SOURCE FUNCTIONS FOR DIFFUSION IN UNIFORM SHEAR FLOW

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Summary

The problem of molecular diffusion in a fluid simultaneously undergoing shear flow is analysed for the simple case of uniform shear. Solutions for instantaneous point and line sources are obtained using Fourier transform methods and the results are presented in some detail. The solutions also describe the diffusion of heat from an instantaneously heated source in a fluid undergoing a similar shear pattern.

I. INTRODUCTION

A study of the transport of matter or heat under simultaneous conditions of diffusion (either laminar or turbulent) and convection is presented. Fortunately, the basic analysis is the same whether the problem is one of dispersion during flow through a porous medium, dispersion in a turbulent flow field, or heat dispersion in laminar or turbulent flow fields.

The problem of steady-state heat transfer in a liquid undergoing laminar flow between parallel plates has been examined by Prins, Mulder, and Schenk (1951) and Dennis and Poots (1956). The dynamic case of diffusion into a turbulent atmosphere of an instantaneous point source has been analysed by Davies (1954) but under conditions of constant horizontal convective velocity. The diffusion of a heated spot in a uniform shear field (different in form from the present analysis) has been studied by Townsend (1951). Novikov (1958) studied the problem of turbulent diffusion in a shear field having a transverse gradient of velocity.

The movement of nutrients in the soil solution to the plant root surface is a combined process of mass flow (convection) with the soil water and diffusion. A recent article by Barber (1962) gives an excellent descriptive outline of the various mechanisms involved. It is not suggested that this solution can be used at present to describe the movement of nutrients in porous media as the boundary conditions governing the microscopic flow are indescribably complex. However, this mathematical approach is a first step in the quantitative study of this problem.

This analysis is restricted to the simple convective case of uniform shear flow. Some of the mathematical techniques employed by Novikov are similar to those developed independently for this study and, where possible, the similarities and differences are indicated.

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II. Fundamental Equations

The general equation describing simultaneous diffusion and convection in a fluid may be written as

$$\partial\theta^*/\partial t^* + \mathbf{V}^* \cdot \nabla^*\theta^* = D\nabla^{*2}\theta^*, \tag{1}$$

where θ^* is the concentration of diffusing substance, D is the (constant) diffusion coefficient, and $\mathbf{V}^*$ is the velocity which, in general, could be a function of the Cartesian coordinates x^*, y^*, z^*, and the time t^*.

In this analysis the velocity is assumed to have only an x^*-component which is given by

$$V^* = ky^*, \tag{2}$$

where k is a positive constant.

Assuming uniform shear flow, (1) may be rewritten as

$$\partial\theta^*/\partial t^* + ky^*\partial\theta^*/\partial x^* = D\nabla^{*2}\theta^*. \tag{3}$$

It is convenient to write (3) in reduced form by means of the following transformation:

$$x = (k/D)^{\frac{1}{2}}x^* ; \quad y = (k/D)^{\frac{1}{2}}y^* ; \quad z = (k/D)^{\frac{1}{2}}z^*, \tag{4}$$

$$t = kt^*, \tag{5}$$

$$\theta = (k/D)^{-3/2}\theta^*. \tag{6}$$

Equation (3) may now be expressed as

$$\partial\theta/\partial t + y\partial\theta/\partial x = \nabla^2\theta, \tag{7}$$

where

$$V \equiv V^*/(kD)^{\frac{1}{2}} = y. \tag{8}$$

III. Solution

We shall consider the problem of diffusion in a fluid undergoing uniform shear flow in a triply infinite medium when the initial concentration is given by

$$\theta = f(x,y,z), \qquad t = 0. \tag{9}$$

The following transformation on the independent variables is similar to that suggested by Novikov and allows us to eliminate the convective term from (7):

$$u = x - yt ; \quad v = y ; \quad w = z ; \tag{10}$$

$$s = t. \tag{11}$$

On substitution, (7) becomes

$$(1+s^2)\partial^2\theta/\partial u^2 - 2s\partial^2\theta/\partial u\partial v + \partial^2\theta/\partial v^2 + \partial^2\theta/\partial w^2 = \partial\theta/\partial s, \tag{12}$$

and the initial condition (9) is now given by

$$f(x,y,z) = f(u,v,w). \tag{13}$$

The triple Fourier transform was found convenient in solving (12) subject to (13). Novikov solved a similar equation by applying a two-sided Laplace transform to the variables u, v, and w.

The notation used for the triple Fourier transform is defined as

$$\Theta(\xi,\eta,\zeta)=(2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta(u,v,w)\exp\left[\mathrm{i}(\xi u+\eta v+\zeta w)\right]\mathrm{d}u\mathrm{d}v\mathrm{d}w. \qquad (14)$$

On multiplying both sides of (12) by $\exp\left[\mathrm{i}(\xi u+\eta v+\zeta w)\right]$ and integrating with respect to u, v, and w from $-\infty$ to ∞, we find that the transform Θ satisfies the first-order ordinary differential equation

$$\mathrm{d}\Theta/\mathrm{d}s=-(\xi^2+\eta^2+\zeta^2)\Theta+2\xi\eta s\Theta-\xi^2s^2\Theta, \qquad (15)$$

subject to the initial condition that

$$\Theta=F(\xi,\eta,\zeta), \quad \text{when } s=0. \qquad (16)$$

The function $F(\xi,\eta,\zeta)$ is therefore the transform of $f(u,v,w)$.

On solution of (15) subject to (16) we have

$$\Theta=F(\xi,\eta,\zeta)\exp\left[-(\xi^2+\eta^2+\zeta^2)s+\xi\eta s^2-\tfrac{1}{3}\xi^2s^3\right]. \qquad (17)$$

The function $G(\xi,\eta,\zeta)$ is defined as follows:

$$G(\xi,\eta,\zeta)\equiv\exp\left[-(\xi^2+\eta^2+\zeta^2)s+\xi\eta s^2-\tfrac{1}{3}\xi^2s^3\right]. \qquad (18)$$

Now the Fourier transform of $G(\xi,\eta,\zeta)$ is given by

$$g(u,v,w)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\xi,\eta,\xi)\exp\left[-\mathrm{i}(\xi u+\eta v+\zeta w)\right]\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta. \qquad (19)$$

Substituting (18) into (19) and performing the integrations gives

$$g(u,v,w)=\sqrt{3}/[4(\pi s)^{3/2}(s^2+12)^{\frac{1}{2}}]\cdot\{\exp-[3(u+\tfrac{1}{2}sv)^2/s(s^2+12)+v^2/4s+w^2/4s]\}. \qquad (20)$$

Equation (17) may now be rewritten as

$$\Theta=F(\xi,\eta,\zeta)G(\xi,\eta,\zeta). \qquad (21)$$

The solution $\theta(u,v,w,s)$ is then given by

$$\theta=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi,\eta,\zeta)G(\xi,\eta,\zeta)\exp\left[-(\xi u+\eta v+\zeta w)\right]\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta. \qquad (22)$$

The Faltung theorem for Fourier transforms extended to three variables (Sneddon 1951) states that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&F(\xi,\eta,\zeta)G(\xi,\eta,\zeta)\exp\left[-\mathrm{i}(\xi u+\eta v+\zeta w)\right]\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u',v',w')g(u-u',v-v',w-w')\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'. \qquad (23)\end{aligned}$$

Utilizing this theorem gives

$$\begin{aligned}\theta(u,v,w,s)=\sqrt{3}/[4(\pi s)^{3/2}(s^2+12)^{\frac{1}{2}}]&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u',v',w')\\ &\cdot\Big\{\exp-\{3[(u-u')+\tfrac{1}{2}s(v-v')]^2/s(s^2+12)+(v-v')^2/4s\\ &+(w-w')^2/4s\}\Big\}\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'. \qquad (24)\end{aligned}$$

Transforming back into the reduced variables gives

$$\theta(x,y,z,t)=\sqrt{3}/[4(\pi t)^{3/2}(t^2+12)^{\frac{1}{2}}]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y',z')$$
$$\cdot\{\exp -\{3[(x-x')-\tfrac{1}{2}t(y+y')]^2/t(t^2+12)+(y-y')^2/4t$$
$$+(z-z')^2/4t\}\}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z', \qquad (25)$$

where the variables of integration have been changed from u',v',w' to x',y',z'.

IV. Instantaneous Point Source

The source function for an instantaneous point source is then given by

$$\theta(x,y,z,t)=\sqrt{3}M/[4(\pi t)^{3/2}(t^2+12)^{\frac{1}{2}}]$$
$$\cdot\{\exp -\{3[(x-x')-\tfrac{1}{2}t(y+y')]^2/t(t^2+12)+(y-y')^2/4t+(z+z')^2/4t\}\}, \qquad (26)$$

where M is the amount of diffusing substance. This is consistent with the solution obtained by Novikov.

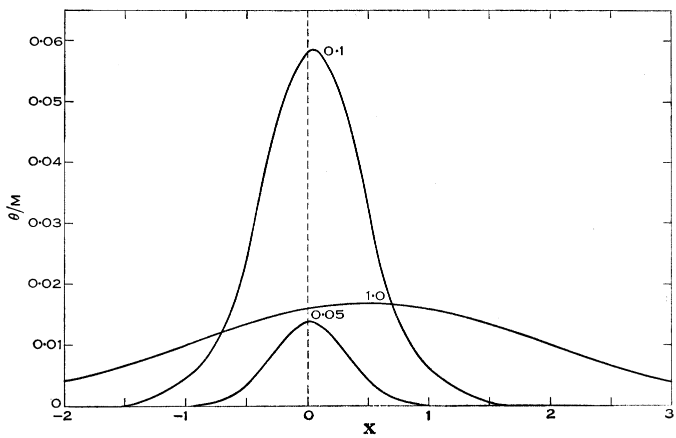

Fig. 1.—Concentration–distance (x-direction) curves for an instantaneous point source located at the origin with y and z held constant at the points 1 and 0 respectively. Numbers on curves denote values of t.

As $t\to 0$ the expression tends to zero at all points except (x',y',z'), where it becomes infinite. The expression also tends to zero as either x, y, or z approach infinity positively or negatively for $t>0$. Equation (26) is symmetrical with respect to the plane parallel to the XY-plane and passing through the point (x',y',z') and if (x',y',z') is at the origin, the expression is then symmetrical about the origin as well.

Also, the total amount of diffusing substance in the infinite region is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta(x,y,z,t)\mathrm{d}x\mathrm{d}y\mathrm{d}z=M. \qquad (27)$$

Thus (26) describes the spreading by diffusion in an infinite medium undergoing uniform shear flow of the form $V=y$ of an amount of substance M deposited at time $t=0$ at a point (x',y',z').

Figure 1 shows typical distributions of θ/M as a function of x with y and z held constant at the points 1 and 0 respectively. The point source is taken to be at the origin. The shift in peak concentration in the positive x-direction is obvious.

The surfaces of constant concentration are diffusion ellipsoids whose size, eccentricity, and direction are time dependent. For ease of presentation, similar data are presented for the line source in the two-dimensional case.

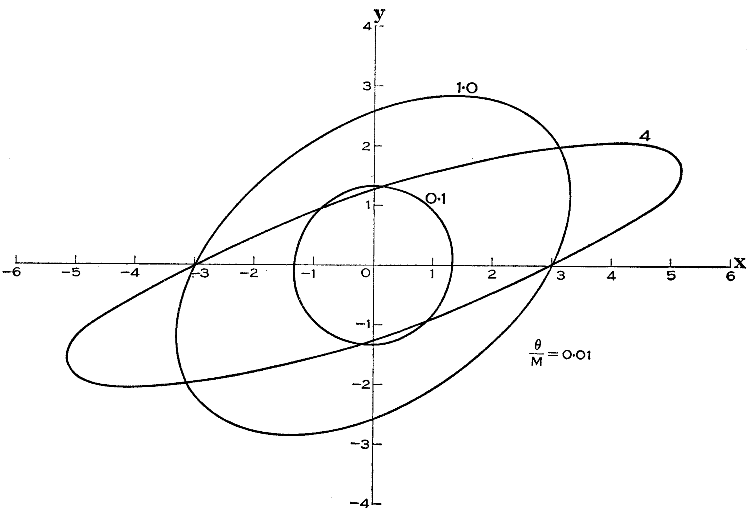

Fig. 2.—Diffusion ellipses of constant concentration where the ratio θ/M is arbitrarily fixed at $0 \cdot 01$. Numbers on curves denote values of t.

V. Instantaneous Line Source

The solution for an instantaneous line source parallel to the z-axis and passing through the point (x',y') can be obtained by integrating (26). Consider a distribution of instantaneous point sources of amount $M\mathrm{d}z'$ at z' along the line. The concentration, obtained by multiplying (26) by $\mathrm{d}z'$ and integrating from $-\infty$ to ∞ is

$$\theta(x,y,t)=\sqrt{3}M/[2\pi t(t^2+12)^{\frac{1}{2}}] \cdot\left\{\exp -\{3[(x-x')-\tfrac{1}{2}t(y+y')]^2/t(t^2+12)+(y-y')^2/4t\}\right\}. \tag{28}$$

Figure 2 shows typical diffusion ellipses of constant concentration for three successive times. The ratio θ/M was arbitrarily fixed at $0 \cdot 01$. Like the point source, the size, eccentricity, and direction of the ellipses are time dependent.

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