = m_0c^2/q$ has the dimensions of electric potential. This " natural unit " of potential is approximately $+930 \times 10^6$ volts for a proton and $-0 \cdot 5 \times 10^6$ volts for an electron. Writing $\beta = (1 - V/V_0)$ equation (2) becomes

$$m_0\beta v^2 \varkappa \mathbf{n} + m_0\beta' v^2 \mathbf{t} + m_0\beta v v' \mathbf{t} = q(\mathbf{E} + v\mathbf{t} \times \mathbf{B}), \tag{5}$$

where we have used $\mathbf{t}' = \varkappa \mathbf{n}$.

The normal component of (5) is

$$\begin{aligned} m_0\beta v^2 \varkappa &= q(E_n + v\mathbf{t} \times \mathbf{B} \cdot \mathbf{n}) \\ &= q(E_n - vB_b), \end{aligned} \tag{6}$$

and the tangential component is

$$m_0\beta' v^2 + m_0\beta v v' = qE_t, \tag{7}$$

where E_t, E_n, and E_b are the tangential, normal, and binormal components of $\mathbf{E}$ etc. (7) is the energy equation while (6) is a trajectory equation.

Making the following substitutions in (6)

$$\left.\begin{aligned} \beta &= (1 - V/V_0) = (1 - \delta), \\ v^2 &= c^2\delta(-2 + \delta)/(1 - \delta)^2, \\ v &= -c\delta^{\frac{1}{2}}(-2 + \delta)^{\frac{1}{2}}/(1 - \delta), \end{aligned}\right\} \tag{8}$$

gives for the curvature

$$\varkappa = \{E_n(1 - \delta) + B_b c\delta^{\frac{1}{2}}(-2 + \delta)^{\frac{1}{2}}\}/V(-2 + \delta). \tag{9}$$

Note that δ is negative for positive kinetic energies. Thus $c\delta^{\frac{1}{2}}$ is imaginary and $\varkappa$ is real.

For low energies $\delta \to 0$ and (9) reduces to

$$\varkappa = -\tfrac{1}{2}\{E_n - \left(\frac{|2qV|}{m_0}\right)^{\frac{1}{2}} B_b\}/V. \tag{10}$$

Arc Rate of Change of Curvature

To find $\varkappa'$, the rate of change of curvature with trajectory arc length, differentiate (5) with regard to s. This gives

$$\begin{aligned} (m_0\beta v^2\varkappa)'\mathbf{n} + (m_0\beta v^2\varkappa)\mathbf{n}' + (m_0\beta' v^2)\mathbf{t}' + (m_0\beta' v^2)'\mathbf{t} + (m_0\beta v v')'\mathbf{t} + (m_0\beta v v')\mathbf{t}' \\ = q\{\mathbf{E}' + v'\mathbf{t} \times \mathbf{B} + v(\mathbf{t}' \times \mathbf{B} + \mathbf{t} \times \mathbf{B}')\}. \end{aligned}$$

The normal component of this is

$$(m_0\beta v^2\varkappa)'+(m_0\beta' v^2)\varkappa+(m_0\beta v v')\varkappa=q\{(\mathbf{E}')_n+v'\mathbf{t}\times\mathbf{B}\cdot\mathbf{n}+v\mathbf{t}\times\mathbf{B}'\cdot\mathbf{n}\}$$
$$=q\{(\mathbf{E}')_n-v'B_b-v(\mathbf{B}')_b\}.$$

Whence

$$\varkappa'=\frac{q}{m_0\beta v^2}\{(\mathbf{E}')_n-v'B_b-v(\mathbf{B}')_b\}-\varkappa\left\{\frac{2\beta'}{\beta}+\frac{3v'}{v}\right\}. \tag{11}$$

Using the relations (8), together with

$$\left.\begin{aligned}\beta'&=E_t/V_0,\\ v'&=-E_t c\delta^{\frac{1}{2}}/\{V(1-\delta)^2(-2+\delta)^{\frac{1}{2}}\},\end{aligned}\right\} \tag{12}$$

in (11) gives

$$\varkappa'=\frac{(\mathbf{E}')_n(1-\delta)}{V(-2+\delta)}+\frac{(\mathbf{B}')_b c\delta^{\frac{1}{2}}}{V(-2+\delta)^{\frac{1}{2}}}-\frac{2E_tB_bc\delta^{\frac{1}{2}}}{V(1-\delta)(-2+\delta)^{\frac{1}{2}}}\left\{\frac{1}{V_0}+\frac{1}{V(-2+\delta)}\right\}$$
$$-\frac{E_tE_n}{V(-2+\delta)}\left\{\frac{2}{V_0}+\frac{3}{V(-2+\delta)}\right\}. \tag{13}$$

Substituting $c\delta^{\frac{1}{2}}=(qV/m_0)^{\frac{1}{2}}$ and allowing $\delta\to 0$ in (13) we have for low energies

$$\varkappa'=\frac{1}{2V}\left\{\left(\frac{|2qV|}{m_0}\right)^{\frac{1}{2}}(\mathbf{B}')_b-(\mathbf{E}')_n\right\}+\frac{E_t}{2V^2}\left\{\left(\frac{|2qV|}{m_0}\right)^{\frac{1}{2}}B_b-\frac{3E_n}{2}\right\}. \tag{14}$$

Torsion

To find an expression for the torsion τ, take the vector product of (5) with $\mathbf{t}$, whence

$$-m_0\beta v^2\varkappa\mathbf{b}=q\{\mathbf{E}\times\mathbf{t}+v(\mathbf{t}\times\mathbf{B})\times\mathbf{t}\}$$
$$=q\{\mathbf{E}\times\mathbf{t}+v\mathbf{B}-vB_t\mathbf{t}\}. \tag{15}$$

Differentiating with regard to s gives

$$(-m_0\beta v^2\varkappa)'\mathbf{b}+m_0\beta v^2\varkappa\tau\mathbf{n}=q\{\mathbf{E}'\times\mathbf{t}+\mathbf{E}\times\mathbf{n}\varkappa+v'\mathbf{B}+v\mathbf{B}'-(vB_t)'\mathbf{t}-vB_t\varkappa\mathbf{n}\}.$$

The normal component of this is

$$m_0\beta v^2\varkappa\tau=q\{\mathbf{E}'\times\mathbf{t}\cdot\mathbf{n}+\varkappa\mathbf{E}\times\mathbf{n}\cdot\mathbf{n}+v'B_n+v(\mathbf{B}')_n-vB_t\varkappa\}$$
$$=q\{(\mathbf{E}')_b+v'B_n+v(\mathbf{B}')_n-vB_t\varkappa\}.$$

Thus either $v^2\varkappa=0$ or

$$\tau=\frac{q}{m_0}\left\{\frac{(\mathbf{E}')_b}{\beta v^2\varkappa}+\frac{v'B_n}{\beta v^2\varkappa}+\frac{(\mathbf{B}')_n}{\beta v\varkappa}-\frac{B_t}{\beta v}\right\}$$
$$=\frac{q}{m_0\beta v\varkappa}\left\{\frac{(\mathbf{E}')_b}{v}+\frac{v'B_n}{v}+(\mathbf{B}')_n\right\}-\frac{qB_t}{m_0\beta v}. \tag{16}$$

Substituting from (8) and (12) in (16) gives

$$\tau=\frac{(\mathbf{E}')_b(1-\delta)-E_tB_nc\delta^{\frac{1}{2}}/\{V(1-\delta)(-2+\delta)^{\frac{1}{2}}\}-(\mathbf{B}')_nc\delta^{\frac{1}{2}}(-2+\delta)^{\frac{1}{2}}}{E_n(1-\delta)+B_bc\delta^{\frac{1}{2}}(-2+\delta)^{\frac{1}{2}}}+\frac{B_tc\delta^{\frac{1}{2}}}{V(-2+\delta)^{\frac{1}{2}}}. \tag{17}$$

For low energies, $\delta \to 0$, we have either

$$\tau = \frac{(\mathbf{E}')_b - \dfrac{E_t B_n}{2V}\left(\dfrac{|2qV|}{m_0}\right)^{\frac{1}{2}} + (\mathbf{B}')_n \left(\dfrac{|2qV|}{m_0}\right)^{\frac{1}{2}}}{E_n - B_b\left(\dfrac{|2qV|}{m_0}\right)^{\frac{1}{2}}} + \frac{B_t}{2V}\left(\frac{|2qV|}{m_0}\right)^{\frac{1}{2}} \tag{18}$$

or $\varkappa = 0$.

Conclusion

In the absence of magnetic fields and for low energies

$$\left.\begin{aligned} \varkappa &= -E_n/2V, \\ \varkappa' &= -(\mathbf{E}')_n/2V - 3E_nE_t/(4V^2), \\ \tau &= (\mathbf{E}')_b/E_n. \end{aligned}\right\} \tag{19}$$

Note that in all the above expressions for $\varkappa$, $\varkappa'$, and τ the potential V is positive or negative according to whether q is negative or positive.

References

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