TWO-STREAM INSTABILITIES IN A PLASMA

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Summary

A phenomenological scattering formula is used to investigate the effect of collision on the two-stream instabilities in a plasma of interstreaming electrons and ions. The result is that the enhanced Landau damping due to thermalized electron energy competes with the growth of the instability.

Introduction

When two interpenetrating streams in a plasma have relative mass motion, instability is produced by a klystron bunching effect. A perturbation in one stream thereby produces a spatial bunching of the second. This bunching can be spatially in phase or out of phase with the perturbing wave, depending on its wavelength and the relative velocity of the streams. This type of instability was first investigated by Haeff (1949) and by Pierce (1949) using a macroscopic moment equation. An examination of this interstreaming instability was also made by Jackson (1960) using the Boltzmann–Vlasov equation. Tidman (1961) considered the effect of a two-body scattering in the instability using the Fokker–Planck scattering formula. In general there exists a range of wave numbers for instability when the relative velocity exceeds the thermal velocity of the streams.

In the present paper we consider the effect of two-body scattering on the instability with the use of a phenomenological scattering formula developed by Bhatnagar, Gross, and Krook (1954). This scattering formula, which does not contain the diffusion nature in velocity space, may account for the gradual heating of the electron stream due to distant encounters with the Coulomb field of the ions at rest. It is shown that, as the kinetic energy of the stream is thermalized, the resulting enhanced Landau damping competes with the growth of the unstable perturbation.

Basic Equations and Assumptions

The Boltzmann equations for the electron distribution function $f(\mathbf{r}, \mathbf{v}, t)$ and ion distribution function $F(\mathbf{r}, \mathbf{v}, t)$ are given by

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}-\frac{e}{m}\mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{v}}=\left(\frac{\delta f}{\delta t}\right)_{\text{collision}}, \tag{1}$$

and

$$\frac{\partial F}{\partial t}+\mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{r}}+\frac{e}{M}\mathbf{E}\cdot\frac{\partial F}{\partial \mathbf{v}}=\left(\frac{\delta F}{\delta t}\right)_{\text{collision}}, \tag{2}$$

where m and M are the electron and ion masses with charge $-e$ and $+e$ respectively.

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The electric field $\mathbf{E}$ is also given by

$$\nabla \,.\, \mathbf{E} = 4\pi e \int (F-f)\, d\mathbf{v}. \tag{3}$$

We put

$$f = f_0 + f_1, \qquad F = F_0 + F_1, \qquad \text{and} \qquad E = E_0 + E_1 \quad \text{with } E_0 = 0, \tag{4}$$

where suffix 1 corresponds to a small perturbation.

Considering first the electron distribution function, this gives, on using the model of Bhatnagar, Gross, and Krook (1954),

$$\left(\frac{\delta f}{\delta t}\right)_{\text{collision}} = \lambda(\phi_1 - f), \tag{5}$$

with $\phi = n_0(m/2\pi k_1 T)^{3/2}\exp(-mv^2/2k_1 T)$, k_1 being the Boltzmann constant, and λ the collision frequency;

$$\frac{\partial f_0}{\partial t} = \lambda(\phi - f_0); \tag{6}$$

and (on neglecting the non-linear terms)

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \,.\, \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m}\mathbf{E}_1 \,.\, \frac{\partial f_0}{\partial \mathbf{v}} = \lambda(\phi_1 - \phi - f_1), \tag{7}$$

where

$$\phi_1 = n(m/2\pi k_1 T)^{3/2}\exp\{-m|\mathbf{v}-\mathbf{q}|^2/2k_1 T\},$$

with

$$\left.\begin{aligned} n &= n_0 + \nu, & \mathbf{q} &= (1/n)\int \mathbf{v} f_1\, d\mathbf{v}, \\ \nu &= \int f_1\, d\mathbf{v}, & \nabla \,.\, \mathbf{E}_1 &= -4\pi e \nu. \end{aligned}\right\} \tag{8}$$

Similar expressions are obtained for the ion distribution function.

The solution of (6) gives

$$f_0(t) = \phi + \{f_0(0) - \phi\}\exp(-\lambda t). \tag{9}$$

From (7), on using (8) and neglecting terms containing products and higher powers of ν, $\mathbf{q}$, $\mathbf{E}_1$, and f_1, we obtain

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \,.\, \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m}\mathbf{E}_1 \,.\, \frac{\partial f_0}{\partial \mathbf{v}} = \lambda\left\{\left(\frac{\nu}{n_0} + \frac{m}{k_1 T}\mathbf{v} \,.\, \mathbf{q}\right)\phi - f_1\right\}. \tag{10}$$

Also, from the condition of charge conservation,

$$\frac{\partial \nu}{\partial t} + n_0 \frac{\partial \,.\, \mathbf{q}}{\partial \mathbf{r}} = 0, \tag{11}$$

where

$$\mathbf{q} = (1/n_0)\int \mathbf{v} f_1\, d\mathbf{v}. \tag{12}$$

Transformed Equations and the Initial Conditions

We use the following Fourier space transform and Laplace time transform defined by

$$A_w = \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{r} \int_0^{\infty} \mathrm{d}t \exp\{\mathrm{i}(wt-\mathbf{k}\,.\,\mathbf{r})\}A, \tag{13}$$

with $\mathrm{Im}(w) > 0$, and find from (10), after using (11) and (12),

$$\begin{aligned} {}_{1w} = \frac{1}{\lambda+\mathrm{i}(kU_x-w)}\Bigg[\bigg\{\int_{-\infty}^{\infty} \exp(-\mathrm{i}\mathbf{k}\,.\,\mathbf{r})f_1(0)\,\mathrm{d}\mathbf{r}\bigg\}_x \\ -\frac{e}{m}\frac{m\phi}{k_1 T}v_x(E_{1w})_x + \frac{e}{m}\Big\{E_{1(w+\mathrm{i}\lambda)}\Big\}_x\bigg\{\frac{\partial f_0(0)}{\partial v}+\frac{m\phi}{k_1 T}v_x\bigg\} \\ +\frac{\lambda\phi}{n_0}\nu_w + \frac{\lambda m\phi}{k_1 T}v_x\bigg\{\frac{w}{n_0 k}\nu_w + \frac{1}{\mathrm{i}kn_0}\bigg(\int \nu(0)\exp(-\mathrm{i}\mathbf{k}\,.\,\mathbf{r})\,\mathrm{d}\mathbf{r}\bigg)_x\bigg\}\Bigg], \end{aligned} \tag{14}$$

where $\mathbf{k}$ is in the x direction and the suffix x denotes the x component.

We use the following initial conditions:

$$f_0(0) = \tfrac{1}{2}n_0\{\delta(\mathbf{v}-\mathbf{U})+\delta(\mathbf{v}+\mathbf{U})\} \tag{15}$$

and

$$f_1(0) = \tfrac{1}{2}n_1\{\delta(\mathbf{v}-\mathbf{U})+\delta(\mathbf{v}+\mathbf{U})\}\exp(\mathrm{i}\mathbf{K}\,.\,\mathbf{r}), \tag{16}$$

where $\mathbf{U}$ is the velocity of the electron stream passing through an ion background, and $n_1 \ll n_0$. With these conditions relation (14) reduces to

$$A(x)\nu_w = B(x)\nu_{w+\mathrm{i}\lambda}+C(x), \tag{17}$$

where

$$A(x) = (x^2+w_\mathrm{p}^2-\lambda^2)/x^2,$$

$$B(x) = w_\mathrm{p}^2\{y^2(3x^2+y^2)/x^2(x^2+y^2)^2\},$$

and

$$C(x) = (2\pi)^3 n_1\,\delta(\mathbf{k}-\mathbf{K})\{x(x^2+y^2)^{-1}+(\lambda^2/x^2)\}, \tag{18}$$

with

$$x = \lambda-\mathrm{i}w, \qquad y^2 = k^2U_x^2, \qquad \text{and} \qquad w_\mathrm{p}^2 = (4\pi e^2 n_0/m). \tag{19}$$

Calculation of $\nu(t)$

In this section we consider two approximate cases.

(a) *Approximation where* $t \to \infty$

In this approximation $\nu_{w+\mathrm{i}\lambda} = 0$, and we subsequently find

$$\nu(t) = \nu(0)\,\frac{\mathrm{e}^{-\lambda t}}{2\pi\mathrm{i}} \int_{\mathrm{C}} \mathrm{d}x\,\mathrm{e}^{xt}\,\frac{x^3+\lambda(x^2+y^2)}{(x^2+y^2)(x^2+w_{\mathrm{p}}^2-\lambda^2)}$$

$$= \nu(0)\mathrm{e}^{-\lambda t}\left[\frac{\lambda}{(w_{\mathrm{p}}^2-\lambda^2)^{\frac{1}{2}}}\sin\{(w_{\mathrm{p}}^2-\lambda^2)^{\frac{1}{2}}t\} + \frac{w_{\mathrm{p}}^2-\lambda^2}{w_{\mathrm{p}}^2-k^2U_x^2-\lambda^2}\cos\{(w_{\mathrm{p}}^2-\lambda^2)^{\frac{1}{2}}t\} - \frac{k^2U_x^2}{w_{\mathrm{p}}^2-k^2U_x^2-\lambda^2}\cos(kU_x t)\right], \tag{20}$$

where we put $\nu(0) = n_1\exp(\mathrm{i}\mathbf{K}\,.\,\mathbf{r})$ and the corresponding inverse operator of (13), that is,

$$(2\pi)^{-4}\int_{-\infty}^{\infty}\mathrm{d}k\int_{\mathrm{C}}\mathrm{d}w\,\exp\{-\mathrm{i}(wt-\mathbf{k}\,.\,\mathbf{r})\} \tag{21}$$

is used. The integral over C goes from $-\infty$ to $+\infty$ above all singularities of ν_w.

(b) *Approximation where* $\nu_{w+\mathrm{i}\lambda}(x) = \nu_w(x)+\mathrm{i}\lambda\{\partial\nu_w(x)/\partial x\}$

Under this assumption, we find from (17)

$$(\partial\nu_w/\partial x)-\alpha(x)\nu_w = \beta(x), \tag{22}$$

where

$$\left.\begin{aligned}\alpha(x) &= -\,\frac{w_{\mathrm{p}}^2x^2(x^2-y^2)+(x^2+y^2)^2(x^2-\lambda^2)}{\mathrm{i}\lambda w_{\mathrm{p}}^2y^2(3x^2+y^2)},\\ \text{and}\qquad \beta(x) &= -(2\pi)^3\delta(\mathbf{k}-\mathbf{K})n_1\,\frac{(x^2+y^2)\{x^3+\lambda(x^2+y^2)\}}{\mathrm{i}\lambda w_{\mathrm{p}}^2y^2(3x^2+y^2)}.\end{aligned}\right\} \tag{23}$$

The solution of (22) is given by

$$\nu_w(x) = \int_x^{\infty}\mathrm{d}x'\,\beta(x')\exp\left\{-\int_x^{x'}\alpha(x'')\,\mathrm{d}x''\right\}. \tag{24}$$

We define

$$(2\pi)^3\delta(\mathbf{k}-\mathbf{K})n_1\sigma(X) = \beta(X)/\alpha(X), \tag{25}$$

and

$$\gamma(X) = 1/\lambda\alpha(X),$$

with $x^2/y^2 = X^2$, $w_{\mathrm{p}}^2/y^2 = W^2$, and $\lambda^2/y^2 = \epsilon^2$.

Thus (24) can be written in the form

$$\nu_w(X) = (2\pi)^3\delta(\mathbf{k}-\mathbf{K})n_1 \int_X^\infty \mathrm{d}X'\, \sigma(X')\, \frac{\mathrm{d}}{\mathrm{d}X'} \left[-\exp\left\{-\int_\infty^{X'} \frac{\mathrm{d}X''}{\lambda\gamma(X'')}\right\}\right]$$

$$= (2\pi)^3\delta(\mathbf{k}-\mathbf{K})n_1 \left[\gamma+\lambda\frac{\partial\sigma}{\partial X}\gamma+\lambda^2\frac{\partial}{\partial X}\left(\frac{\partial\sigma}{\partial X}\gamma\right)\gamma + \ldots + \lambda^n \frac{\partial}{\partial X}\left\{\ldots\frac{\partial}{\partial X}\left(\frac{\partial\sigma}{\partial X}\underbrace{\gamma\Big)\gamma\ldots\Big\}\gamma}_{n\ \gamma\text{'s}}+\ldots\right]. \tag{26}$$

In all orders of λ, poles are at zeros of

$$W^2X^2(X^2-1)+(X^2+1)^2(X^2-\epsilon^2) = 0, \tag{27}$$

that is,

$$X_1^2 = 1-(4/W^2)[1+O(W^{-2},\epsilon^2)],$$

$$X_2^2 = -W^2[1+(3/W^2)\{1+O(W^{-2},\epsilon^2)\}],$$

$$X_3^2 = -(\epsilon^2/W^2)[1+O(W^{-2},\epsilon^2)],$$

with $W > 1$ and $\epsilon < 1$.

Now we restrict to $t > 1$ and this will correspond to the pole

$$X = 1-(2/W^2)[1+O(W^{-2},\epsilon^2)]. \tag{28}$$

Define $s = X-1$, so that one gets on using (28)

$$\sigma(s) = \frac{1+2\epsilon}{W^2}\{1+O(W^{-2},\epsilon^2)\}\left[\frac{1}{s}+\text{const.}+Os+\ldots\right],$$

and

$$\gamma(s) = \left[\frac{2}{(1+W^{-2})}\{1+O(\epsilon^2W^{-2},W^{-4})\}\right]\frac{1}{s}(1-s+Os^2\ldots). \tag{29}$$

Thus we find

$$\nu_w(s) = (2\pi)^3\delta(\mathbf{k}-\mathbf{K})n_1\left(\frac{1+2\epsilon}{W^2}\right)\left[\frac{1}{s}+\epsilon\left\{\frac{(-1)}{s^3}\frac{2}{1+W^{-2}}(1-s+\ldots)\right\} + \epsilon^2\left\{\frac{(-1)(-3)}{s^5}\left(\frac{2}{1+W^{-2}}\right)^2\left(1-(1+\tfrac{2}{3})s\ldots\right)\right\}+\ldots\right]$$

$$= (2\pi)^3\delta(\mathbf{k}-\mathbf{K})n_1\left(\frac{1+2\epsilon}{W^2}\right)\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{2\epsilon}{1+W^{-2}}\right)^n\frac{(2n-1)!}{s^{2n+1}} - \sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)!}{3}\left(\frac{2\epsilon}{1+W^{-2}}\right)^n\frac{1}{s^{2n}} + O\sum_{n=1}^{\infty}\frac{1}{s^{2n+1}}\right]. \tag{30}$$

Finally, we get with the use of the inverse operator (21) and after performing the integrations

$$\nu(t) = \nu(0)(k^2U_x^2/w_p^2)\{1-(4/3)\lambda^2 kU_x t^3+O(\lambda^4)\ldots\}\exp\{(kU_x)t-(kU_x\lambda)t^2\}. \tag{31}$$

For the other limit, that is, $t < 1$, we have

$$\nu(t) = \nu(0)\frac{1}{2\pi i}\int_C \frac{e^{Xt}(X^2+1)\{X^3+\epsilon(X^2+1)\}}{(X+1)(X-1)(X^2+W^2)(X^2+\epsilon^2/W^2)}\,dX\,\{1+O(\epsilon)\}$$

$$= \nu(0)\left[\frac{2k^2U_x^2}{w_p^2}\cosh kU_x t+\left\{1+\frac{\lambda}{w_p^2}\tan w_p t\right\}\cos w_p t\right.$$

$$\left.-\frac{kU_x}{w_p}\sin\frac{\lambda}{w_p}kU_x t\left\{1-\frac{\lambda^2k^2U_x^2}{w_p^4}\cos\frac{\lambda kU_x}{w_p}t\right\}\right]. \tag{32}$$

Discussion

The expansion in terms of electron collision frequency leads to the terms in t. We now consider in general the expression (31). It can be seen that the term,

$$\{1-(4/3)\lambda^2 kU_x t^3+O(\lambda^4)\ldots\},$$

which results from the interaction between the electric field of the wave and the scattered electrons in the stream, will oppose the growth of the instability. Thus the wave will lose energy to these particles, and so this process can be assumed to be an enhanced Landau damping.

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