AN ELECTROMAGNETIC FORCE THEOREM FOR QUASI-STATIONARY CURRENTS

By W. E. SMITH*

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Summary

It is shown that the time-averaged magnetic forces produced in a system excited by quasi-stationary currents may be expressed in terms of derivatives of effective inductance parameters in a manner similar to that of magnetostatics. Practical applications of the theorem include electromagnetic levitation, the absolute calibration of electrical measuring instruments, and the assessment of mechanical stresses in magnet assemblies.

I. INTRODUCTION

In magnetostatic theory the following equation for the generalized force F_x resulting from the action of a number of current-carrying loops is well known (being derived ultimately from the Ampere law of force†):

$$F_x = \tfrac{1}{2}\sum_{n,m} j_n\, j_m\, \partial L_{nm}/\partial x, \tag{1}$$

where j_n are loop currents, $L_{nm} = L_{mn}$ are coefficients of self or mutual inductance, and x is the appropriate generalized coordinate. For a system excited by a single current j, equation (1) reduces to

$$F_x = \tfrac{1}{2}j^2\, \partial L/\partial x, \tag{2}$$

where L is the self inductance. Some previous investigations (Smith 1960, 1961, 1964) showed how equations (1) and (2) could be extended for finding time-averaged generalized forces in systems excited by varying currents. Simple results were obtained for loss-free systems and for some restricted lossy systems.

A lossy system of some practical interest not included by these extensions occurs in the theory of electromagnetic levitation (Okress *et al.* 1952). Here, eddy currents are induced in a conducting body by the alternating magnetic field from an excitation coil. The average force exerted between these eddy currents and the original excitation current is used to levitate the body. An examination (Smith 1965) of the problem of a ball on the axis of an axially symmetric coil system using a detailed field solution (Brisley and Thornton 1963; Fomin 1964) showed that

* Department of Applied Mathematics, University of New South Wales, Kensington, N.S.W.

† Most treatments use energy principles to obtain expressions for generalized forces (e.g. Smythe 1950, p. 320) although this is not necessary.

the resulting axial levitation force $\bar{F}$ could be written in the form, equivalent to equation (2),

$$\bar{F} = \tfrac{1}{2} I^* I \, \partial L / \partial h,$$

where I is the complex root-mean-square (r.m.s.) excitation current of angular frequency ω, L is the effective inductance at this frequency, and h is the separation of the ball and coil. Further work using a circuit-theory representation for the induced eddy currents (Smith, unpublished data) has indicated that the above result is a special case of a general extension of the magnetostatic equations (1) and (2) to quasi-stationary eddy-current systems. The general analysis to be presented confirms this view.

The following generalization of equation (2) will be shown to hold. The average generalized force $\bar{F}_x$ exerted by a current system in the presence of fixed conducting bodies may be written

$$\bar{F}_x = \tfrac{1}{2} I^* I \, \partial L / \partial x, \tag{3}$$

where x is a generalized coordinate of the system, I is the complex r.m.s. coil current of angular frequency ω, and L is the effective inductance of the coil at this frequency. When there are several independent currents I_m, the generalization of equation (1) is

$$\bar{F}_x + \bar{F}'_x = \sum_{n,m} I_n^* (\partial L_{nm} / \partial x) I_m, \tag{4}$$

where the L_{nm} are effective coefficients of inductance, and $\bar{F}'_x$ is the average force under the excitation I_m^* rather than I_m (i.e. the relative phases of different currents are reversed). Changes in the generalized coordinate x may involve either gross movements or internal distortions of the coil system, or both. According to the nature of x, equations (3) and (4) may give levitation-type forces, internal loop forces, or combinations.

Equations (3) and (4) were deduced for a circuit-theory model of eddy currents (Smith, unpublished data). In the present work they are obtained as reductions of the following general field equation:

$$\bar{F}_x + \bar{F}'_x = (\partial / \partial x)_I \, \mathrm{Re}\left(\int_V \mathbf{A} . \mathbf{J}^* \, d\tau \right),$$

where $\mathbf{A}$ is the complex r.m.s. vector potential, in a fixed medium, arising from current sources described by the complex r.m.s. current density $\mathbf{J}$, and the differentiation is to be carried through in a specified way corresponding to constant current for current loops. The result is derived from the usual quasi-stationary approximation (neglect of the Maxwell displacement current) for fields in the presence of material of high conductivity (electric field negligible except for the production of conduction currents).

Initially a single excitation frequency is considered and the results are extended to general time dependence.

II. Basic Field Equations and the Green's Tensor Solution

All field quantities are taken as having the time dependence $\exp(\mathrm{i}\omega t)$ corresponding to a single frequency of excitation. It will be convenient to work with r.m.s. amplitudes throughout. The entire field distribution is produced by the current density $\mathbf{J}$ as source. Eddy currents and magnetization currents may then be induced in the material medium present by these fields.

We neglect the Maxwell displacement current (retardation effects) and make the further approximation appropriate to eddy-current systems that the electric field is produced by changes in the magnetic induction and is negligible except in so far as it gives rise to conduction currents. In this approximation the Maxwell equations become (in MKS rationalized units)

$$\nabla\times\mathbf{H} = \mathbf{J}^{(\mathrm{t})}, \tag{5a}$$

$$\nabla.\mathbf{B} = 0, \tag{5b}$$

where $\mathbf{B}$ is the complex magnetic induction, $\mathbf{H}$ is the complex magnetic intensity, and $\mathbf{J}^{(\mathrm{t})}$, the total current density, is given by

$$\mathbf{J}^{(\mathrm{t})} = \mathbf{J}+\mathbf{J}^{(\mathrm{e})}, \tag{6}$$

where $\mathbf{J}^{(\mathrm{e})}$ is the induced conduction-current density, the induced magnetization currents having already been included in $\mathbf{H}$. Introduction of the vector potential $\mathbf{A}$, by

$$\mathbf{B} = \nabla\times\mathbf{A}, \tag{7a}$$

$$\nabla.\mathbf{A} = 0, \tag{7b}$$

gives (Stratton 1941, p. 346; Smythe 1950, Ch. IX)

$$\mathbf{J}^{(\mathrm{e})} = \sigma\mathbf{E} = -\mathrm{i}\omega\sigma\mathbf{A}, \tag{8}$$

where $\mathbf{E}$ is the electric field and σ the conductivity. Further the usual linear constitutive equation

$$\mathbf{B} = \mu\mathbf{H} \qquad (\mu \neq 0) \tag{9}$$

is assumed.

For mathematical convenience any discontinuous material boundary surfaces are supposed to be replaced by regions having large but finite gradients of the material constants.

The entire field is expressible in terms of the vector potential $\mathbf{A}$. By use of equations (6), (7a), (8), and (9) in equation (5a) the following differential equation for $\mathbf{A}$ in terms of the source currents $\mathbf{J}$ is obtained:

$$\nabla\times(1/\mu)\nabla\times\mathbf{A}+\mathrm{i}\omega\sigma\mathbf{A} = \mathbf{J}. \tag{10}$$

The differential equation (10) determines $\mathbf{A}$ when a suitable surface boundary condition, amounting to sufficiently rapid vanishing of $\mathbf{A}$ at infinity, is imposed ($\mathbf{A}\to 0$ as $1/r$ and $\nabla\times\mathbf{A}\to 0$ as $1/r^2$, or faster). The material constants μ and σ are functions of position and may also depend on frequency. The solution to equation (10) is supposed to be expressible in terms of a Green's tensor $\mathbf{G}_k(\mathbf{r}, \mathbf{r}')$ which satisfies the equation

$$\nabla\times(1/\mu)\nabla\times\mathbf{G}_k(\mathbf{r}, \mathbf{r}')+\mathrm{i}\omega\sigma\mathbf{G}_k(\mathbf{r}, \mathbf{r}') = \mathbf{e}_k\,\delta(\mathbf{r}-\mathbf{r}'), \tag{11}$$

where $\mathbf{e}_k$ is a unit vector of the orthogonal coordinate system and $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function. $\mathbf{G}_k$ has the same surface boundary condition as $\mathbf{A}$. Clearly, $\mathbf{G}_k(\mathbf{r}, \mathbf{r}')$ is the solution of equation (10) for a point source in the direction $\mathbf{e}_k$ at the point $\mathbf{r} = \mathbf{r}'$, so the solution may be written

$$\mathbf{A}(\mathbf{r}) = \sum_k \int_V \mathbf{G}_k(\mathbf{r}, \mathbf{r}')\,J_k(\mathbf{r}')\,\mathrm{d}\tau', \tag{12}$$

where $J_k(\mathbf{r}')$ is the kth component of $\mathbf{J}(\mathbf{r}')$.

(a) *Reciprocity and the Symmetry of the Green's Tensor*

The Green's tensor of equation (11) possesses an important symmetry property expressing the reciprocity of the system. This symmetry is required for the evaluation of the average force $\overline{F}_x$ in Section III. Consider two Green's tensors, $\mathbf{G}_k(\mathbf{r}, \mathbf{r}')$ satisfying equation (11) and $\mathbf{G}_l(\mathbf{r}, \mathbf{r}'')$ satisfying

$$\nabla\times(1/\mu)\nabla\times\mathbf{G}_l(\mathbf{r}, \mathbf{r}'')+\mathrm{i}\omega\sigma\mathbf{G}_l(\mathbf{r}, \mathbf{r}'') = \mathbf{e}_l\,\delta(\mathbf{r}-\mathbf{r}'').$$

Then,

$$\mathbf{G}_k(\mathbf{r}, \mathbf{r}').\nabla\times(1/\mu)\nabla\times\mathbf{G}_l(\mathbf{r}, \mathbf{r}'')+\mathrm{i}\omega\sigma\mathbf{G}_k(\mathbf{r}, \mathbf{r}').\mathbf{G}_l(\mathbf{r}, \mathbf{r}'') = \mathbf{e}_l.\mathbf{G}_k(\mathbf{r}, \mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}'') \tag{13}$$

and,

$$\mathbf{G}_l(\mathbf{r}, \mathbf{r}'').\nabla\times(1/\mu)\nabla\times\mathbf{G}_k(\mathbf{r}, \mathbf{r}')+\mathrm{i}\omega\sigma\mathbf{G}_k(\mathbf{r}, \mathbf{r}').\mathbf{G}_l(\mathbf{r}, \mathbf{r}'') = \mathbf{e}_k.\mathbf{G}_l(\mathbf{r}, \mathbf{r}'')\,\delta(\mathbf{r}-\mathbf{r}'). \tag{14}$$

Subtraction of (14) from (13) and an integration over the volume V of the system yields

$$\int_V \{\mathbf{G}_k(\mathbf{r}, \mathbf{r}').\nabla\times(1/\mu)\nabla\times\mathbf{G}_l(\mathbf{r}, \mathbf{r}'')-\mathbf{G}_l(\mathbf{r}, \mathbf{r}'').\nabla\times(1/\mu)\nabla\times\mathbf{G}_k(\mathbf{r}, \mathbf{r}')\}\,\mathrm{d}\tau$$
$$= \mathbf{e}_l.\mathbf{G}_k(\mathbf{r}'', \mathbf{r}')-\mathbf{e}_k.\mathbf{G}_l(\mathbf{r}', \mathbf{r}''). \tag{15}$$

But the following vector identity may be noted:

$$\nabla.\{a(\nabla\times\mathbf{C})\times\mathbf{B}\} = \mathbf{B}.\nabla\times a\nabla\times\mathbf{C}-a(\nabla\times\mathbf{C}).(\nabla\times\mathbf{B}),$$

whence

$$\nabla.\{a(\mathbf{C}\times\nabla\times\mathbf{B}-\mathbf{B}\times\nabla\times\mathbf{C})\} = \mathbf{B}.\nabla\times a\nabla\times\mathbf{C}-\mathbf{C}.\nabla\times a\nabla\times\mathbf{B}.$$

Application of the Gauss divergence theorem gives the following vector Green's theorem which is a generalization of that used by Stratton (1941, Sec. 4.14),

$$\int_V (\mathbf{B}.\nabla\times a\nabla\times\mathbf{C}-\mathbf{C}.\nabla\times a\nabla\times\mathbf{B})\,\mathrm{d}\tau = \int_S a(\mathbf{C}\times\nabla\times\mathbf{B}-\mathbf{B}\times\nabla\times\mathbf{C}).\mathrm{d}\mathbf{S},$$

where the surface S encloses the volume V. Use of this theorem in equation (15) gives

$$\int_S \{(1/\mu)\mathbf{G}_l(\mathbf{r},\mathbf{r}'')\times\nabla\times\mathbf{G}_k(\mathbf{r},\mathbf{r}')-(1/\mu)\mathbf{G}_k(\mathbf{r},\mathbf{r}')\times\nabla\times\mathbf{G}_l(\mathbf{r},\mathbf{r}'')\}.\mathrm{d}\mathbf{S}$$
$$= \mathbf{e}_l.\mathbf{G}_k(\mathbf{r}'',\mathbf{r}')-\mathbf{e}_k.\mathbf{G}_l(\mathbf{r}',\mathbf{r}''),$$

and the surface integral vanishes from the boundary conditions imposed on equation (10) as the surface S becomes remote. Therefore

$$\mathbf{e}_l.\mathbf{G}_k(\mathbf{r}'',\mathbf{r}') = \mathbf{e}_k.\mathbf{G}_l(\mathbf{r}',\mathbf{r}''), \tag{16}$$

which is the required symmetry result. Physically equation (16) corresponds to the reciprocity of an interchange of source and field point. If $k \neq l$ there is a further vector symmetry expressed by equation (16). Equation (16) is an example of very general reciprocity principles for irreversible electromagnetic systems (Meixner 1963).

III. The Average Generalized Force

(a) *Work Done in an Adiabatic Displacement* δx

The average generalized force $\bar{F}_x$ is defined by saying that $\bar{F}_x\,\delta x$ is the work done in a small virtual displacement generated by the change δx of the generalized coordinate x. This small displacement is to be thought of as being carried out sufficiently slowly to be dynamically adiabatic. The displacement of the current system generated by δx is specified at each point by the vector function $\delta\mathbf{r}(\mathbf{r})$. That is, the current at $\mathbf{r}$ is displaced $\delta\mathbf{r}(\mathbf{r})$ during the displacement δx. (There may also be some other change in $\mathbf{J}$ required in conjunction with this displacement to satisfy the conditions of the theorem.) The vector function $\delta\mathbf{r}$ specifies the displacement field defining the nature of the coordinate variation δx and the corresponding generalized force F_x. For example, if $\delta\mathbf{r}$ is a constant vector the current system undergoes a pure translation, and the corresponding generalized force is an actual force in the direction of the translation.

Electric forces are assumed negligible so that the average force density operating on the current system is $\mathrm{Re}(\mathbf{J}^*\times\mathbf{B})$. The work done by the generalized force during the displacement δx then becomes

$$\delta x\,\bar{F}_x = \mathrm{Re}\left(\int_V \delta\mathbf{r}.\mathbf{J}^*\times\mathbf{B}\,\mathrm{d}\tau\right), \tag{17}$$

where V is the whole volume in which source currents flow. Equation (17) may also be written

$$\delta x\, \overline{F}_x = \mathrm{Re}\left(\int_V \delta\mathbf{r}\times\mathbf{J}^*.\mathbf{B}\,\mathrm{d}\tau\right)$$

$$= \mathrm{Re}\left(\int_V \delta\mathbf{r}\times\mathbf{J}^*.\nabla\times\mathbf{A}\,\mathrm{d}\tau\right), \qquad \text{using (7a),}$$

$$= \mathrm{Re}\left(\int_V [\mathbf{A}.\nabla\times(\delta\mathbf{r}\times\mathbf{J}^*)+\nabla.\{\mathbf{A}\times(\delta\mathbf{r}\times\mathbf{J}^*)\}]\,\mathrm{d}\tau\right)$$

$$= \mathrm{Re}\left(\int_V \mathbf{A}.\nabla\times(\delta\mathbf{r}\times\mathbf{J}^*)\,\mathrm{d}\tau + \int_S \mathbf{A}\times(\delta\mathbf{r}\times\mathbf{J}^*).\mathrm{d}\mathbf{S}\right),$$

by Gauss' theorem, where S encloses the volume V,

$$= \mathrm{Re}\left(\int_V \mathbf{A}.\nabla\times(\delta\mathbf{r}\times\mathbf{J}^*)\,\mathrm{d}\tau\right), \tag{18}$$

since the surface integral vanishes,

$$= \mathrm{Re}\left(\int_V \mathbf{A}.\{(\mathbf{J}^*.\nabla)\delta\mathbf{r}-(\delta\mathbf{r}.\nabla)\mathbf{J}^*-\mathbf{J}^*\,(\nabla.\delta\mathbf{r})\}\,\mathrm{d}\tau\right), \tag{19}$$

where use has been made of the fact that $\mathbf{J}$ satisfies the continuity equation $\nabla.\mathbf{J} = 0$. For the moment suppose $\delta\mathbf{r}$ to be constant, representing a pure translation of the current system. The only term which contributes to the right-hand side of equation (19) is the one involving $\{-(\delta\mathbf{r}.\nabla)\mathbf{J}^*\}$, which may be written $\delta\mathbf{J}^*$ where $\delta\mathbf{J}$ is the change of source current density generated by translation of the current system *without change of current magnitudes*. For currents confined to conducting loops, this condition corresponds to constant current in the loops during the displacement. Equation (19) is then simply

$$\delta x\, \overline{F}_x = \mathrm{Re}\left(\int_V \mathbf{A}.\delta\mathbf{J}^*\,\mathrm{d}\tau\right). \tag{20}$$

The concept of constant current may be broadened so that equation (20) is appropriate to general deformations of the current system. The required generalization is that the flux of $\mathbf{J}$ through any surface moving with the displaced current system shall be constant.

To show this, observe that $\mathbf{J}$ is always solenoidal, so that

$$\nabla.\delta\mathbf{J} = 0,$$

where $\delta\mathbf{J}$ is the change of current density generated by δx. Consequently $\delta\mathbf{J}$ may be derived from the curl of a vector $\delta\boldsymbol{\xi}$, thus,

$$\delta\mathbf{J} = \nabla\times\delta\boldsymbol{\xi}.$$

For constant flux of $\mathbf{J}$ through an arbitrary surface S,

$$\delta\int_S \mathbf{J}.\mathrm{d}\mathbf{S} = 0$$

or
$$\int_S (\nabla\times\delta\boldsymbol{\xi}).\mathrm{d}\mathbf{S}+\int_{\delta S}\mathbf{J}.\mathrm{d}\mathbf{S}=0,$$
where the integral $\int_{\delta S}$ takes account of the change of the surface S during the deformation. Then by Stokes' theorem,
$$\oint_s \delta\boldsymbol{\xi}.\mathrm{d}\mathbf{s}+\int_{\delta S}\mathbf{J}.\mathrm{d}\mathbf{S}=0, \tag{21}$$
where s is a closed path bounding the surface S. However, the second integral in equation (21) may be written
$$\oint_s \mathbf{J}.(\delta\mathbf{r}\times\mathrm{d}\mathbf{s}) = -\oint_s \delta\mathbf{r}\times\mathbf{J}.\mathrm{d}\mathbf{s},$$
therefore,
$$\oint_s (\delta\boldsymbol{\xi}-\delta\mathbf{r}\times\mathbf{J}).\mathrm{d}\mathbf{s}=0, \qquad \text{for all closed paths } s,$$
and $\delta\boldsymbol{\xi}$ and $\delta\mathbf{r}\times\mathbf{J}$ differ only by the gradient of a scalar function, giving
$$\nabla\times\delta\boldsymbol{\xi}=\nabla\times(\delta\mathbf{r}\times\mathbf{J}),$$
that is
$$\delta\mathbf{J}=\nabla\times(\delta\mathbf{r}\times\mathbf{J}).$$
Substitution in equation (18) gives
$$\bar{F}_x\,\delta x=\mathrm{Re}\left(\int_V \mathbf{A}.\delta\mathbf{J}^*\,\mathrm{d}\tau\right), \tag{22}$$
where $\delta\mathbf{J}^*$ has the generalized constant current interpretation. For currents in loops, distortions and displacements of the loops may be performed but with constant loop currents.

(*b*) *Evaluation of the Generalized Force*

Equation (22) is still not in a suitable form for the evaluation of the generalized force. The Green's tensor symmetry of Section II will be used to express equation (22) in a more suitable form. Consider
$$\delta\int_V \mathbf{A}.\mathbf{J}^*\,\mathrm{d}\tau=\int_V \delta\mathbf{A}.\mathbf{J}^*\,\mathrm{d}\tau+\int_V \mathbf{A}.\delta\mathbf{J}^*\,\mathrm{d}\tau. \tag{23}$$
From equation (12), we have
$$\delta\mathbf{A}(\mathbf{r})=\sum_k\int_V \mathbf{G}_k(\mathbf{r},\mathbf{r}')\,\delta J_k(\mathbf{r}')\,\mathrm{d}\tau',$$
$\delta\mathbf{G}_k$ being zero since the configuration of the material medium is not altered by the change δx. Therefore,
$$\int_V \delta\mathbf{A}.\mathbf{J}^*\,\mathrm{d}\tau=\int_V\left\{\mathbf{J}^*(\mathbf{r}).\sum_k\int_V \mathbf{G}_k(\mathbf{r},\mathbf{r}')\,\delta J_k(\mathbf{r}')\,\mathrm{d}\tau'\right\}\mathrm{d}\tau$$
$$=\int_V \mathbf{A}'.\delta\mathbf{J}\,\mathrm{d}\tau,$$

where

$$\mathbf{A}'(\mathbf{r}) = \sum_k \int_V \mathbf{G}_k(\mathbf{r}, \mathbf{r}')\, \mathbf{J}_k^*(\mathbf{r}')\, \mathrm{d}\tau',$$

using the reciprocity of the Green's tensor (equation (16)). $\mathbf{A}'$ is the vector potential resulting from the current system $\mathbf{J}^*$ rather than $\mathbf{J}$. Since $\mathbf{G}_k$ is complex in general, $\mathbf{A}'$ and $\mathbf{A}^*$ are not equal. Equation (23) becomes

$$\delta \int_V \mathbf{A}.\mathbf{J}^*\, \mathrm{d}\tau = \int_V \mathbf{A}'.\delta\mathbf{J}\, \mathrm{d}\tau + \int_V \mathbf{A}.\delta\mathbf{J}^*\, \mathrm{d}\tau, \tag{24}$$

and comparison of equations (22) and (24) gives

$$\delta x(\overline{F}_x + \overline{F}'_x) = \mathrm{Re}\left(\delta \int_V \mathbf{A}.\mathbf{J}^*\, \mathrm{d}\tau \right), \tag{25}$$

where $\overline{F}'_x$ is the average generalized force produced by the current sources $\mathbf{J}^*$ rather than $\mathbf{J}$, and the variation of the right-hand side is such as to preserve the source current flux through displaced circuits. Alternatively,

$$\overline{F}_x + \overline{F}'_x = (\partial/\partial x)_I\, \mathrm{Re}\left(\int_V \mathbf{A}.\mathbf{J}^*\, \mathrm{d}\tau \right). \tag{26}$$

Equations (25) and (26) are the basic field results of the paper.

The usual application relates to currents I_n flowing through closed loops, in which case equations (25) and (26) reduce to a simple form involving inductance coefficients. Thus

$$\delta \int_V \mathbf{A}.\mathbf{J}^*\, \mathrm{d}\tau = \sum_n I_n^*(\delta \oint \mathbf{A}.\mathrm{d}\mathbf{s}_n),$$

where $\oint \mathrm{d}\mathbf{s}_n$ denotes integration around the nth loop, and using Stokes' theorem,

$$\delta \int_V \mathbf{A}.\mathbf{J}^*\, \mathrm{d}\tau = \sum_n I_n^*\left(\delta \int_{S_n} \nabla\times\mathbf{A}.\mathrm{d}\mathbf{S} \right),$$

where S_n caps the nth loop,

$$= \sum_n I_n^*\left(\delta \int_{S_n} \mathbf{B}.\mathrm{d}\mathbf{S} \right)$$

$$= \sum_n I_n^*\, \delta\phi_n, \qquad \left(\phi_n = \int_{S_n} \mathbf{B}.\mathrm{d}\mathbf{S}\right) \tag{27}$$

where ϕ_n is the complex r.m.s. flux of magnetic induction through the nth loop. Notice that changes in size and shape of each loop are allowed for in the variation $\delta\phi_n$ of equation (27). But

$$\phi_n = \sum_m Q_{nm}\, I_m,$$

where the Q_{nm} are circuit coupling coefficients, the real parts L_{nm} of which are coefficients of effective self or mutual inductance. Hence, for constant current variations,

$$\delta\phi_n = \sum_m \delta Q_{nm} I_m.$$

It is easily shown from the results of Section II(a) that the usual reciprocity $Q_{nm} = Q_{mn}$ applies, therefore,

$$\mathrm{Re}\left(\delta \int_V \mathbf{A}.\mathbf{J}^* \,\mathrm{d}\tau\right) = \sum_{n,m} I_n^* \delta L_{nm} I_m,$$

and equation (26) becomes

$$\overline{F}_x + \overline{F}'_x = \sum_{n,m} I_n^* (\partial L_{nm}/\partial x) I_m, \tag{28}$$

which is the required result. For coils excited by a single current I,

$$\overline{F}_x = \overline{F}'_x$$

and

$$\overline{F}_x = \tfrac{1}{2} I^* I \,\partial L/\partial x. \tag{29}$$

If x is the actual separation of a body from a coil system, $\overline{F}_x$ becomes the separating force in this direction giving the force used for electromagnetic levitation (Smith 1965, and unpublished data).

Equation (28) and its antecedent, equation (26), involve the time-symmetric combination $(\overline{F}_x + \overline{F}'_x)$. The individual forces $\overline{F}_x$ and $\overline{F}'_x$ are not usually expressible as simple terminal forms like equation (29) but involve internal details of the system (Smith 1960, and unpublished data). Equation (22) gives

$$\begin{aligned}(\overline{F}_x - \overline{F}'_x)\,\delta x &= \int_V (\mathbf{A}^* - \mathbf{A}').\delta\mathbf{J}\,\mathrm{d}\tau \\ &= \int_V \left\{\delta\mathbf{J}(\mathbf{r}).\sum_k \int_V \{\mathbf{G}_k^*(\mathbf{r}, \mathbf{r}') - \mathbf{G}_k(\mathbf{r}, \mathbf{r}')\} J_k^*(\mathbf{r}')\,\mathrm{d}\tau'\right\}\mathrm{d}\tau,\end{aligned}$$

by use of equation (12), and the difference in the forces depends directly on the imaginary part of the Green's tensor $\mathbf{G}_k$. For loss-free systems having $\sigma = 0$ the Green's tensor is real and $\overline{F}_x = \overline{F}'_x$.

IV. Discussion

The results obtained are basically generalizations of the magnetostatic results of equation (1) to alternating current systems in which eddy currents may flow. The magnetostatic results are recovered from equation (28) by considering the limit of zero frequency ($\omega \to 0$). The magnetostatic results obtained in this way do not rely on field energy conservation. However, there is a restriction on the type of generalized coordinate contemplated, the case of deformation of magnetic or conducting material being excluded from the present derivation.

The limit $\sigma \to 0$ gives the loss-free results already established (Smith 1960, 1961). It should be observed that the present derivation allows for the dispersion present when μ and σ are functions of frequency. This situation was not covered by the time-averaging of magnetostatic results employed in the author's initial paper (Smith 1960).

The extension to arbitrary time dependence is trivial. The average force for sinusoidal excitation is quadratic in the current amplitudes so that it follows from Parseval's formula of Fourier analysis that the forces from individual Fourier components are additive. The total force is thus computed by making a spectral resolution of the excitation currents and summing over all frequencies.

The results obtained have applications in electromagnetic levitation (Smith 1965, and unpublished data) and in the absolute calibration of electrical measuring instruments (Smith 1960, and unpublished data). They may also be useful for finding mechanical stresses in inductor or magnet assemblies.

Equation (28) was obtained for a circuit-theory model of electromagnetic eddy-current systems using the time-averaged magnetostatic results (Smith, unpublished data). The appropriateness of this coupled-circuit approach to eddy currents in this case has been verified by the present derivation. The concept of constant current now developed has clarified the requirements to be imposed on the excitation currents. At the same time systems containing dispersive material have been included quite naturally.

V. References

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