REMARKS ON THE TRANSPORT PROPERTIES OF A FULLY IONIZED GAS WITH APPLICATION TO A RADIALLY CONSTRICTED GAS DISCHARGE

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Summary

The correction of Vaughan-Williams and Haas to Marshall's λ' set of thermal conductivities for a fully ionized hydrogen isotope in the presence of a magnetic field is extended to the λ and θ (or K) sets, and the high-field limits for the components perpendicular to the magnetic field are included.

The results of Kihara, Midzuno, and Kaneko for the rate of entropy production due to irreversible processes in an anisotropic plasma are then used to obtain the Onsager expressions for the reduced heat flow and the electric field respectively, when the magnetic field is zero. From these Onsager expressions Spitzer's expressions for the absolute heat flow and conduction current density are obtained, with attendant derivation of relations between the scalar transport coefficients referred to earlier by Spitzer, Kaneko, and Seymour.

Comparison of results obtained by Kaneko, Marshall, and Landshoff for the 2nd approximation to the scalar electrical, thermal, and thermoelectric transport coefficients of an isotropic deuterium plasma shows close agreement. The corresponding results of Spitzer and Härm in infinite approximation are also included for purpose of comparison.

Finally, the influence of Marshall's corrected thermal conductivities on the radial heat flow approximation in Seymour's earlier analyses of a radially constricted deuterium plasma between electrodes is considered, and it is shown that the original basic approximation of perfect thermal insulation at the free boundary surface of the plasma can still be made more plausible in the limit of a strong external guiding magnetic field, provided that this magnetic field corresponds to the more stringent inequality $\omega^2\tau^2 \gg 1400$ (ω the electron gyrofrequency, τ the electron collision time) for highly ionized deuterium, rather than the inequality $\omega\tau \gg 1$ which appeared in the earlier analyses.

I. Introduction

During 1961 the author published papers on the estimation of the maximum temperature in a radially constricted deuterium discharge between electrodes (Seymour 1961*a*), the influence of thermoelectric effects on the maximum temperature in such a discharge (Seymour 1961*b*), and the stability of this particular discharge configuration (Seymour 1961*c*). Since a region of ionized gas is made anisotropic by the presence of a magnetic field, it was convenient in the first of these papers to choose as basic equations for the constricted discharge analysis the forms given in Marshall's detailed report (1957) for the current density vector, **j**, and the heat flux vector, **q**, in a fully ionized gas having atomic number, $Z = 1$. The

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expression for $\mathbf{j}$ contains the components σ^{I}, σ^{II}, and σ^{III} of the electrical conductivity tensor, and the components ϕ^{I}, ϕ^{II}, and ϕ^{III} of the thermal diffusion tensor. Likewise, the expression for $\mathbf{q}$ contains the components K^{I}, K^{II}, and K^{III} of the thermal conductivity tensor when electric currents are flowing (θ^{I}, θ^{II}, and θ^{III} in Marshall's notation) and the components ξ^{I}, ξ^{II}, and ξ^{III} of the tensor accounting for the contribution to the heat flux from the electric field $\mathbf{E}$. Marshall's σ, ϕ, K, and ξ transport coefficients are given for an actual gas in terms of a collision time for electrons, τ, which varies as $T^{3/2}$, where T is the temperature. To obtain the values of these transport coefficients, Marshall adopts a successive approximation technique, which he limits to the second approximation because he considers this sufficient.

For an ionized gas not in the presence of a magnetic field Spitzer and Härm (1953) obtain the isotropic forms of $\mathbf{j}$ and $\mathbf{q}$ in terms of the scalar transport coefficients σ, K, α, and β, where α and β are the thermoelectric coefficients. Each of these coefficients for an actual ionized gas is given as the product of the corresponding coefficient for a Lorentz gas and an appropriate correcting transport coefficient. For the transport coefficients along the magnetic field $\mathbf{H}$, Spitzer and Härm's results for $Z = 1$ are equivalent to Marshall's procedure taken to infinite approximation.

When $\mathbf{H}$ is reduced to zero in Marshall's work, the 3×3 matrices corresponding to the tensor transport coefficients reduce to diagonal form, with equal elements on the diagonal. In this isotropic form, this author compared Marshall's results with those of Spitzer and Härm, to obtain the following results:

$$\left.\begin{aligned} K_{\mathrm{M}}/K_{\mathrm{S,H}} &= 0\cdot 939, & \sigma_{\mathrm{M}}/\sigma_{\mathrm{S,H}} &= 0\cdot 979, \\ \xi/\beta &= -1\cdot 01, & \phi/\alpha &= 1\cdot 12. \end{aligned}\right\} \tag{1.1}$$

In view of this close correspondence, and the particular form of Spitzer and Härm's transport coefficients, they were adopted in the author's earlier papers (Seymour 1961*a*, 1961*b*, and 1961*c*), where it transpired that significant results, such as the expression for the maximum temperature in the discharge and that for the voltage between the electrodes, depended upon the scalar, and not the tensor, form of the transport coefficients.

Since publication of the constricted discharge papers, the author's attention was, in October 1961, directed by W. B. Thompson* to a note by Vaughan-Williams and Haas (1961) on an error in the thermal conductivities. Recalling that Marshall (1957, pp. 42, 43) introduces three sets of thermal conductivity coefficients, namely, λ^{I}, λ^{II}, and λ^{III}, the thermal conductivity coefficients; λ'^{I}, λ'^{II}, and λ'^{III}, the "true" thermal conductivity coefficients if thermal diffusion effects were absent; and K^{I}, K^{II}, and K^{III}, already mentioned, it is to be noted that Vaughan-Williams and Haas give corrected forms of λ'^{II} and λ'^{III}, λ'^{I} being obtained as the low-field limit of λ'^{II}.

In this paper we further present the corrected forms of K^{I}, K^{II}, and K^{III}; and for completeness the corrected forms of λ^{I}, λ^{II}, and λ^{III}, obtained by a relatively quick method. Appropriate high-field limits are also given. While, for reasons

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given above, the correction of K^{I} along the magnetic field will not affect the numerical values of the significant results derived in the author's earlier papers, the correction of K^{II} and K^{III} does make it necessary to review the simplifying assumptions which led to neglect of radial heat flow in these earlier analyses. Prior to this review in Section VI, some results obtained from consideration of the thermodynamics of irreversible processes in a plasma are employed to derive a useful relation between α, β, and σ, given by Spitzer (1956), and also an inequality involving these three transport coefficients and K.

II. The Corrected Thermal Conductivity Coefficients

In their paper Vaughan-Williams and Haas mention that Marshall's last collision integral was found to be in error (Marshall 1957, p. 81), and that their correction left Marshall's parameters a_1^0 and a_1^1 unchanged, but that a_2^1 (Marshall 1957, p. 42) was varied from its original form. As a consequence, Vaughan-Williams and Haas give corrected forms of λ'^{II} and λ'^{III} as shown in Table 1, where n is the plasma total number density, k is Boltzmann's constant, m_1 is the mass of the electron, m_2 is the ion mass, $M_1 \approx m_1/m_2$, ω is the electron gyrofrequency, and T and τ have already been defined in Section I. We also give in Table 1 the low-field limit λ'^{I}, gained from λ'^{II} when $\omega\tau \ll 1$. It remains to obtain expressions for the λ and K sets; the σ, ϕ, and ξ sets are left unaltered by the change in a_2^1.

From Marshall (1957, p. 29) we simplify equations (3.87) by taking $(m_2-m_1)/(m_2+m_1) \sim 1$, since $m_2 \gg m_1$, whence use of the corrected λ' set of Table 1 and the ϕ set (Marshall 1957, p. 42) readily yields the corrected K^{I}, K^{II}, and K^{III}, as shown in Table 1.

In an attempt to obtain the λ set from equations (3.80), (3.81), and (3.82) of Marshall's report (1957, p. 28), one encounters extremely tedious algebra. The author discovered that it was simpler to use the results (3.4) derived earlier (Seymour 1961*a*, p. 132). Using the corrected K set, together with the ϕ set and the transport coefficients, ψ^{I}, ψ^{II}, and ψ^{III}, giving the heat flux due to electric currents, equations (3.4) yield λ^{I}, λ^{II}, and λ^{III}. Like the ϕ set, the ψ set (Marshall 1957, p. 44) is not affected by the change in a_2^1. Table 1 is completed by inclusion of the corrected λ set.

III. Low and High-field Limits of the Thermal Conductivity Coefficients

In Table 1, λ'^{I} was obtained indirectly as the low-field limit ($\omega\tau \ll 1$) of λ'^{II}, whereas K^{I} and λ^{I} were obtained directly from expressions derived by Marshall and Seymour respectively. Consistently, the low-field limits of K^{II} and λ^{II} agree with the K^{I} and λ^{I} results obtained in the direct manner mentioned above.

As will be understood from Seymour (1961*a*, pp. 135–6) the high-field limits of the σ and K sets for conditions of extreme anisotropy are of importance, and so in the high-field limit

$$\omega^2\tau^2 \gg 3\cdot 394 M_1^{-1/2} + 0\cdot 32 M_1^{-1} + 9, \tag{3.1}$$

the new values of λ'^{II}, λ'^{III}, λ^{II}, λ^{III}, K^{II}, and K^{III} are given in Table 2. From Tables 1 and 2 it will be seen that the λ^{I}, λ^{II}, and λ^{III} given by Thompson (1961) require revision.

IV. Relations between the Transport Coefficients

The steady non-equilibrium state to be found in an ionized gas when electric current and heat flows are present can be analysed by application of the principles of the thermodynamics of irreversible processes. These principles stem from a theorem due to Onsager (1931), and essentially permit choice of the so-called proper "forces" which give rise to the "fluxes" in the irreversible processes. (Callen 1948, 1961; De Groot 1951; Bosworth 1956; Landau and Lifshitz 1960). In De Groot's notation, with X_i $(i = 1, 2, \ldots, n)$ as the forces and J_i $(i = 1, 2, \ldots, n)$ as the fluxes, Onsager's linear relations for small departures from system equilibrium are

Table 1

Components of the thermal conductivity tensors

$$\lambda'^{\mathrm{I}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{1\cdot035+\frac{3M_1+0\cdot566M_1^{1/2}}{9M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$\lambda'^{\mathrm{II}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{\frac{1\cdot866\omega^2\tau^2+0\cdot966}{\omega^4\tau^4+6\cdot282\omega^2\tau^2+0\cdot933}+\frac{3M_1+0\cdot566M_1^{1/2}}{(\omega^2\tau^2+9)M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$\lambda'^{\mathrm{III}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{\frac{-\omega\tau(\omega^2\tau^2+1\cdot9)}{\omega^4\tau^4+6\cdot282\omega^2\tau^2+0\cdot933}+\frac{\omega\tau M_1}{(\omega^2\tau^2+9)M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$\lambda^{\mathrm{I}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{0\cdot536+\frac{3M_1+0\cdot566M_1^{1/2}}{9M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$\lambda^{\mathrm{II}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{\frac{1\cdot866}{\omega^2\tau^2+3\cdot48}+\frac{3M_1+0\cdot566M_1^{1/2}}{(\omega^2\tau^2+9)M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$\lambda^{\mathrm{III}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{\frac{-\omega\tau}{\omega^2\tau^2+3\cdot48}+\frac{\omega\tau M_1}{(\omega^2\tau^2+9)M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$K^{\mathrm{I}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{2\cdot588+\frac{3M_1+0\cdot566M_1^{1/2}}{9M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$K^{\mathrm{II}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{\frac{0\cdot366\omega^2\tau^2+2\cdot415}{\omega^4\tau^4+6\cdot282\omega^2\tau^2+0\cdot933}+\frac{3M_1+0\cdot566M_1^{1/2}}{(\omega^2\tau^2+9)M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$K^{\mathrm{III}} = \tfrac{5}{4}\frac{nk^2T\tau}{m_1}\left\{\frac{-\omega\tau(\omega^2\tau^2+6\cdot199)}{\omega^4\tau^4+6\cdot282\omega^2\tau^2+0\cdot933}+\frac{\omega\tau M_1}{(\omega^2\tau^2+9)M_1+3\cdot394M_1^{1/2}+0\cdot32}\right\}$$

$$J_i = \sum_{k=1}^{n} L_{ik}X_k, \tag{4.1}$$

where the phenomenological coefficients L_{ik} $(i,k = 1, 2, \ldots, n)$ give, for example, the electrical and thermal conductivities when $i = k$, and the kinetic coefficients coupling the various flows when $i \neq k$. Onsager's reciprocity theorem then advances that for properly chosen X_i and J_i

$$L_{ik} = L_{ki} \qquad \text{for } \mathbf{H} = 0, \tag{4.2}$$

where $i, k = 1, 2, \ldots, n$.

TABLE 2

COMPONENTS IN THE HIGH-FIELD LIMIT $\omega^2\tau^2 \gg 3 \cdot 394 M_1^{-1/2} + 0 \cdot 32 M_1^{-1} + 9$

$\lambda'^{\mathrm{II}} \approx \frac{5}{4} \frac{nk^2T\tau}{m_1}\left\{\frac{4 \cdot 866 + 0 \cdot 566 M_1^{-1/2}}{\omega^2\tau^2}\right\} \sim 0$
$\lambda'^{\mathrm{III}} \approx 0$
$\lambda^{\mathrm{II}} \approx \frac{5}{4} \frac{nk^2T\tau}{m_1}\left\{\frac{4 \cdot 866 + 0 \cdot 566 M_1^{-1/2}}{\omega^2\tau^2}\right\} \sim 0$
$\lambda^{\mathrm{III}} \approx 0$
$K^{\mathrm{II}} \approx \frac{5}{4} \frac{nk^2T\tau}{m_1}\left\{\frac{3 \cdot 366 + 0 \cdot 566 M_1^{-1/2}}{\omega^2\tau^2}\right\} \sim 0$
$K^{\mathrm{III}} \approx 0$

The selection of the proper forces and fluxes can be readily made by means of the expression for the rate of entropy production. For a plasma close to thermal equilibrium the rate of entropy production due to irreversible processes has been obtained by Kihara (1959) and Kihara, Midzuno, and Kaneko (1960) as

$$T\left(\frac{\partial s}{\partial t}\right)_{\mathrm{irr}} = -\mathbf{Q} \cdot \operatorname{grad} \ln T + \sum_j n_j \overline{\mathbf{V}}_j \cdot \{\mathbf{F}_j - (\operatorname{grad} \mu_j)_T\}, \tag{4.3}$$

where
$$\mathbf{F}_j = e_j\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_0 \times \mathbf{H}\right) + m_j\left\{\mathbf{g} - \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \operatorname{grad}\right)\mathbf{v}_0\right\},$$

$(\operatorname{grad} \mu_j)_T = \operatorname{grad} \mu_j + s_j T \operatorname{grad} \ln T$,

T is temperature,

s is the entropy per unit volume of the plasma,

s_j is the entropy per particle,

t is time,

$\mathbf{Q}$ is the reduced heat flux vector,

n_j is the number density of the jth component,

$m_j n_j$ is the mass density of the jth component,

$e_j n_j$ is the charge density of the jth component,

$\overline{\mathbf{V}}_j$ is the flow velocity of the jth component,

$\mathbf{v}_0 = \sum_j n_j m_j \overline{\mathbf{v}_j} / \sum_j m_j n_j$ is the mean mass velocity,

$\overline{\mathbf{V}}_j = \overline{\mathbf{v}}_j - \mathbf{v}_0$ is the mean peculiar velocity of the jth component,

μ_j is the chemical potential per particle,

$\mathbf{g}$ is the gravitational acceleration,

$\mathbf{E}$ is the electric field.

In cases where we can neglect gravitation and inertia, and mutual diffusion we have respectively

$$\left.\begin{aligned} \mathbf{g} - \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right)\mathbf{v}_0 &= 0, \\ \text{and} \qquad \sum_j n_j \overline{\mathbf{V}}_j \cdot (\text{grad } \mu_j)_T &= 0 \ . \end{aligned}\right\} \tag{4.4}$$

Then, for $\mathbf{H} = 0$, the result (4.3) assumes the simple form

$$T\left(\frac{\partial s}{\partial t}\right)_{\text{irr}} = -\mathbf{Q} \cdot \text{grad ln } T + \mathbf{j} \cdot \mathbf{E}, \tag{4.5}$$

where $\mathbf{j} = \sum_j n_j e_j \overline{\mathbf{V}}_j$ is the conduction current density.

From (4.5) the proper forces are identified as—grad ln T and $\mathbf{E}$, while the fluxes are $\mathbf{j}$ and $\mathbf{Q}$. In mixed form, the linear relations between these quantities may be written as

$$\left.\begin{aligned} \mathbf{Q} &= -T\,\lambda \text{ grad ln } T + A_{12}\mathbf{j}, \\ \text{and} \qquad \mathbf{E} &= -A_{21} \text{ grad ln } T + \mathbf{j}/\sigma, \end{aligned}\right\} \tag{4.6}$$

where the Onsager reciprocity is now expressed (Casimir 1945) as

$$A_{12} = -A_{21} = \rho \quad \text{say}, \tag{4.7}$$

and λ and σ correspond respectively to Marshall's λ^{I} and σ^{I}. Finally we have

$$\left.\begin{aligned} \mathbf{Q} &= -T\,\lambda \text{ grad ln } T + \rho\mathbf{j}, \\ \text{and} \qquad \mathbf{E} &= \rho \text{ grad ln } T + \mathbf{j}/\sigma. \end{aligned}\right\} \tag{4.8}$$

Inserting the results (4.8) into (4.5) we obtain

$$T\left(\frac{\partial s}{\partial t}\right)_{\text{irr}} = T\,\lambda(\text{grad ln } T)^2 + \mathbf{j}^2/\sigma. \tag{4.9}$$

For an irreversible process $(\partial s/\partial t)_{\text{irr}} > 0$, and since $T > 0$, it follows that the quadratic form in (4.9) is positive definite, so that

$$\lambda > 0, \qquad \sigma > 0. \tag{4.10}$$

In terms of the particle velocity $\mathbf{v}_j$ of the jth component and the mean velocity

$$\overline{\mathbf{v}} = \frac{1}{n}\sum_j n_j \overline{\mathbf{v}}_j, \qquad \text{with } n = \sum_j n_j, \tag{4.11}$$

relative to which there is no flux of particles, the reduced heat flux vector due to conduction alone is expressed as

$$\mathbf{Q} = \sum_j \tfrac{1}{2} n_j m_j \overline{(\mathbf{v}_j - \overline{\mathbf{v}})^2 (\mathbf{v}_j - \overline{\mathbf{v}})}. \tag{4.12}$$

Enskog's analysis (Chapman and Cowling 1952) shows that (4.12) can be readily converted to the form

$$\mathbf{Q} = \mathbf{q} - \tfrac{5}{2} kT \sum_j n_j \overline{\mathbf{V}}_j, \tag{4.13}$$

where

$$\mathbf{q} = \sum_j \tfrac{1}{2} n_j m_j \overline{(\mathbf{v}_j - \mathbf{v}_0)^2 (\mathbf{v}_j - \mathbf{v}_0)}. \tag{4.14}$$

In (4.14) the thermal flow and thermal energy are referred to the mean mass velocity $\mathbf{v}_0$, relative to which there is a flux of particles $\sum_j n_j \overline{\mathbf{V}}_j$. Hence (4.13) shows that the heat flux vector $\mathbf{q}$ is the sum of the reduced heat flux vector $\mathbf{Q}$ and a heat flux vector consisting of a flux of particles $\sum_j n_j \overline{\mathbf{V}}_j$, each particle of which possesses enthalpy of $5kT/2$.

It is of interest to introduce a further heat flux vector,

$$\mathbf{q}_0 = \sum_j \tfrac{1}{2} m_j n_j \overline{\mathbf{v}_j^2 \mathbf{v}_j}. \tag{4.15}$$

Here the thermal flow and thermal energy are referred to a fixed coordinate frame, relative to which there is a flux of particles $\sum_j n_j \overline{\mathbf{v}}_j$. By an analysis similar to that of Enskog, and use of (4.13), we convert (4.15) to the form

$$\mathbf{q}_0 = \mathbf{Q} + \tfrac{5}{2} kT \sum_j n_j \overline{\mathbf{V}}_j + \tfrac{5}{2} kTn\mathbf{v}_0, \tag{4.16}$$

with neglect of a small quadratic term $\frac{1}{2}\mathbf{v}_0^2 \mathbf{v}_0 \sum_j n_j m_j$. Since $\overline{\mathbf{V}}_j + \mathbf{v}_0 = \overline{\mathbf{v}}_j$, we see from (4.16) that the heat flux vector $\mathbf{q}_0$ is the sum of $\mathbf{Q}$ and a heat flux vector consisting of a particle flux $\sum_j n_j \overline{\mathbf{v}}_j$, each particle of which possess enthalpy of $5kT/2$. For plasmas at rest, or for plasmas having a sufficiently small mean mass velocity $\mathbf{v}_0$, we shall neglect the convection term $5kTn\mathbf{v}_0/2$ in (4.16), so that $\mathbf{q}_0 = \mathbf{q}$.

It is to be noted that (4.16) is in agreement with Kihara's (1959, p. 130) definition of the reduced heat flux vector, if his enthalpy $h_j = \mu_j + s_j T$ is set equal to $5kT/2$, and his absolute heat flux vector $\mathbf{q}^*$ is identified as our $\mathbf{q}_0$.

By simple manipulation it is possible to write (4.13) as

$$\mathbf{Q} = \mathbf{q} - \tfrac{5}{2} kT \frac{m_2 - m_1}{e_1 m_2 - e_2 m_1} \mathbf{j}. \tag{4.17}$$

With $m_2 \gg m_1$, $e_1 = -e$, $e_2 = e$, (4.17) simplifies to

$$\mathbf{Q} = \mathbf{q} + \frac{5kT}{2e} \mathbf{j}, \tag{4.18}$$

which is in agreement with Kaneko's (1960) result for a plasma at rest.

From (4.8) and (4.18) we readily obtain

$$\left.\begin{aligned} \mathbf{q} &= -K \operatorname{grad} T - \beta \mathbf{E}, \\ \mathbf{j} &= \alpha \operatorname{grad} T + \sigma \mathbf{E}, \end{aligned}\right\} \tag{4.19}$$

where

$$\alpha = -\sigma\rho/T, \tag{4.20}$$

$$\beta = \alpha T + \frac{5kT}{2e}\sigma, \tag{4.21}$$

$$K = \lambda + \alpha\beta/\sigma. \tag{4.22}$$

The equations (4.19) are in the form given by Spitzer (1956, p. 87), who also quotes the relationship (4.21). The result (4.22) for K is in agreement with earlier results obtained by Seymour (1961*a*, p. 132). In view of the inequalities of (4.10), (4.22) yields the interesting result

$$K\sigma - \alpha\beta > 0. \tag{4.23}$$

V. Comparison of Results for Zero Magnetic Field

Using the Chapman-Enskog method up to the 6th approximation, Kaneko (1960, pp. 1685–96) has calculated the electrical and thermal conductivities and the coefficient of thermal diffusion of a plasma in a magnetic field. Similarly, Landshoff (1949, 1951) has carried out calculations for the 3rd approximation with $\mathbf{H}$ finite, and up to the 5th approximation for $\mathbf{H} = 0$.

Restricting attention to zero magnetic field at this stage, in effect Kaneko gives σ, α, and Marshall's λ', where λ' is given by Kaneko's equation (34), which applies to the equation for $\mathbf{Q}$ immediately preceding his equation (25). Unfortunately, Kaneko uses the same symbol for this thermal conductivity and for the thermal conductivity appearing in the first of his equations (20), where, in fact, Marshall's λ should appear, as can be seen from the first of our equations (4.8).

From our equations (4.8), and equation (4.20) it is readily found that

$$\lambda = \frac{25}{8}\left\{b^{\mathrm{I}(-1)} + \left(\frac{m_1}{m_2}\right)^{1/2} b^{\mathrm{I}(1)} - \frac{1}{e^{\mathrm{I}(0)}}(b^{\mathrm{I}(0)})^2\right\}\frac{nk^2T\tau}{m_1}, \tag{5.1}$$

since

$$\sigma = \frac{ne^2\tau}{2m_1}e^{\mathrm{I}(0)}, \tag{5.2}$$

and

$$\alpha = -\frac{5nke\tau}{4m_1}b^{\mathrm{I}(0)}, \tag{5.3}$$

as given by Kaneko. Here $b^{\mathrm{I}(-1)}$, $b^{\mathrm{I}(1)}$, $b^{\mathrm{I}(0)}$, and $e^{\mathrm{I}(0)}$ are the parameters given by Kaneko in 2nd through to 6th approximation, for atomic number $Z = 1$ and $\omega\tau = 0$.

To express K in terms of Kaneko's parameters we use (4.21), (4.22), (5.1), (5.2), and (5.3) to obtain

$$K = \frac{25}{8}\left\{b^{\mathrm{I}(-1)} + \left(\frac{m_1}{m_2}\right)^{1/2} b^{\mathrm{I}(1)} - b^{\mathrm{I}(0)}\right\}\frac{nk^2T\tau}{m_1}. \tag{5.4}$$

From (4.21), (5.2), and (5.3) we have

$$\beta = \frac{5nkeT\tau}{4m_1}(e^{\mathrm{I}(0)} - b^{\mathrm{I}(0)}). \tag{5.5}$$

Using Kaneko's equation (31) for the electron collision time (which is in close agreement with Marshall's result for τ), Spitzer and Härm's (1953) equations (33), (34), and (36) can be written in the form of our equations (5.2), (5.3), (5.4) (with omission of the term $(m_1/m_2)^{1/2}b^{\mathrm{I}(1)}$), to yield the results

$$\left.\begin{aligned} b^{\mathrm{I}(0)} &= -(32/5\pi)\gamma_T, \\ b^{\mathrm{I}(-1)} &= (256/15\pi)\delta_T + b^{\mathrm{I}(0)}, \\ e^{\mathrm{I}(0)} &= (32/3\pi)\gamma_E, \end{aligned}\right\} \tag{5.6}$$

in infinite approximation. Spitzer and Härm (1953) give values of γ_T, δ_T, γ_E in their Table III for various Z-values, together with $\delta_E = \frac{1}{8}(5\gamma_E + 3\gamma_T)$. For $Z = 1$ we have from (5.6)

$$\left.\begin{aligned} b^{\mathrm{I}(0)} &= -0{\cdot}5555, \\ b^{\mathrm{I}(-1)} &= 0{\cdot}6679, \\ e^{\mathrm{I}(0)} &= 1{\cdot}9747, \end{aligned}\right\} \tag{5.7}$$

in agreement with the figures for infinite approximation appearing in Kaneko's tables.

Using Kaneko's tables for $b^{\mathrm{I}(-1)}$, $b^{\mathrm{I}(1)}$, $b^{\mathrm{I}(0)}$, and $e^{\mathrm{I}(0)}$ in 2nd approximation ($\mathbf{H} = 0$, $Z = 1$), and equations (5.2), (5.3), (5.4), and (5.5) for deuterium (with $(m_1/m_2)^{\frac{1}{2}} \approx 1/61$), we compare Kaneko's numerical coefficients for σ, α, K, and β with those of Marshall and Landshoff in the same approximation, and with those of Spitzer and Härm in infinite approximation, as in Table 3, where A, B, C, D are defined by the equations

$$\left.\begin{aligned} \sigma &= A\,\frac{ne^2\tau}{m_1}, & \alpha &= B\,\frac{nke\tau}{m_1}, \\ K &= C\,\frac{nk^2T\tau}{m_1}, & \beta &= D\,\frac{nkeT\tau}{m_1}. \end{aligned}\right\} \tag{5.8}$$

Table 3 shows very satisfactory agreement between the results of Kaneko, of Marshall, and of Landshoff in 2nd approximation.

While the inequality (4.23) is easily satisfied in 2nd and higher approximations, the quantity $4K_0\sigma_0 - (\alpha_0 + \beta_0)^2 = \mu$, say, which appeared in Seymour's analysis (1961*b*, p. 283) is negative in 2nd approximation, but positive in all higher approximations (cf. Landshoff 1951; Spitzer and Härm 1953). It is of interest to note that in 2nd approximation $\mu < 0$ leads to hyperbolic forms for V and T given respectively by (3.2.20) and (3.2.21) of Seymour (1961*b*), but in fact the trigonometrical forms of (3.2.20) and (3.2.21), which were obtained by use of Spitzer and Härm's infinite approximations to σ, α, K, and β, are the valid results.

VI. Influence of Marshall's Corrected K^{II}, K^{III} on the Radial Heat Flow Approximation

In Seymour's earlier papers (1961*a*, 1961*b*, 1961*c*) the analyses were made possible by introduction of the basic simplifying assumption that perfect thermal insulation exists at the plasma boundary surface when it is pinched away from the walls of the discharge table during the containment period, an insulation condition which cannot be realized in practice. However, this assumption became more plausible in the limit of a strong external guiding magnetic field, because the original form of Marshall's K^{II} and K^{III} showed that for $\omega\tau \gg 1$, the heat flow across the radially constricted gas discharge could be neglected compared to the heat flow along the discharge.

From Table 2 we see that in the high-field limit (now a more stringent inequality which approximates to $\omega^2\tau^2 \gg 1400$ for deuterium),

Table 3

Results for numerical coefficients A, B, C, D, defined by equations (5·8)

Authors	A	B	C	D
Kaneko	0·966	0·777	3·272	3·192
Marshall	0·966	0·777	3·270	3·191
Landshoff	0·966	0·777	3·236*	3·192
Spitzer and Härm	0·987	0·6944	3·823*	3·163

* These results exclude the correction term $(25/8)(m_1/m_2)^{1/2}b^{\mathrm{I}(1)}$, which for deuterium amounts to 0·036 in 2nd approximation and closely 0·046 in infinite approximation.

$$K^{\mathrm{II}} \approx \frac{5nk^2T\tau}{4m_1}\left(\frac{0\cdot 566M_1^{-1/2}}{\omega^2\tau^2}\right), \tag{6.1}$$

where we recall that ω and τ apply to the electrons.

For the ions we have

$$\tau_+ \approx M_1^{-1/2}\tau, \tag{6.2}$$

and

$$\omega_+ = -M_1\omega, \tag{6.3}$$

and so (6.1) can be written

$$K^{\mathrm{II}} \approx \frac{5nk^2T\tau_+}{4m_2}\left(\frac{0\cdot 566}{\omega_+^2\tau_+^2}\right). \tag{6.4}$$

Noting from Table 2 that for $\omega^2\tau^2 \gg 1400$, $K^{\mathrm{III}} \approx 0$, it is found from (6.1) that K^{II}, which now arises mainly from ion-ion collisions (see Vaughan-Williams and Haas 1961, p. 165), is nearly an order of magnitude greater than Marshall's original result for K^{II}, here considered for $\omega^2\tau^2 \gg 1400$ to facilitate comparison. Hence the heat flow across the discharge will be correspondingly enhanced. We can estimate the importance of this effect on the earlier radially constricted deuterium discharge analyses as follows. From (4·26) (Seymour 1961*a*) we have, for $\omega\tau \gg 1$,

$$K^{\text{II}} \approx K^{\text{I}} \frac{1\cdot 369}{\omega^2\tau^2}, \tag{6.5}$$

$$K^{\text{III}} \approx 0, \tag{6.6}$$

for Marshall's original K's perpendicular to a magnetic field. It follows immediately from (6.5) and (6.6) that our original approximation of neglect of heat flow across **H** compared to that along **H** for comparable temperature gradients was reasonable in the high-field limit $\omega\tau \gg 1$.

Using the corrected Marshall K^{I} given in Table 1 and (6.1), we have for deuterium, with $\omega^2\tau^2 \gg 1400$,

$$K^{\text{II}} \approx K^{\text{I}}\ 13\cdot 2/\omega^2\tau^2, \tag{6.7}$$

where the corrected K^{I} is about 90% of the original K^{I}, and as mentioned above,

$$K^{\text{III}} \approx 0. \tag{6.8}$$

Thus we see from (6.7) and (6.8) that it is still possible to neglect heat flow across **H** compared to that along **H** in the more stringent high-field limit $\omega^2\tau^2 \gg 1400$. We therefore conclude that the basic approximation of perfect thermal insulation at the free boundary surface of the plasma can still be made more plausible in the limit of a strong external guiding magnetic field, provided that this magnetic field now corresponds to the more stringent inequality $\omega^2\tau^2 \gg 1400$ for a highly ionized deuterium discharge, rather than the inequality $\omega\tau \gg 1$ which gave the high-field limit in the case of Marshall's original results for the perpendicular components of the thermal conductivity tensors.

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