AN EXACT SOLUTION TO A PROBLEM IN HEAT TRANSFER*

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In order to determine the complete temperature history in problems concerning the transfer of heat from a moving fluid to a solid body one is obliged, in general, to solve both the flow and heat transport equations for the fluid and the heat conduction equation for the solid. Analytical solutions to this problem cannot in general be given owing to the (non-linear) way in which the flow and heat transport equations for the fluid are coupled.

However, there is a class of problems of this kind for which an exact unsteady-state solution can be given. These problems involve a symmetry which is such

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that in the equation for the fluid the convection terms vanish identically. The result is that these equations are uncoupled in the sense that the flow velocity is independent of the temperature distribution and the temperature depends on the flow only through the viscous dissipation function. Under these conditions the problem reduces to solving the heat conduction problem for a composite body with heat generation in one body, the fluid.

The problem we solve here concerns the parallel flow of fluid over a heat conducting half space, the motion being started impulsively. Other problems such as flow between parallel walls and flow round cylinders could be considered but not in the same way as the present problem.

The statement of problem is as follows: The region $z>0$ is occupied by a viscous heat-conducting incompressible constant property fluid. The region $z<0$ is occupied by a heat-conducting solid. The initial temperature of the fluid is T_0, while that of the solid is zero. To determine the temperature distribution in the system subsequent to the uniform impulsive motion of the fluid in a direction parallel to the plane $z=0$.

In the fluid, the pressure gradients being zero, we have

$$\partial u/\partial t-\nu\partial^2 u/\partial z^2=0, \qquad z>0,\ t>0, \tag{1}$$

$$\rho_1 c_1\partial T_1/\partial t-k_1\partial^2 T_1/\partial z^2=\mu\left(\frac{\partial u}{\partial z}\right)^2, \quad z>0,\ t>0, \tag{2}$$

and in the solid,

$$\rho_2 c_2\partial T_2/\partial t-k_2\cdot\partial^2 T_2/\partial z^2=0, \qquad z<0,\ t>0, \tag{3}$$

with

$$u=0, \qquad z=0,\ t>0, \tag{4}$$

$$u=U, \qquad z>0,\ t=0, \tag{5}$$

$$T_1=T_2, \qquad z=0,\ t>0, \tag{6}$$

$$k_1\partial T_1/\partial z=k_2\partial T_2/\partial z, \quad z=0,\ t>0, \tag{7}$$

$$T_1=T_0, \qquad z>0,\ t=0, \tag{8}$$

$$T_2=0, \qquad z<0,\ t=0, \tag{9}$$

with T_1, T_2, and u everywhere bounded.

The subscript "1" refers to the fluid and the subscript "2" to the solid. The remaining symbols have their usual meanings.

Introducing the dimensionless variable ξ, $\xi=z\,.\,(4\nu_1 t)^{-\frac{1}{2}}$, the solution to equation (1) with (4) and (5) is

$$u=\operatorname{erf}\xi. \tag{10}$$

Writing $v_1=T_1/T_0$ and using (10), equation (2) becomes

$$\mathrm{d}^2 v_1/\mathrm{d}\xi^2+2P.\xi.\mathrm{d}v_1/\mathrm{d}\xi=-b\,.\,\exp(-2\xi^2), \tag{11}$$

where $P=\nu/k_1$ is the Prandtl number and $b=(4/\pi)PE$, where the Eckert number E is given by $E=U^2/(c_1T_0)$.

Integrating equation (11) twice we find

$$v_1(\xi)=-d\int_{\eta=\xi}^{\infty}\exp(-P\eta^2)\,.\,\operatorname{erfc}\{\eta(2-P)^{\frac{1}{2}}\}\mathrm{d}\eta-A\,.\,\tfrac{1}{2}(\pi/P)^{\frac{1}{2}}\,.\,\operatorname{erfc}(\xi P^{\frac{1}{2}})+B,\quad (12)$$

where $d=\frac{1}{2}\,.\,b\,.\,(\pi/(2-P))^{\frac{1}{2}}$ and A and B are constants of integration. In order to satisfy equation (8) we must have $B=1$.

If $v_2=T_2/T_0$ we have from equation (3)

$$v_2(\xi)=C\,.\,\operatorname{erfc}\{-\xi(\nu/k_2)^{\frac{1}{2}}\},\quad \xi<0,\quad (13)$$

where C is a constant to be determined.

The constants A and C are to be determined from the two conditions (6) and (7), which in terms of the variable ξ become

$$\left.\begin{aligned} v_1&=v_2, && \xi=0,\\ k_1\frac{\mathrm{d}v_1}{\mathrm{d}\xi}&=k_2\frac{\mathrm{d}v_2}{\mathrm{d}\xi}, && \xi=0.\end{aligned}\right\}\quad (14)$$

Substituting (12) and (13) in (14) and solving for A and C we find

$$A=\frac{1-d\,.\,I(0,P)-d\varepsilon^{-1}}{\frac{1}{2}(\pi/P)^{\frac{1}{2}}+\varepsilon^{-1}},\quad (15)$$

$$C=\frac{1+d[\frac{1}{2}(\pi/P)^{\frac{1}{2}}-I(0,P)]}{1+\varepsilon\,.\,\frac{1}{2}(\pi/P)^{\frac{1}{2}}},\quad (16)$$

where

$$\varepsilon=\frac{2}{\pi^{\frac{1}{2}}}\cdot\frac{k_2}{k_1}\cdot\left(\frac{K_1}{K_2}\right)^{\frac{1}{2}}\cdot P^{\frac{1}{2}},$$

and

$$I(\xi,P)=\int_{\eta=\xi}^{\infty}\exp(-P\eta^2)\,.\,\operatorname{erfc}\{\eta(2-P)^{\frac{1}{2}}\}\mathrm{d}\eta.\quad (17)$$

We thus have

$$v_1(\xi)=1-A\,.\,\tfrac{1}{2}\,.\,(\pi/P)^{\frac{1}{2}}\operatorname{erfc}(\xi P^{\frac{1}{2}})-d\,.\,I(\xi,P).\quad (18)$$

The wall temperature is given by

$$v_1(0)=\frac{1+d[\frac{1}{2}(\pi/P)^{\frac{1}{2}}-I(0,P)]}{1+\varepsilon\,.\,\frac{1}{2}(\pi/P)^{\frac{1}{2}}},\quad (19)$$

which is a constant.

The integral defining $I(\xi,P)$ cannot be evaluated in terms of elementary functions but the integral for $I(0,P)$ can be so evaluated. We find

$$I(0,P)=\left(\frac{1}{\pi P}\right)^{\frac{1}{2}}\left[\tfrac{1}{2}\pi-\tan^{-1}\left(\frac{2-P}{P}\right)^{\frac{1}{2}}\right],\quad (20)$$

whence

$$v_1(0)=\frac{1+d\,.\,(1/\pi P)^{\frac{1}{2}}\,.\,\tan^{-1}\{(2-P)/P\}^{\frac{1}{2}}}{1+\varepsilon\,.\,\frac{1}{2}(\pi/P)^{\frac{1}{2}}}.\quad (21)$$

On differentiating (18) it can be seen that the flux $Q(t)$ across the boundary $z=0$ is proportional to $t^{-\frac{1}{2}}$.

The parameter ε expresses the interaction between the solid and the fluid. Generally k_2/k_1 is large compared to unity and so ε will be large ($10^3 \sim 10^4$). The large value of k_2/k_1 implies that when a given quantity of heat crosses the

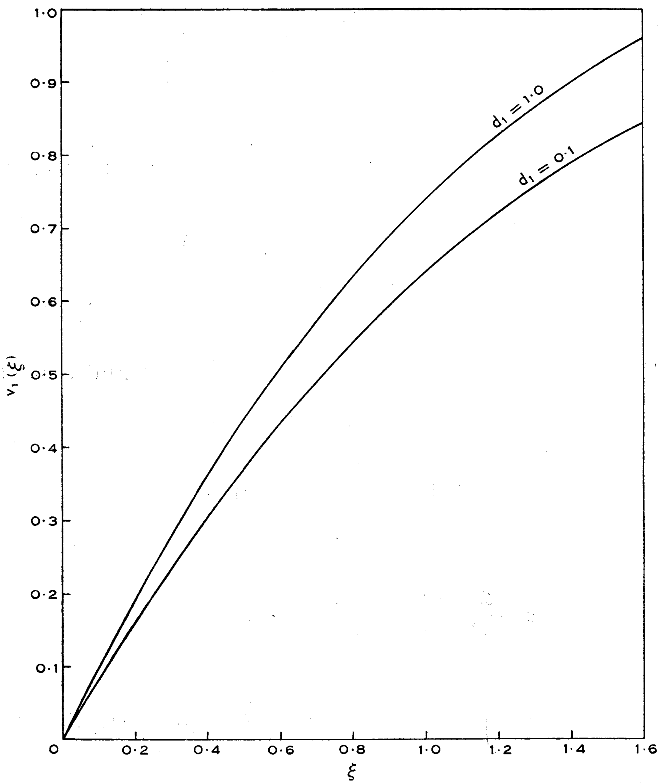

Fig. 1.—Graph of $v_1(\xi)$ as a function of ξ for $P=0\cdot7$.

boundary its effect after a given time is felt in the gas at a greater distance from the boundary than in the solid. The parameter d is usually much less than unity.

Neglecting terms involving ε^{-1} we have for

$$v_1(\xi)=1-[1-d\,.\,I(0,P)]\ \mathrm{erfc}\ (\xi P^{\frac{1}{2}})-d\,.\,I(\xi,P), \tag{22}$$

where $I(\xi, P)$ is given by (17). In Figure 1 we give curves for $v_1(\xi)$ for $P=0\cdot7$, for large ε, and for two values of d.

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