RAY PATHS IN INHOMOGENEOUS ANISOTROPIC MEDIA*

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Summary

The paths of rays in inhomogeneous anisotropic media are studied, starting from Fermat's Principle of Stationary Time. The media considered have a refractive index which depends on the angle between the direction of propagation and an axis of symmetry in the medium, and on several scalar parameters; both these and the direction of the axis are functions of position. Expressions are developed for the curvature of a ray path and are applied to (*a*) radio rays in an ionized medium pervaded by a non-uniform magnetic field and (*b*) acoustical rays in a moving fluid.

I. Introduction

The expression for the curvature of a ray path at a point in an inhomogeneous *isotropic* medium is well known:

$$K=\frac{1}{\mu}\,\frac{\partial\mu}{\partial x_T}, \tag{1}$$

where μ is the refractive index of the medium and x_T is the coordinate in the direction of the projection of $\nabla\mu$ on the plane perpendicular to the ray.‡ This formula, which is useful in ray tracing, applies equally to various types of media (optical, radio, acoustical, etc.) whenever "geometrical optics" is valid as an approximation to the full solution of the equations of propagation; that is, whenever the properties of the medium vary by only a small amount over a distance of the order of a wavelength of the disturbance.

In an *anisotropic* inhomogeneous medium certain complications arise. Firstly, the ray (giving the direction of energy flow) is in general inclined to the wave-normal (direction of phase propagation) and, secondly, the value of the phase refractive index μ depends not only on position in the medium but, at each point, on the direction of the ray relative to one or more axes in the medium; and these axes may themselves change in direction from point to point. In this paper we shall be concerned to develop the appropriate generalization of (1) for anisotropic media. We shall do this mainly in the context of the propagation of radio waves through an ionized medium in the presence of a non-uniform magnetic field. This is an important and fairly general case, but the analysis and many of the results obtained will apply with only minor changes to propagation in other anisotropic media; thus a particularly simple application is provided by the propagation of acoustical rays in an inhomogeneous moving fluid.

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‡ This direction serves also to define the osculating plane of the ray path.

Suppose the *phase* refractive index μ of the medium is given by

$$\mu=\mu(X,Y,\theta), \tag{2}$$

where X, Y are scalar functions of position and θ is the angle between the direction of the *wave-normal* and some direction in space associated with the medium, which may itself be a function of position. That is,

$$\cos\theta=\mathbf{n}\,.\,\mathbf{H}/H, \tag{3}$$

where $\mathbf{n}$ is the unit vector along the wave-normal and $\mathbf{H}$ is some vector function of position. (In the radio case X and Y are the well-known Appleton parameters of magneto-ionic theory, and $\mathbf{H}$ is the magnetic field-vector.) The ray (direction of mean energy flow) is in general inclined to the wave-normal (direction of phase propagation) at an angle given by

$$\tan\alpha=-\frac{1}{\mu}\frac{\partial\mu}{\partial\theta}, \tag{4}$$

angles being taken as positive in the sense of rotation from the direction of $\mathbf{H}$ to the direction of the wave-normal. It is clear by symmetry from (2) and (3) that the directions of the ray, the wave-normal, and $\mathbf{H}$ are co-planar. We shall write

$$\varphi=\alpha+\theta \tag{5}$$

for the angle between the ray and $\mathbf{H}$, and

$$m=\mu\cos\alpha \tag{6}$$

for the *ray* refractive index. It is then easy to show that

$$\tan\alpha=-\frac{1}{m}\frac{\partial m}{\partial\varphi}. \tag{7}$$

II. Ray Paths in Two Dimensions

Initially we shall restrict attention to the problem of propagation in two dimensions, so that the trajectories of the rays lie in one plane : in general this requires that there should be no variation in the properties of the medium perpendicular to this plane, and that $\mathbf{H}$ should lie always in the plane. Here as always we also suppose (*a*) that the variation of X, Y and of the direction of $\mathbf{H}$ is small over any interval of the order of the wavelength considered, namely, that the medium is " slowly varying ", and (*b*) that the radii of curvature of the surfaces of equal phase (eiconels) in the region of propagation are everywhere large compared with the wavelength. These two assumptions assure that the application of geometrical optics is justified.

Let the direction of $\mathbf{H}$ at any point be specified by the angle, ψ say, which it makes with some direction fixed in space. We shall express the curvature K of the ray path at any point in terms of the quantities X, Y, and θ (or φ) and of the *gradients* of X, Y, and ψ. In particular problems some of these six quantities may be constant, but in general they are all functions of position.

The trajectory of the ray between two points A, B, is specified by Fermat's principle of stationary time in the form

$$\delta\int_A^B m\mathrm{d}s=0, \tag{8}$$

where $\mathrm{d}s$ is an element of length along the ray. If (x, y) are Cartesian coordinates this becomes

$$\delta\int_A^B m\sigma\mathrm{d}x=0, \tag{9}$$

where

$$\sigma^2=1+y'^2. \tag{10}$$

Thus by Euler's condition in the calculus of variations, the differential equation of the ray is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial(m\sigma)}{\partial y'}\right]-\frac{\partial(m\sigma)}{\partial y}=0. \tag{11}$$

This equation is intractable except in a few simple cases. It may, however, be used to calculate the curvature of the ray path in even the most general case.

At the point O on the ray take axes Oxy with Ox tangent to the ray path. Then at this point

$$y'=0, \quad \sigma=1, \quad \mathrm{d}\sigma/\mathrm{d}x=y'y''/\sigma=0,$$
$$\partial\sigma/\partial y=\partial\sigma/\partial y'=0,$$

and

$$y''=K,$$

the curvature of the ray path. Writing

$$\beta=\tan^{-1} y',$$

we have

$$\begin{aligned}\frac{\partial m}{\partial y'}&=\frac{\partial m}{\partial \beta}\cdot\frac{\partial\beta}{\partial y'}\\ &=\frac{\partial m}{\partial\varphi}\cdot\frac{1}{1+y'^2}\\ &=-\mu\sin\alpha/\sigma^2,\end{aligned}$$

using (7). Then, writing

$$M=\mu\sin\alpha,$$

(11) becomes

$$K=\frac{1}{m}\left[\left(\frac{\partial m}{\partial y}\right)_{x,y'}+\frac{\mathrm{d}M}{\mathrm{d}x}\right]. \tag{12}$$

Now

$$\begin{aligned}\left(\frac{\partial m}{\partial y}\right)_{y'}&=\left(\frac{\partial m}{\partial y}\right)_{\varphi}+\frac{\partial m}{\partial\varphi}\nabla_T\psi\\ &=\left(\frac{\partial m}{\partial y}\right)_{\varphi}-m\tan\alpha\,\nabla_T\psi,\end{aligned} \tag{13}$$

denoting by $\nabla_T\psi$ the y-component of $\boldsymbol{\nabla}\psi$.

Also

$$\frac{\mathrm{d}M}{\mathrm{d}x}=\left(\frac{\partial M}{\partial x}\right)_{y'}+\frac{\partial M}{\partial \beta}\cdot\frac{\partial \beta}{\partial x}$$
$$=\left(\frac{\partial M}{\partial x}\right)_{\varphi}+\frac{\partial M}{\partial \varphi}\nabla_L\psi+K\frac{\partial M}{\partial \beta},$$

denoting the x-component of $\nabla\psi$ by $\nabla_L\psi$. Now

$$\frac{\partial M}{\partial \beta}=\frac{\partial M}{\partial \varphi}=\frac{\partial}{\partial \varphi}(m\tan\alpha)$$
$$=m\left(\sec^2\alpha\frac{\partial \alpha}{\partial \varphi}-\tan^2\alpha\right),$$

using (7). Hence

$$K=\left(1-\frac{\partial \alpha}{\partial \varphi}\right)^{-1}\frac{\cos\alpha}{\mu}\left[\left(\frac{\partial m}{\partial y}\right)_{\varphi}+\left(\frac{\partial M}{\partial x}\right)_{\varphi}+\frac{\partial M}{\partial \varphi}\nabla_L\psi-M\nabla_T\psi\right]. \tag{14}$$

Also

$$\left(\frac{\partial}{\partial x}\right)_{\varphi}\equiv\left(\frac{\partial}{\partial X}\right)_{\varphi,Y}\nabla_L X+\left(\frac{\partial}{\partial Y}\right)_{\varphi,X}\nabla_L Y,$$
$$\left(\frac{\partial}{\partial y}\right)_{\varphi}\equiv\left(\frac{\partial}{\partial X}\right)_{\varphi,Y}\nabla_T X+\left(\frac{\partial}{\partial Y}\right)_{\varphi,X}\nabla_T Y, \tag{15}$$

and, employing a well-known transformation (Goursat 1904, p. 76),

$$\left(\frac{\partial \mu}{\partial X}\right)_{\varphi}=\left(\frac{\partial \mu}{\partial X}\right)_{\theta}+\left(\frac{\partial \mu}{\partial \varphi}\right)_{X}\left(\frac{\partial \theta}{\partial X}\right)_{\varphi}\bigg/\frac{\partial \theta}{\partial \varphi}$$
$$=\left(\frac{\partial \mu}{\partial X}\right)_{\theta}+\mu\tan\alpha\frac{\partial \alpha}{\partial X}\bigg/\frac{\partial \theta}{\partial \varphi}, \tag{16}$$

using (5) and (7). A similar expression holds for $(\partial\mu/\partial Y)_{\varphi}$. Hence we obtain for the curvature

$$K=\sum_{X,Y,\psi}\{Z_L^X\nabla_L X+Z_T^X\nabla_T X\}, \tag{17}$$

where

$$Z_L^X=\left(1+\frac{\partial \alpha}{\partial \theta}\right)\frac{\sin 2\alpha}{2\mu}\left(\frac{\partial \mu}{\partial X}\right)_{\theta}+\left(\frac{\partial \alpha}{\partial X}\right)_{\theta},$$
$$Z_T^X=\left(1+\frac{\partial \alpha}{\partial \theta}\right)\frac{\cos^2\alpha}{\mu}\left(\frac{\partial \mu}{\partial X}\right)_{\theta},$$
$$Z_L^Y=\left(1+\frac{\partial \alpha}{\partial \theta}\right)\frac{\sin 2\alpha}{2\mu}\left(\frac{\partial \mu}{\partial Y}\right)_{\theta}+\left(\frac{\partial \alpha}{\partial Y}\right)_{\theta},$$
$$Z_T^Y=\left(1+\frac{\partial \alpha}{\partial \theta}\right)\frac{\cos^2\alpha}{\mu}\left(\frac{\partial \mu}{\partial Y}\right)_{\theta},$$
$$Z_L^\psi=\left(1+\frac{\partial \alpha}{\partial \theta}\right)\cos^2\alpha-1,$$
$$Z_T^\psi=-\tfrac{1}{2}\left(1+\frac{\partial \alpha}{\partial \theta}\right)\sin 2\alpha. \tag{18}$$

Here μ and α are given as function of X, Y, and θ by (2) and (4) respectively, and $\nabla_L X$, $\nabla_T X$, etc. denote the components of $\boldsymbol{\nabla} X$ etc. respectively along and transverse to the direction of the ray.*

III. Ray Path in Three Dimensions

So far we have been considering the case when the ray paths are confined to a plane. This requires that there should be no variation of the properties of the medium, and no component† of the vector **H**, perpendicular to the plane. In general this condition will not be satisfied and the trajectories of the rays will be twisted curves in three dimensions.

These trajectories are again determined by Fermat's principle (8), from which the equations of the ray are easily found:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial(m\sigma_1)}{\partial y'}\right]-\frac{\partial(m\sigma_1)}{\partial y}=0,$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial(m\sigma_1)}{\partial z'}\right]-\frac{\partial(m\sigma_1)}{\partial z}=0, \tag{19}$$

where $\sigma_1^2=1+y'^2+z'^2$.

These equations will serve to derive an expression for the curvature of the ray analogous to (17). There is, however, an initial difficulty: the gradient of the angle, ψ say, between a fixed direction and the direction of the vector **H**, is no longer, in the three-dimensional case, independent of the fixed direction itself. That is, the vector $\nabla\psi$ in (17) etc. is no longer determined by the vector field **H** alone. Suppose that (ν_1, ν_2, ν_3) are the direction cosines of the fixed direction, and define the tensor

$$T_{ij}=\frac{1}{H}\frac{\partial H_i}{\partial x_j},\quad (i, j=1, 2, 3). \tag{20}$$

Then, with the usual summation convention,

$$H\cos\psi=\nu_i H_i, \tag{21}$$

so that

$$\frac{\partial H}{\partial x_j}\cos\psi-H\sin\psi\frac{\partial\psi}{\partial x_j}=\nu_i\frac{\partial H_i}{\partial x_j}.$$

Now

$$H^2=H_i^2\equiv H_1^2+H_2^2+H_3^2.$$

Hence

$$2H\frac{\partial H}{\partial x_j}=2H_i\frac{\partial H_i}{\partial x_j},$$

and we obtain

$$\nabla_j\psi=\frac{\partial\psi}{\partial x_j}=\left(\frac{\cos\psi}{H}H_i-\operatorname{cosec}\psi\,\nu_i\right)T_{ij}. \tag{22}$$

* Some attention must be paid to the convention of signs. It is perhaps best always to take angles as positive in the sense of rotation from the direction of **H** to the direction of the ray, even if this should involve left-hand axes.

† The case where **H** is everywhere perpendicular to the plane of propagation is excluded as trivial, since the medium is then essentially isotropic for propagation in this plane.

In our applications the relevant direction (ν_1, ν_2, ν_3) will, of course, be that of the ray.

Now take right-hand coordinates (x_L, x_T, x_P) with x_L along the ray and x_T in the plane determined by the directions of the ray and **H**. Let K_1, K_2 be the curvatures of the projections of the ray trajectory on the $x_L x_T$ and the $x_L x_P$ planes respectively. Then, by analysis similar to that used to obtain (17), we find

$$K_1 = \sum_{X,Y,\psi} \{Z_L^X \nabla_L X + Z_T^X \nabla_T X\}, \tag{23}$$

$$K_2 = \frac{\sin\varphi \cos\alpha}{\mu \sin\theta} \left\{ \left(\frac{\partial\mu}{\partial X}\right)_\theta \nabla_P X + \left(\frac{\partial\mu}{\partial Y}\right)_\theta \nabla_P Y \right\} + \frac{\sin\alpha}{\sin\theta} T_{PL}, \tag{24}$$

where the Z's are those defined under (18). Thus (23) is the same expression as (17).

The curvature K of the ray path is given by

$$K^2 = K_1^2 + K_2^2, \tag{25}$$

and the angle, Ω say, between the osculating plane of the ray path and the $x_L x_T$ plane (namely, the plane containing the directions of the ray and the field) is given by

$$\tan\Omega = K_2/K_1. \tag{26}$$

IV. Radio Ray Tracing

In applying the curvature analysis to the propagation of radio waves through an ionized medium in the presence of a magnetic field, X and Y are interpreted as the Appleton parameters usual in magneto-ionic theory:

$$X = Ne^2/\varepsilon_0 m p^2, \quad Y = eB_0/mp, \tag{27}$$

where N is the electron density, p is the wave frequency, B_0 is the induction of the magnetic field, e and m are the charge and mass of the electron, and ε_0 is the permittivity of free space. The phase refractive index neglecting collisions is then given by the Appleton-Hartree expression

$$\mu^2 = 1 - \frac{2X(1-X)}{2(1-X) - Y^2\sin^2\theta \pm [Y^4 \sin^4\theta + 4Y^2(1-X)^2\cos^2\theta]^{\frac{1}{2}}}, \tag{28}$$

where θ represents the angle between the wave-normal and the magnetic field.

When **H** represents the magnetic field vector and Y has the meaning given above, the expression for the curvature of a *plane* ray path may be simplified by using the field equations to express $\nabla\psi$ in terms of ∇Y. In fact, in a region where there are no macroscopic currents, these two vectors are always perpendicular, as we now show.

Suppose that $(\cos\omega, \sin\omega)$ is some fixed direction and ψ represents the angle between this direction and **H**. Then

$$H\cos\psi = H_1\cos\omega + H_2\sin\omega,$$

whence

$$\nabla_1\psi = \frac{\partial\psi}{\partial x_1} = \frac{1}{H\sin\psi}\left\{\frac{\partial H}{\partial x_1}\cos\psi - \frac{\partial H_1}{\partial x_1}\cos\omega - \frac{\partial H_2}{\partial x_1}\sin\omega\right\}.$$

Also, since $H^2=H_1^2+H_2^2$,

$$H\frac{\partial H}{\partial x_1}=H_1\frac{\partial H_1}{\partial x_1}+H_2\frac{\partial H_2}{\partial x_1}.$$

Now choose axes Ox_1x_2 so that Ox_1 lies along the direction of the field. In this coordinate system,

$$H_1=H, \quad H_2=0, \quad \omega=\psi$$

and we find

$$\nabla_1\psi=-\frac{1}{H}\frac{\partial H_2}{\partial x_1},$$

and similarly

$$\left.\begin{aligned}\nabla_1\psi&=-\frac{1}{H}\frac{\partial H_2}{\partial x_1},\\ \nabla_2\psi&=-\frac{1}{H}\frac{\partial H_2}{\partial x_2}.\end{aligned}\right\}\tag{29}$$

Also, since $Y=(e\mu_0/mp)H$, we have

$$\left.\begin{aligned}\nabla_1Y&=(e\mu_0/mp)\frac{\partial H}{\partial x_1}=(e\mu_0/mp)\frac{\partial H_1}{\partial x_1},\\ \nabla_2Y&=(e\mu_0/mp)\frac{\partial H}{\partial x_2}=(e\mu_0/mp)\frac{\partial H}{\partial x_2}.\end{aligned}\right\}\tag{30}$$

For a region in which no macroscopic currents flow, the field equations give

$$\begin{aligned}\partial H_2/\partial x_1-\partial H_1/\partial x_2&=0,\\ \partial H_1/\partial x_1+\partial H_2/\partial x_2&=0,\end{aligned}$$

and hence with (29) and (30)

$$\left.\begin{aligned}\nabla_1\psi&=-\frac{1}{Y}\nabla_2Y,\\ \nabla_2\psi&=\frac{1}{Y}\nabla_1Y,\end{aligned}\right\}\tag{31}$$

so that $\boldsymbol{\nabla}Y$ is perpendicular to $\boldsymbol{\nabla}\psi$ and Y times its magnitude. Thus, in particular

$$\nabla_L\psi=-\frac{1}{Y}\nabla_TY=-\nabla_T\log Y,$$

$$\nabla_T\psi=\frac{1}{Y}\nabla_LY=\nabla_L\log Y.\tag{32}$$

This enables the curvature to be expressed entirely in terms of $\boldsymbol{\nabla}X$ and $\boldsymbol{\nabla}Y$, for using (17)

$$K=Z_L^X\nabla_LX+Z_T^X\nabla_TX+\zeta_L^Y\nabla_LY+\zeta_T^Y\nabla_TY,\tag{33}$$

where

$$\left.\begin{aligned}\zeta_L^Y&=Z_L^Y+\frac{1}{Y}Z_T^\psi,\\ \zeta_T^Y&=Z_T^Y-\frac{1}{Y}Z_L^\psi.\end{aligned}\right\}\tag{34}$$

(a) The Quasi-longitudinal and Quasi-transverse Approximations

Explicit expressions for the Z coefficients obtained by substituting the value (28) for μ in (4) and (18) are naturally complex. The well-known quasi-longitudinal and quasi-transverse approximations for the refractive index may, however, be employed in many cases to give a good idea of the behaviour of the ray path.

The quasi-longitudinal approximation, valid when

$$\left|\frac{Y\sin^2\theta}{2(1-X)\cos\theta}\right| \ll 1, \tag{35}$$

is

$$\mu^2 = 1 - \frac{X}{1 \pm Y\cos\theta}, \tag{36}$$

the upper (lower) sign referring to the ordinary (extraordinary) mode with $X<1 (X<1-Y)$.

We obtain

$$\left.\begin{aligned} Z_L^X &= \pm arY_T(2a-qX), \\ Z_T^X &= -2a^2brq, \\ Z_L^Y &= rX\sin\theta\{(qX-2a-2b)Y_L \pm 2ab\}, \\ Z_T^Y &= \pm 2abrqX\cos\theta, \\ Z_L^\psi &= 4a^2b^2rq-1, \\ Z_T^\psi &= \mp 2abrqXY_T, \end{aligned}\right\} \tag{37}$$

where

$$\begin{aligned} a &= 1 \pm Y_L, \\ b &= 1 - X \pm Y_L, \\ r &= (4a^2b^2 + X^2Y_T^2)^{-1}, \\ q &= 1 + 2rX\{(a+b)Y_T^2 \pm abY_L\}, \end{aligned}$$

and Y_L, Y_T denote respectively $Y\cos\theta$ and $Y\sin\theta$.

The quasi-transverse approximation is valid when

$$\left|\frac{Y\sin^2\theta}{2(1-X)\cos\theta}\right| \gg 1. \tag{38}$$

For the ordinary mode the correct expression for the refractive index is (Whitehead 1952)

$$\mu^2 = (1-X)/(1-X\cos^2\theta), \tag{39}$$

whence we obtain

$$\left.\begin{aligned} Z_L^X &= \eta\sin 2\theta[\zeta X\sin^2\theta/(X-1)+2], \\ Z_T^X &= 2\xi\eta\sin^2\theta/(X-1), \\ Z_L^Y &= Z_T^Y = 0, \\ Z_L^\psi &= 4\xi^2\eta\zeta - 1, \\ Z_T^\psi &= -2\xi\eta\zeta X\sin 2\theta, \end{aligned}\right\} \tag{40}$$

where

$$\begin{aligned} \xi &= 1 - X\cos^2\theta, \\ \eta &= (4\xi^2 + X^2\sin^2 2\theta)^{-1}, \\ \zeta &= 1 + 2\eta(2\xi X\cos 2\theta - X^2\sin^2 2\theta). \end{aligned}$$

It is notable that in this approximation the gradient of the field-strength has no effect on the ray path, though the curvature of the lines of force does exert an influence.

For the extraordinary mode, the quasi-transverse approximation gives

$$\mu^2=1-X(1-X)/(1-X-Y^2\sin^2\theta), \tag{41}$$

and this, too, can be used to calculate the coefficients of curvature; but the resulting expressions, which are somewhat complex, will be omitted here.

V. Acoustical Ray Tracing

A moving fluid, such as a windy atmosphere, is essentially anisotropic for the propagation of sound. The phase refractive index μ is given by

$$\mu=1/(c+w\cos\theta), \tag{42}$$

where c is the speed of sound in the fluid at rest, w the speed of the fluid flow or wind, and θ the angle between the direction of the wind and the wave-normal. The angle between the wave-normal and the ray is then, by (4),

$$\tan\alpha=-\mu w\sin\theta. \tag{43}$$

Hence, in terms of the angle $\varphi=\theta+\alpha$ between the wind and the ray,

$$\sin\alpha=-(w/c)\sin\varphi. \tag{44}$$

The expressions (42) and (43) may be used to calculate the coefficients of curvature (18). Here we shall treat the special case where the velocity w of the fluid is small compared with the velocity c, so that

$$W\equiv w/c\ll 1. \tag{45}$$

With this condition we have approximately

$$\left.\begin{aligned}
Z_L^c&=\frac{2}{c}W\sin\varphi,\\
Z_T^c&=\frac{1}{c}(2W\cos\varphi-1),\\
Z_L^w&=\frac{\sin\varphi}{c}(2W\cos\varphi-\sin\varphi),\\
Z_T^w&=\frac{1}{c}[(1+\cos^2\varphi)W-\cos\varphi],\\
Z_L^\psi&=-W\cos\varphi,\\
Z_T^\psi&=W\sin\varphi.
\end{aligned}\right\} \tag{46}$$

The component of $\nabla\psi$ in the direction of the wind is, at any point, the curvature, K_w say, of the streamline of the wind through that point. Hence

$$\begin{aligned}
Z_L^\psi\nabla_L\psi+Z_T^\psi\nabla_T\psi&=-W(\cos\varphi\nabla_L\psi-\sin\varphi\nabla_T\psi)\\
&=-WK_w.
\end{aligned} \tag{47}$$

It will usually be the case that when w/c is small $|\nabla w|/c$ is small also, so that, to the first order of small quantities, the curvature of a plane sound-ray in a light wind is

$$K=K_0+\frac{2W}{c}(\sin\varphi\nabla_L c+\cos\varphi\nabla_T c)-WK_w-\frac{1}{c}\cos\varphi\nabla_T w, \tag{48}$$

where K_0 is the value of the ray curvature for $w=0$, namely,

$$K_0=-(1/c)\nabla_T c. \tag{49}$$

If $\nabla c\equiv 0$ so that the velocity of the sound relative to still air is constant, we have

$$K=-(1/c)(WK_w+\cos\varphi\partial w/\partial x_T), \tag{50}$$

exhibiting the bending of the ray path by the wind alone.

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