THE MECHANICAL IMPULSE METHOD FOR DETERMINING DYNAMIC ELASTIC MODULI AND INTERNAL FRICTION OF SOLIDS

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[*Manuscript received September* 9, 1963]

Summary

The theory governing the excitation of a linear mechanical system by a delta function impulse is summarized. In the case of a finite elastic rod excited by an impulse, the Fourier transform of the output function enables the frequency spectrum of the rod to be obtained. A spectrometer for accomplishing this operation is described and a comparative study is made between the impulse and resonance responses of a cylindrical rod.

A discussion of the basic relationships involved in computing the dynamic elastic moduli and internal friction is presented, together with some typical data obtained by the mechanical impulse method.

I. INTRODUCTION

The elastic constants of both single crystals and polycrystalline materials may be determined from a knowledge of the velocity of plane elastic waves in the material. Important conclusions regarding the internal structure of the material may be drawn from measurements of the attenuation coefficient. The most common methods for measuring the velocity and attenuation coefficient are shown in Figure 1. The traditional method for measuring the dynamic elastic constants has been to employ a resonance method in which a standing wave pattern is established in the specimen. A number of different techniques may be employed for generating and detecting the waves including electrostatic, electrodynamic, and piezoelectric methods.

In recent years, a number of pulse methods have been developed (Mason 1958) which involve the direct timing of the passage of a wave packet as it is transmitted through the material or as it is reflected multiply within the material. The wave packets are usually introduced into the specimen and detected by means of quartz or barium titanate crystals cemented to the specimen. Whereas resonance methods are particularly successful at low frequencies where the vibration modes are easily identified, pulse methods have been found to be most suited to frequencies above 1 MHz. One important advantage to be gained in the high frequency region is that small samples may be employed, since the wavelength is then of the order of 1 mm or less.

The mechanical impulse method, to be described below, is a pulse method that operates successfully at low frequencies. Short-duration (wide frequency band) mechanical impulses are applied to one end of a specimen in the form of a rod or plate. The normal mode response of the system may then be obtained by Fourier analysis of the resulting motion. A capacitative detector is normally mounted

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near the opposite end of the specimen. Measured values of the phase velocity enable the appropriate elastic modulus to be computed. Consequent advantages of this system are (i) apart from the supports, no physical contact with the specimen is required following excitation, (ii) all the desired normal modes of the specimen may be excited simultaneously, (iii) the decay of free vibrations for a given normal mode may be measured easily and accurately.

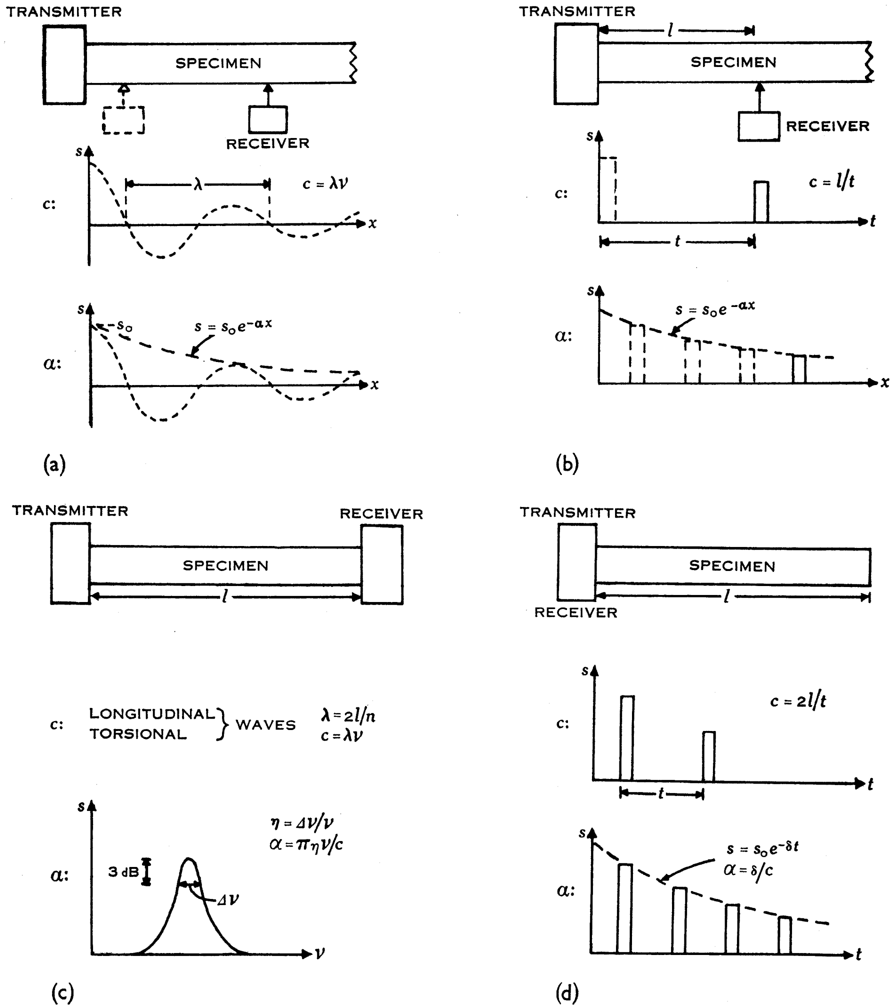


Fig. 1.—Basic methods for measuring velocity and attenuation. (*a*) Travelling wave method; (*b*) single-pulse transit method; (*c*) standing wave (resonance) method; (*d*) reverberation (pulse echo) method.

II. Mechanical Impulse Method—Summary of Theory

Provided the time interval over which the impulse is applied to the system is short compared with the response times involved in the system, the impulse may be treated as equivalent to a delta function. In this case, as shown by Pollard (1962*a*), the actual form of the impulse is immaterial.

When an impulse is applied to a linear system, the analysis of the response of the system may be carried out in terms of either time or frequency. From the experimental standpoint it is preferable to work in terms of frequency. Thus, if a known function $f(t)$ is applied to a linear system, the input spectrum function, $F(\mathrm{i}\omega)$, may be found as the Fourier transform of $f(t)$. When $f(t)$ is a delta function it is readily shown that $|F(\mathrm{i}\omega)| = 1$. That is, the spectrum of a delta function has a value of unity at all frequencies. In the case of an arbitrary pulse of short duration for which $\int_{-\infty}^{\infty} f(t)\mathrm{d}t = S$, it may be shown that $|F(\mathrm{i}\omega)| \to S$ as the pulse duration tends to zero. S is sometimes referred to as the strength of the pulse.

The output spectrum function, $G(\mathrm{i}\omega)$, of the system may be found (Lanczos 1957) from

$$G(\mathrm{i}\omega) = F(\mathrm{i}\omega) . H(\mathrm{i}\omega), \tag{1}$$

where $H(\mathrm{i}\omega)$ is the transfer function of the system. That is, each component of the input spectrum when multiplied by the corresponding component of the transfer function gives a component of the output spectrum.

(*a*) *Spectrum of a Mechanical Impulse*

Consider a mechanical impulse produced by the impact on the system of a mass m_0 moving with velocity v_0 just prior to impact. If the impact is an elastic one then the momentum transferred to the system is $m_0 v_0' = m_0 v_0(1+e)$, where e is the coefficient of restitution. Then

$$S = \int_{-\infty}^{\infty} f(t)\mathrm{d}t = m_0 v_0'. \tag{2}$$

In the limit of very short impact time, $|F(\mathrm{i}\omega)| = m_0 v_0'$ and equation (1) becomes

$$G(\mathrm{i}\omega) = m_0 v_0' H(\mathrm{i}\omega). \tag{3}$$

If the output velocity function $u(t)$ of a system is measured or its Fourier transform, the velocity spectrum function $U(\mathrm{i}\omega)$, then, in equation (3), $G(\mathrm{i}\omega)$ is replaced by $U(\mathrm{i}\omega)$. In this case, $H(\mathrm{i}\omega)$ is identical with the mechanical admittance of the system, that is, the reciprocal of the mechanical impedance $Z(\mathrm{i}\omega)$, since by definition $Z(\mathrm{i}\omega) = F(\mathrm{i}\omega)/U(\mathrm{i}\omega)$. Thus, equation (3) may be written

$$U(\mathrm{i}\omega) = m_0 v_0'/Z(\mathrm{i}\omega). \tag{4}$$

From equation (4), the velocity spectrum may be computed for any mechanical system for which $Z(\mathrm{i}\omega)$ is known.

(*b*) *Mechanical Impulse applied to an Elastic Rod*

In order to determine the response of a given mechanical system when excited by a mechanical impulse, the following method may be employed (Pollard 1962*a*):

(i) an analogous electrical circuit may be devised to represent the mechanical system,

(ii) the appropriate transfer function for this circuit may be computed using steady-state analysis,

(iii) the response of the system may then be determined when excited by an impulse of strength $S = m_0 v_0'$.

Consider, then, the impact of a small elastic sphere on the end face of a semi-infinite elastic rod so that a longitudinal disturbance is produced in the rod. The mechanical impedance for this system is

$$Z(\mathrm{i}\omega) = R_0 + \mathrm{i}\omega m_0 - \omega^2 m_0 R_0 C_m, \tag{5}$$

where $R_0 = A(E\rho)^{\frac{1}{2}}$ is the characteristic impedance (a pure resistance in the case of an infinite rod), A the area of cross section, E Young's modulus, ρ the density of the rod, and C_m the compliance. Substitution in equation (4) and rearrangement yields the relation, in non-dimensional form,

$$(R_0/m_0 v_0')|U(\mathrm{i}\omega)| = [1+(\omega m_0/R_0)^2(1+\epsilon)]^{-\frac{1}{2}}, \tag{6}$$

where

$$\epsilon = (R_0^2 C_m/m_0)(\omega^2 m_0 C_m - 2).$$

The velocity spectrum will therefore remain uniform over the range of frequencies for which the right-hand side of equation (6) is unity. At higher frequencies, $|U(\mathrm{i}\omega)|$ falls off more rapidly as m_0 and C_m are increased. The high frequency dependency is shown graphically by Pollard (1962*a*).

(*c*) *The Finite Elastic Rod*

The complete low frequency resonance response of a finite rod may be computed by the application of a transmission-line theory (Pollard 1962*b*). However, near resonant peaks, the theory may be simplified and may be shown to be in agreement with normal mode analysis. A mode impedance Z_n may then be introduced (Skudrzyk 1958) that governs the response of the nth normal mode of the system and is defined by

$$Z_n = R_n + \mathrm{i}\omega M_n + 1/\mathrm{i}\omega C_n, \tag{7}$$

where R_n, M_n, and C_n are the equivalent resistance, mass, and compliance associated with the nth normal mode. As defined by Skudrzyk, these parameters depend on the type of force distribution (point source, line source, etc.) and on the relative locations of the point of observation and the point of application of the force. At each resonant peak, $Z_n = R_n$. Then, the mechanical impedance of the system at resonance when excited by a mechanical impulse is (Pollard 1962*a*)

$$Z(\mathrm{i}\omega) = R_n + \mathrm{i}\omega m_0 - \omega^2 m_0 C_m R_n. \tag{8}$$

As might be expected, equation (8) is similar to equation (5) with R_n replacing R_0. Substituting equation (7) in equation (4) as before, yields an expression for the velocity spectrum similar to equation (6) with R_n replacing R_0. In the frequency region for which the right-hand side of equation (6) is unity,

$$|U(\mathrm{i}\omega)| = m_0 v_0'/R_n. \tag{9}$$

The particular resonant modes of a system that are excited will depend on the existence of spectral components of the force at the correct frequencies. In the range where the spectrum is essentially constant, a mechanical impulse will excite all the resonant modes of the system. By means of a suitable spectrometer the set of resonances generated by a mechanical impulse may be recorded.

III. The Impulse Spectrometer

(a) *Longitudinal Waves*

The experimental arrangement for determining the response of a rod when excited longitudinally is shown in Figure 2. The feed mechanism consists of a supply

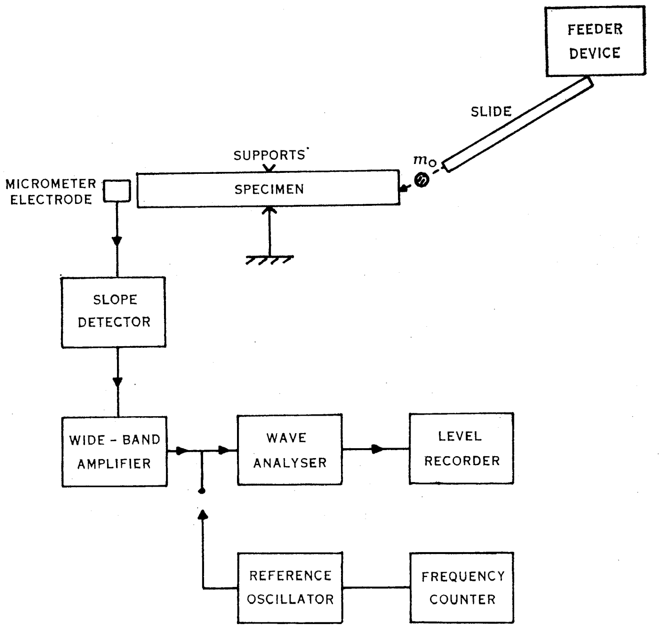


Fig. 2.—Mechanical impulse spectrometer for determining the response of a rod to longitudinal excitation.

tube terminated by a metal gate which may be operated by a relay. Steel balls are released at a controlled rate and are directed against the end of the rod. The detector consists of the variable capacitor formed between the other end of the rod and a fixed electrode. The capacitor is connected into a resonant circuit which forms part of a crystal-controlled oscillator operating as a slope detector (Richardson 1954). For convenience in adjusting the capacitor gap, a micrometer screw with flat end face is used as the fixed electrode. In the case of non-metallic specimens, the end face of the specimen may be coated with aquadag.

The displacement, $s(i\omega)$, of the end face of the specimen corresponding to a resonant peak may be found from the expression for the velocity, equation (9).

Since $s(\mathrm{i}\omega) = U(\mathrm{i}\omega)/\mathrm{i}\omega$,

$$|s(\mathrm{i}\omega)| = m_0 v_0'/\omega R_n. \tag{10}$$

Following amplification, the output of the detector is fed into a wave analyser having a frequency range from 0 to 16 kHz. The response of the wave analyser is

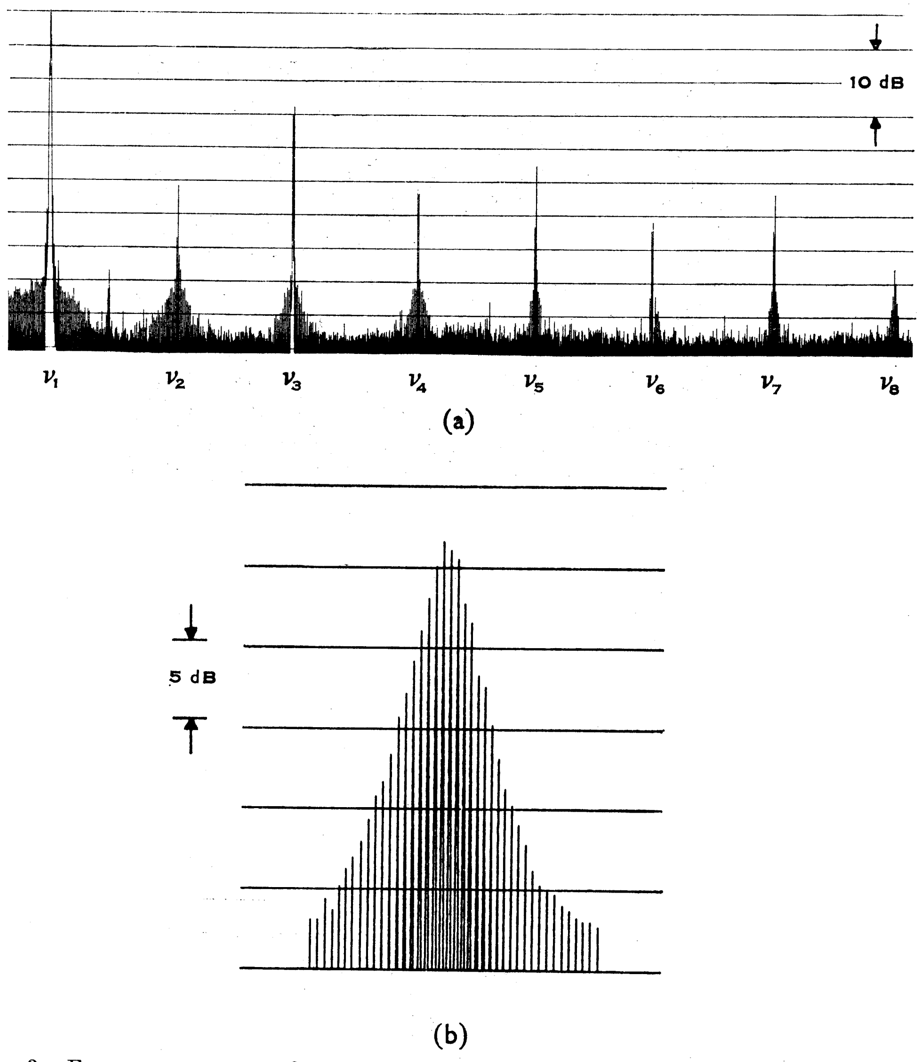


Fig. 3.—Frequency response of a brass rod (alloy 305: length 75·76 cm, radius 0·477 cm, density 8·48 g/cm³, centre clamp) using mechanical impulses. (*a*) Fast scan showing the first eight longitudinal modes; (*b*) slow scan of fundamental longitudinal mode. Frequency interval between impulses is 1 Hz.

conveniently recorded on a high-speed level recorder. The frequencies of the resonant modes of the specimen may now be located by driving the wave analyser slowly through the appropriate frequency range while periodic impulses are applied to the specimen. For the accurate measurement of individual resonances the wave analyser may be tuned manually to the peak response. An auxiliary oscillator is then switched into the wave analyser and adjusted to give maximum response at the same setting of the analyser. The frequency setting of the oscillator is then read accurately on the frequency counter. An example of a recorded spectrum is shown in Figure 3 together with a detailed scan of the first resonant peak.

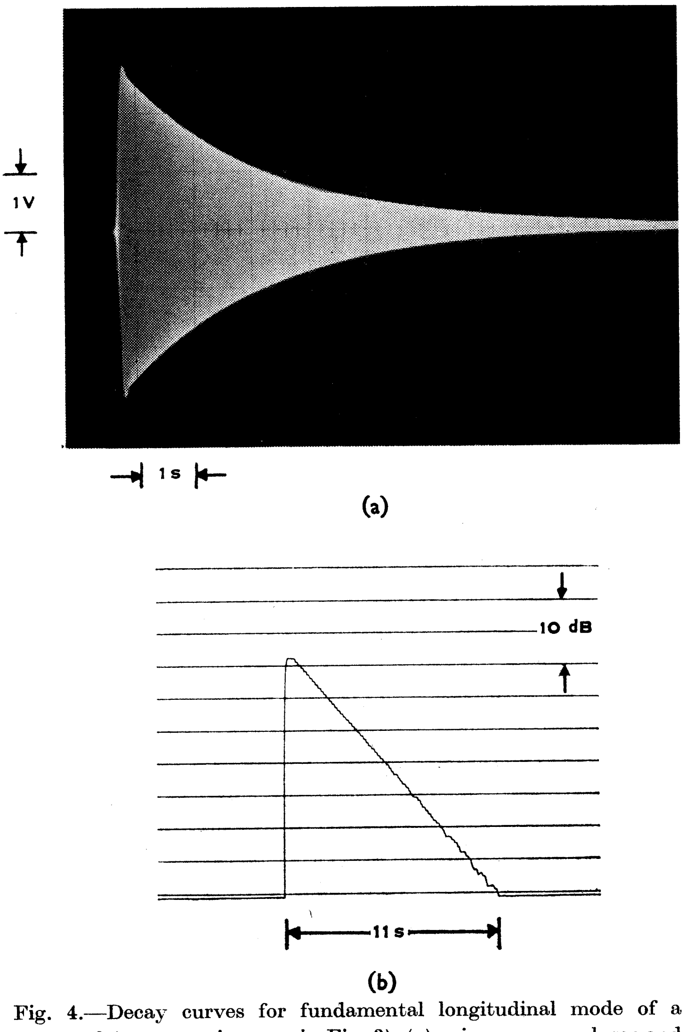


Fig. 4.—Decay curves for fundamental longitudinal mode of a brass rod (same specimen as in Fig. 3), (*a*) using wave analyser and oscilloscope, (*b*) using wave analyser and level recorder.

The decay rate at a given resonant frequency is easily found by exciting the specimen with a single impulse and then recording the decay curve on the level recorder. Because of the logarithmic scale of the recorder, a straight line decay is obtained if the system has an exponential decay rate. In Figure 4 is shown the decay curve for the fundamental longitudinal mode of the specimen shown in Figure 3, (*a*) recorded by means of the wave analyser and oscilloscope, and (*b*) recorded by means of the wave analyser and level recorder. When the decay rate becomes too short for accurate measurements it is then more satisfactory to measure the bandwidth of the resonant curve.

(*b*) *Torsional and Flexural Waves*

For the generation of torsional and flexural waves an arrangement similar to that shown in Figure 5 may be employed. A steel ball is allowed to fall onto a projecting lug attached to one end of the specimen while the detector electrode is mounted vertically over a similar lug attached either to the same or to the other end of the specimen. With this arrangement both torsional and flexural waves may be generated and detected. The torsional modes are clearly identified since, like the longitudinal modes, the frequency difference between resonant modes is constant whereas for flexural waves the frequency difference increases with frequency.

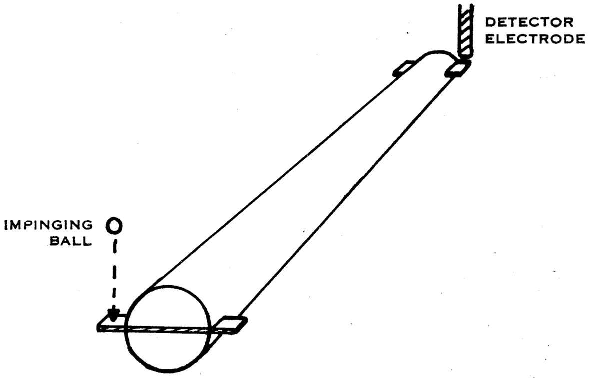


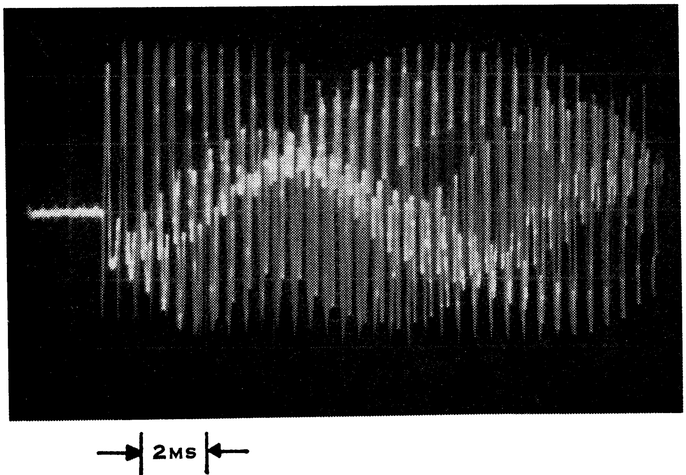
Fig. 5.—Impulse excitation and detection of torsional and flexural waves in a rod.

(*c*) *Examination of Impacts*

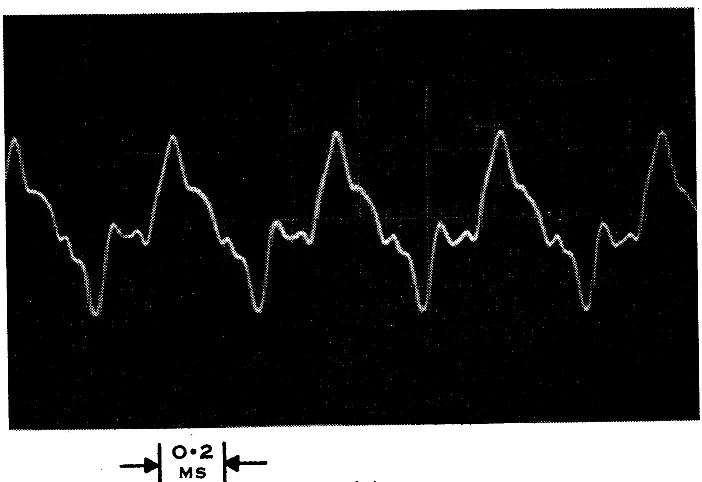
In Figure 6 are shown oscillograms of a single impact on the end face of a brass rod. The time base of the oscilloscope was triggered externally by a signal derived from a photosensitive transistor when the ball interrupted a light beam just prior to impact. The unfiltered response is initially very complicated since the impulse, applied to a small area of the end of the rod, generates flexural waves as well as longitudinal. This is particularly so if the impact does not take place at the centre of the end face. There is also some evidence that surface waves are generated by the impact (McMillen 1946). However, the mounting conditions favour longitudinal waves, and after a short time the unwanted vibrations decay.

IV. Comparison of the Resonance and Pulse Responses of a Cylindrical Rod

It is of interest to compare the behaviour of a specimen in the form of a rod, (i) when driven continuously into resonance and (ii) when excited by a mechanical impulse. A theoretical discussion given in a previous paper (Pollard 1962*b*) suggests that the response of the rod should be identical in the two cases. The specimen selected for the experiment was a cylindrical rod of drawn brass (Alloy 305: 61% Cu, 36% Zn, 3% Pb) of length 2·910 m, mean radius 0·477 cm, computed density 8·48 g/cm³.



(a)



(b)

Fig. 6.—Initiation of vibrations in a brass rod (same specimen as in Fig. 3) following longitudinal impact. (*a*) Total output from slope detector—flexural modes below 1700 Hz attenuated by high-pass filter, ν_1 (longitudinal) = 1994 Hz; (*b*) as in (*a*) but with time base of oscilloscope delayed by 100 ms after triggering.

Continuous excitation of the specimen was achieved by an electrostatic drive system. One end of the specimen and a fixed electrode form a capacitor with a thin piece of mica as dielectric. When an alternating voltage is applied between the specimen and the electrode, vibrations are induced in the specimen. The detector at the other end of the specimen is the same as that described in Section III(a).

Measurements were made on the odd harmonics 1 to 21, for which the specimen was clamped at its centre. Measurements may be made using even harmonics when the specimen is clamped at two displacement nodes. In the case of the odd harmonics, because of the simpler mounting conditions, it is found that the accuracy of measurement is greater. If even harmonics are used, the location of the clamping points is very critical. All measurements were made at 22°C.

The mean value ν/n, where ν is the frequency and n is the mode number, for the odd harmonics $n = 1$ to $n = 21$ by the two methods are shown below together with the standard deviation of the measurements (for the number of measurements quoted, the standard deviation has the same value as the range of error found by applying the 't' test for a 99% confidence level).

Electrostatic Drive	*Impulse Excitation*
$517 \cdot 4 \pm 0 \cdot 1$ Hz	$517 \cdot 5 \pm 0 \cdot 2$ Hz

With both methods, the main limiting factor is the stability of the oscillator employed. With the impulse method there is in addition the limit imposed by the minimum detectable change in frequency setting of the wave analyser.

Further measurements on other metallic and non-metallic specimens confirmed the identity of the longitudinal resonant modes when measured by impulse excitation and by continuous excitation.

V. Measurement of Dynamic Elastic Moduli and Internal Friction

(a) *Longitudinal Waves*

Solution of the wave equation for an infinitely long, isotropic rod thin enough to be regarded as a one-dimensional medium shows that the phase velocity of plane longitudinal waves, c_{L}, is related to the dynamic Young's modulus E by

$$E = \rho c_{\mathrm{L}}^2. \tag{11}$$

Equation (11) thus ignores lateral motions of the rod produced by "Poisson coupling".

The measured or effective phase velocity, c_{L}', of longitudinal waves in a rod of radius a is found to depend on the ratio a/λ where λ is the wavelength. A typical dispersion curve is shown in Figure 7 for a brass rod (alloy 305), the highest harmonic included being the 60th for which $\nu_{60} = 113\ 370$ Hz.

For some problems it is sufficient to compute an effective Young's modulus E', given by

$$E' = \rho(c_{\mathrm{L}}')^2, \tag{12}$$

but E' will now depend on the geometrical configuration of the specimen as well as on the material itself. In order to compute E, the phase velocity c_{L} for a negligibly thin specimen must be determined. This may be done using a relationship first derived by Rayleigh (1894), namely,

$$c_{\mathrm{L}} = c_{\mathrm{L}}'[1 + \sigma^2\pi^2(a/\lambda)^2], \tag{13}$$

where σ is Poisson's ratio.

Since the wavelength for a given mode of vibration is fixed by the length of the specimen, equation (13) may also be written

$$\nu_n = \nu'_n[1+\sigma^2\pi^2(a/\lambda_n)^2], \tag{14}$$

where ν'_n is the effective frequency of the nth normal mode in a rod of radius a and ν_n is the frequency of the nth normal mode in an infinitely thin rod of the same length. Thus, in terms of a finite cylindrical rod, equation (11) may be written

$$E = \rho c_L^2 = \rho(\lambda_n\nu_n)^2 = \frac{4\rho l^2(\nu'_n)^2}{n^2}\left[1+\left(\frac{n\pi\sigma a}{2l}\right)^2\right]^2, \tag{15}$$

where l is the length of the rod. This expression for E holds for small values of

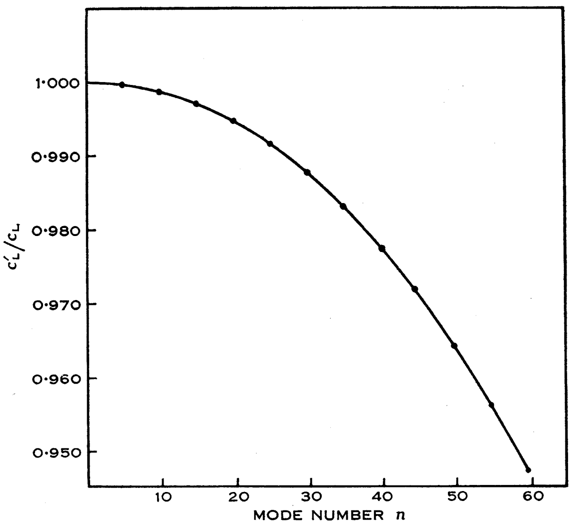


Fig. 7.—Longitudinal wave velocity dispersion curve for a brass rod (alloy 305: length 75·76 cm, radius 0·477 cm). The abcissae may alternatively be stated in terms of a/λ since $a/\lambda = 0\cdot00315n$ for this specimen.

a/λ. When a/λ approaches unity, a more precise calculation of c_L involves the application of the Pochhammer–Chree theory, as discussed by Bancroft (1941). It may be noted, however, that, in order to compute E from equation (15), the value of Poisson's ratio must be known.

(b) *Poisson's Ratio*

From equation (14), it is seen that Poisson's ratio may be found if ν'_n is measured for a number of different values of a/λ. It is then convenient to determine σ from the slope of the graph of $(1-\nu'_n/n\nu_1)$ versus $(a/\lambda)^2$. In Figure 8 is shown such a graph for brass (alloy 305), the plotted points corresponding to the 15th, 25th, 35th, and 45th harmonics. Some uncertainty arises as to the correct value of ν_1. If an incorrect value is assumed, the straight line does not pass through the origin. A new line drawn parallel to the original one applies the necessary correction to the value of ν_1 without altering the value of σ, which only depends on the slope. Although measurements at only two harmonics are sufficient, additional check points are useful as a/λ increases, in order to determine whether Rayleigh's relationship is still valid. The value of σ calculated from Figure 8 is $0 \cdot 377$. The corresponding values of c_L and E are: $c_L = 3020$ m/s, $E = 7 \cdot 72 \times 10^{10}$ N/m^2.

Largely because of the number of squared quantities involved, values of the elastic moduli will be in error by about $\pm 1\%$. The velocity c_L' may be measured to within $\pm 0 \cdot 05\%$ but the corrected velocity c_L will be in error by about $\pm 0 \cdot 1\%$.

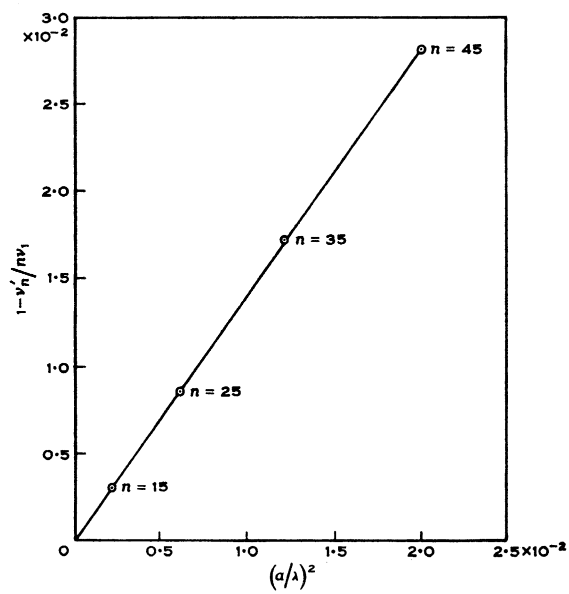


Fig. 8.—Graphical determination of Poisson's ratio: $(1-\nu_n'/n\nu_1)$ versus $(a/\lambda)^2$ for the 15th, 25th, 35th, and 45th harmonics of the brass rod quoted in Figure 7.

In Table 1 are shown values of σ, c_L, and E for a number of different specimens, obtained by the impulse method. Commercial drawn alloys show considerable variations in values of c_L and E, depending on such variables as the annealing treatment and whether the specimen is taken from the beginning or the end of the drawn material.

(c) *Torsional Waves*

The phase velocity of torsional waves, c_T, in a rod is governed by the rigidity or shear modulus G. Solution of the wave equation for an infinitely long rod yields the relation

$$G = \rho c_T^2. \tag{16}$$

Since there is no configurational dispersion for the fundamental torsional mode in a finite cylindrical rod, equation (16) may be written

$$G = 4\rho l^2 \nu_n^2 / n^2. \tag{17}$$

The following values were measured for the brass rod referred to in Section V(b):

$$\nu_1 = 1516 \text{ Hz}, \; c_T = 2297 \text{ m/s}, \; G = 4\cdot47\times10^{10} \text{ N/m}^2.$$

If a given specimen is isotropic, Poisson's ratio may be found from E and G by means of the equation (Landau and Lifshitz 1959)

$$\sigma = E/2G-1. \tag{18}$$

TABLE 1

VALUES OF POISSON'S RATIO, LONGITUDINAL WAVE VELOCITY, AND YOUNG'S MODULUS

Specimen	Poisson's Ratio	c_L (m/s)	E (N/m²)
Brass alloy 305	0·377	3020	$7\cdot72\times10^{10}$
Brass alloy 210	0·384	3050	7·83
Brass alloy 303	0·356	3520	10·4
Copper alloy 101	0·398	3240	9·40
Aluminium	0·326	5130	7·10
Polystyrene	0·360	1870	0·366

Specimen Data:

Specimen	Composition	Length (cm)	Mean Radius (cm)	Density (g/cm³)
Alloy 305	61% Cu, 36% Zn, 3% Pb	75·76	0·477	8·48
Alloy 210	62% Cu, 38% Zn	75·24	0·477	8·41
Alloy 303	58% Cu, 39% Zn, 3% Pb	75·30	0·478	8·42
Alloy 101	Approx. 98% Cu with some Ag	74·98	0·479	8·96
Aluminium	—	75·05	0·525	2·71
Polystyrene	—	73·00	0·644	1·04

However, in practice, most metal rods exhibit some degree of anisotropy introduced by the manufacturing processes. Values of σ found from equation (18) may then be in serious error. The anisotropy may be taken into account by a method devised by Bradfield and Pursey (1953), who introduced an anisotropy index α, defined as $\alpha = (\sigma'-\sigma)/\sigma$, where σ' is the value of Poisson's ratio found by applying equation (18) and σ is the value found from dispersion measurements, as in Section V(b) above.

(d) *Damping Measurements*

One of the advantages of the mechanical impulse method is the ease with which the damping of free vibrations of a normal mode may be measured. Since several measures are commonly used for the damping, the relationships between them have been summarized in Appendix I. The values of the loss factor, $\eta = Q^{-1}$, for the first longitudinal mode of the specimens referred to in Table 1 are as follows:

Alloy 305	$8 \cdot 40 \times 10^{-5}$
alloy 210	$5 \cdot 35$
alloy 303	$4 \cdot 93$
alloy 101	$20 \cdot 2$
aluminium	$0 \cdot 66$
polystyrene	530

The quantity most easily measured by the mechanical impulse method is the decay rate D, expressed in decibels per second. The precision of measurement of D depends mainly on the error arising from the scale of the level recorder. Using an input potentiometer with a 50 decibel range, the uncertainty in reading the level is $\pm 0 \cdot 5$ dB. The error in the time scale is of the order of ± 3 in 10^3, so that the overall uncertainty in measuring D is of the order of ± 1 in 10^2. The upper practical limit for the direct measurement of D on the level recorder is determined by the fastest paper speed available, which sets the limit at approximately 570 dB/s (corresponding to a bandwidth of 21 Hz).

For highly damped materials, it is necessary to make a direct determination of the bandwidth. The main source of error is now the uncertainty in determining the points on the resonant curve for which the displacement is $1/\sqrt{2}$ times the maximum displacement (corresponding to a reduction in signal level of 3 dB). Using an input potentiometer with a 10 dB range, the estimated error in determining these points is $\pm 10\%$. In addition, the uncertainty in measuring a frequency difference using the smallest bandwidth of the wave analyser is ± 1 Hz. Thus for, bandwidths of the order of 20 Hz, the overall error may be as high as $\pm 15\%$. As the bandwidth increases, the precision of measurement improves.

The effect of absorption on the measured values of phase velocity and elastic moduli is considered in Appendix II.

(e) *Mode Parameters*

To complete the description of each normal mode of the specimen, the mode parameters may be computed (Skudrzyk 1958) from the measured values of resonant frequency and damping. The required relationships are summarized in Appendix III. The mode parameters are required for the computation of the mode impedance, given by equation (7) and for the velocity or displacement amplitude (equations (9) and (10)).

VI. Conclusions

The computation of the response of a linear mechanical system to impulse excitation is considerably simplified if the impulse may be treated as a delta function. Application of Fourier analysis allows the frequency spectrum to be evaluated. In the case of a finite elastic rod, a further simplification arises from consideration of the normal modes of the rod.

An impulse spectrometer has been devised in which the specimen is excited by the impact of a small steel ball. Following detection using a capacitative transducer, the signal is examined by means of a wave analyser and level recorder. Longitudinal, torsional, and flexural waves may readily be generated and detected. From the measured dimensions and resonant frequencies, the appropriate wave velocities and elastic moduli may be computed.

It is suggested that, in order to determine the dynamic elastic moduli with maximum accuracy, measurements should be made at one or more of the higher harmonics. With a high harmonic, a large number of nodes and loops are introduced into the specimen. When anisotropy is present, as occurs with drawn metals and alloys, the measured modulus thus represents an average value taken over all the loops present in the specimen.

The strain level, which depends on the momentum imparted to the specimen, is controllable and may be varied from approximately 10^{-7} to 10^{-5} with the present arrangement. One of the main advantages of the impulse method is the ease with which the decay rate for free vibrations of a normal mode may be measured.

VII. Acknowledgments

Gratitude is expressed to the Australian Atomic Energy Commission for supporting this work as part of a research contract; to the Austral Bronze Co. Pty. Ltd. for the donation of specimens; to Professor C. J. Milner, Head of the School of Physics, for encouraging this research; to R. W. Harris for helpful discussions; and to A. Parts for assistance in designing and constructing apparatus.

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Appendix I

Damping Measures

The most basic measures are those that involve only energy changes and hence do not depend on a knowledge of the detailed mechanism by which the energy is dissipated. The internal friction, $\Delta W/W$, is the ratio of ΔW, the energy dissipated per unit volume when the specimen is taken through a stress cycle, to W, the elastic energy per unit volume stored in the specimen when the strain is a maximum. As shown in standard texts (e.g. Morse 1948),

$$\Delta W/W = 2\pi/Q = 2\pi\eta, \tag{A1}$$

where Q is the quality factor and η is the loss factor or dissipation function.

Equation (A1) may also be written

$$\eta = T\Delta S/2\pi W, \tag{A2}$$

where T is the absolute temperature and ΔS is the irreversible entropy generated per unit volume per stress cycle.

Other measures depend on the postulated mechanism causing loss of energy. The simplest system possessing an exponential decay of energy is the free linear oscillator with resistance proportional to velocity, for which the equation of motion is

$$m\ddot{s}+R\dot{s}+s/C = 0, \tag{A3}$$

where m is the mass, R is the mechanical resistance, C is the mechanical compliance, and s the displacement. The solution for the decay of amplitude is

$$s = s_0 \exp(-\delta t), \tag{A4}$$

and for the decay of the stored energy

$$W = W_0 \exp(-2\delta t). \tag{A5}$$

The damping coefficient δ is given by

$$\delta = R/2m. \tag{A6}$$

Equation (A5) may be written in terms of energy levels with the decibel as unit

$$L = L_0 - Dt, \tag{A7}$$

where L and L_0 are the final and initial energy levels respectively. D is the decay rate and is related to δ by

$$D = 8\cdot 68\delta. \tag{A8}$$

The logarithmic decrement Λ is defined as the natural logarithm of the ratios of the values of s at two times differing by one period. Thus,

$$\Lambda = \delta T = \pi\eta, \tag{A9}$$

since the quality factor, $Q = 1/\eta$, for a linear oscillator is defined as $Q = \omega m/R$. The internal friction for the linear oscillator may be shown to be given by

$$\Delta W/W \approx 2\delta T = 2\pi\eta. \tag{A10}$$

It is also useful to relate Q and η to D, whence

$$Q = \pi\nu/\delta = 27\cdot 28\nu/D. \tag{A11}$$

In terms of the bandwidth $\Delta\nu$ of a resonant curve with centre frequency ν, $Q = \nu/\Delta\nu$ and

$$\Delta\nu = D/27\cdot 28. \tag{A12}$$

Finally, the spatial attenuation coefficient a for a travelling wave is related to δ, η, and D by

$$a = \delta/c = \omega\eta/2c = 0\cdot 1152D/c, \tag{A13}$$

where c is the phase velocity of the wave.

Appendix II

Effect of Absorption on Phase Velocity and Elastic Moduli

When a plane wave travels along an isotropic cylindrical rod in the $+x$ direction and allowance is made for the attenuation of the wave as it progresses, a solution of the wave equation in terms of displacement may be written

$$\mathbf{s}_x = \mathbf{A} \exp \mathrm{i}(\omega t - \mathbf{k}x), \tag{B1}$$

where $\mathbf{s}_x$ is the complex displacement, $\mathbf{A}$ the complex amplitude, ω the angular frequency, and $\mathbf{k}$ the complex wave vector. $\mathbf{k}$ may be written as $\mathbf{k} = k - \mathrm{i}\alpha$ where $k = \omega/c$, c is the phase velocity of the wave, and α is the attenuation coefficient.

In order to take into account the effects of attenuation, the phase velocity may be regarded as being complex, so that

$$\mathbf{c} = \omega/\mathbf{k} = \omega/(k - \mathrm{i}\alpha). \tag{B2}$$

Provided α^2 can be ignored compared with k^2, it may be shown that

$$\mathbf{c} = (\omega/k)(1 - \mathrm{i}\alpha/k) = c(1 - \mathrm{i}1/2Q), \tag{B3}$$

since $\alpha = \omega/2cQ = k/2Q$. Hence, to make a change in $|\mathbf{c}|$ of 1 part of 10^4, Q must be ≈ 35; for a change of 1 part in 10^3, Q must be ≈ 10 and for a change of 1%, Q must be $\approx 3\cdot5$. Absorption therefore does not seriously affect the phase velocity except for highly absorbing materials.

The effect of absorption on the elastic modulus is somewhat more marked. Taking the case of plane longitudinal waves, Young's modulus may now be regarded as being complex:

$$\mathbf{E} = \rho \mathbf{c}_{\mathrm{L}}^2 = E(1 - \mathrm{i}/Q). \tag{B4}$$

For the absorption to make a 1% change in $|\mathbf{E}|$, Q must be ≈ 70. For a $0\cdot1\%$ change in $|\mathbf{E}|$, Q must be ≈ 225. Similar considerations apply to the shear modulus.

Appendix III

Mode Parameters

For the longitudinal resonant modes of a rod, with driver and receiver at opposite ends of the specimen

$$\left.\begin{aligned} M_n &= \tfrac{1}{2}M, \\ R_n &= \omega_n M_n \eta_n, \\ C_n &= 1/\omega_n^2 M_n, \end{aligned}\right\} \tag{C1}$$

where M is the total mass, M_n, R_n, and C_n are the effective mass, resistance, and compliance respectively of the system for the nth normal mode. In addition, the characteristic impedance Z_0 of the medium is given by

$$Z_0 = \rho c_{\mathrm{L}} A, \tag{C2}$$

where A is the area of cross section.

For torsional waves in a rod, the corresponding parameters are

$$\left.\begin{aligned} I_n &= \tfrac{1}{2}I, \\ R_n &= \omega_n I_n \eta_n, \\ C_n &= 1/\omega_n^2 I_n, \end{aligned}\right\} \tag{C3}$$

where I is the moment of inertia about the axis of the rod and $C_n = \theta/\tau$ is an angular compliance, θ is the angular displacement, τ is the torque. The characteristic impedance in this case is

$$Z_0 = \rho c_{\mathrm{T}} A. \tag{C4}$$

For flexural waves in a rod, the mode parameters are given by expressions similar to those in equations (C1) for the longitudinal mode. However, the characteristic impedance is different, namely,

$$Z_0 = \{(1+\mathrm{i})/l\}(\omega_n c_{\mathrm{F}} K)^{\frac{1}{2}} M, \tag{C5}$$

where c_{F} is the flexural wave velocity and K is the radius of gyration of cross section about an axis perpendicular to the displacement.

As an illustration of the above relationships, the following parameters have been computed for the first longitudinal and first torsional mode of alloy 305 (see Table 1):

Longitudinal mode: $\nu_1 = 1994$ Hz, $\eta_1 = 8\cdot40 \times 10^{-5}$,
$M_1 = 0\cdot229$ kg, $R_1 = 0\cdot241$ ohm,
$C_1 = 27\cdot8 \times 10^{-9}$ m/N.

Torsional mode: $\nu_1 = 1516$ Hz, $\eta_1 = 5\cdot24 \times 10^{-5}$,
$I_1 = 5\cdot30 \times 10^{-6}$ kg m^2, $R_1 = 0\cdot264 \times 10^{-6}$ ohm,
$C_1 = 20\cdot8 \times 10^{-4}$ N^{-1}.