VECTOR TETRAD IN COSMOLOGICAL RED SHIFT

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Summary

The tetrad formalism is applied to certain cosmological models and it is found that the vector C_ν has the time component associated with Hubble's expansion.

I. Introduction

Mathematical formulation of the general theory of relativity and cosmology has been given by vector tetrad. This terminology has been used by McCrea and Mikhail (1956) to admit creation of matter in a steady-state universe. Singh and Verma (1962) have applied the formalism to the Schwarzschild interior and exterior solutions. We have used the contracted form of a third-order tensor, $\gamma^\mu_{\nu\sigma}$, a tensor associated with Ricci's coefficient of rotation, in certain cosmological models and found its correspondence with the gravitational red shift.

II. $\gamma^\mu_{\nu\sigma}$ in Tetrad

Four mutually orthogonal vectors are said to form an orthogonal tetrad. The vectors of a tetrad may be denoted by $\lambda^\mu_{(m)}$, where μ is the contravariant tensor index, and m (0, 1, 2, 3) is a tetrad suffix. With each event in space–time are associated four such contravariant vectors, and μ runs from 0 to 3. The suffix (m) is just a label and has nothing to do with the covariance or contravariance of the vector.

The cofactor of the contravariant vector $\lambda^\mu_{(m)}$ in the determinant $|\lambda^\mu_{(m)}|$ gives the covariant vector $\lambda^{(m)}_\mu$. The following relations are of importance.

$$\left.\begin{aligned}\lambda^\mu_{(m)}\,\lambda^{(n)}_\mu &= \delta^n_m,\\ \lambda^\mu_{(m)}\,\lambda^{(m)}_\nu &= \delta^\mu_\nu,\end{aligned}\right\}\tag{1}$$

where δ^n_m and δ^μ_ν are Kronecker delta symbols. The metric tensor in space–time is

$$\left.\begin{aligned}g_{\mu\nu} &= \lambda^{(m)}_\mu\,\lambda^{(m)}_\nu\\ \text{and}\qquad g^{\mu\nu} &= \lambda^\mu_{(m)}\,\lambda^\nu_{(m)}.\end{aligned}\right\}\tag{2}$$

A third-order tensor $\gamma^\mu_{\nu\sigma}$ is defined as

$$\begin{aligned}\gamma^\mu_{\nu\sigma} &= \lambda^\mu_{(m)}\,\lambda^{(m)}_{\nu}{}_{|\sigma}\\ &= \lambda^\mu_{(m)}[\lambda^{(m)}_{\nu,\sigma}-\Gamma^\rho_{\nu\sigma}\,\lambda^{(m)}_\rho],\end{aligned}\tag{3}$$

where ${}_{|\sigma}$ represents covariant differentiation, and

$$\Gamma^\rho_{\nu\sigma} = \tfrac{1}{2}g^{\rho a}[g_{\nu a,\sigma}+g_{a\sigma,\nu}-g_{\nu\sigma,a}].\tag{4}$$

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Using equations (4), (2), and (1), equation (3) gives

$$\gamma^{\mu}_{\nu\sigma} = \tfrac{1}{2}\lambda^{\mu}_{(m)}[\lambda^{(m)}_{\nu,\sigma} - \lambda^{(n)}_{a,\sigma}\lambda^{(n)}_{\nu}\lambda^{a}_{(m)} - \lambda^{(n)}_{a,\nu}\lambda^{(n)}_{\sigma}\lambda^{a}_{(m)}$$
$$-\lambda^{(m)}_{\sigma,\nu} + \lambda^{(n)}_{\nu,a}\lambda^{(n)}_{\sigma}\lambda^{a}_{(m)} + \lambda^{(n)}_{\nu}\lambda^{(n)}_{\sigma,a}\lambda^{a}_{(m)}]. \tag{5}$$

Contraction between the suffixes μ and ν of the tensor $\gamma^{\mu}_{\nu\sigma}$ gives

$$\gamma^{\mu}_{\mu\sigma} = 0, \tag{6}$$

while the contraction between μ and σ gives

$$C_{\nu} = \gamma^{\mu}_{\nu\mu} = \lambda^{\mu}_{(m)}[\lambda^{(m)}_{\nu,\mu} - \lambda^{(m)}_{\mu,\nu}]. \tag{7}$$

The vector C_{ν} has been used by Levi-Civita (1929), McCrea and Mikhail (1956), and Singh and Verma (1962), and it is important in the present work also.

III. Values of C_{ν} in Important Cosmological Models

(a) *Space–Time Conformally Flat*

The metric is given by (Synge 1960)

$$\mathrm{d}s^2 = [\omega(t)]^2(\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2), \tag{8}$$

where ω is a function of time. Then

$$g_{\mu\nu} = \mathrm{diag}\{[\omega(t)]^2;\ -[\omega(t)]^2;\ -[\omega(t)]^2;\ -[\omega(t)]^2\},$$
$$\lambda^{(n)}_{\nu} = \mathrm{diag}\{\omega(t);\ \mathrm{i}\omega(t);\ \mathrm{i}\omega(t);\ \mathrm{i}\omega(t)\},$$
$$\lambda^{\mu}_{(m)} = \mathrm{diag}\{1/\omega(t);\ -\mathrm{i}/\omega(t);\ -\mathrm{i}/\omega(t);\ -\mathrm{i}/\omega(t)\},$$

and

$$C_{\nu} = \{-3\omega'(t)/\omega(t);\ 0;\ 0;\ 0\}, \tag{9}$$

where i is the normal complex operator.

The cosmological red shift is given by (Synge 1960)

$$\rho = r_0\,\omega'(t)/\omega(t), \tag{10}$$

where r_0 is the optical distance for all observations made at t. The constant of proportionality comes out to be $-\frac{1}{3}C_0$, that is, $\frac{1}{3}$ times the value of the time component of the vector C_{ν} in the tetrad.

(b) *Spatially Isotropic Universe*

If we assume that the vector C_{ν} in the tetrad, is such that the time component is related to the red shift and the other three components have special isotropy, we should be able to explain the phenomena of a spatially isotropic universe in which the density of the stars is the same in all directions. Let us take the metric for such a universe as (Weber 1961)

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - y^2(1+ar^2)^{-2}[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2], \tag{11}$$

where y is a function of time alone, and a is a constant. Then

$$g_{\mu\nu} = \mathrm{diag}\{1;\ -y^2/(1+ar^2)^2;\ -y^2/(1+ar^2);\ -y^2/(1+ar^2)\},$$

$$\lambda_{\mu}^{(m)} = \mathrm{diag}\{1; iy/(1+ar^2); iy/(1+ar^2); iy/(1+ar^2)\},$$

and

$$C_{\nu} = \left\{-3\left(\frac{1}{y}\frac{\partial y}{\partial x^0}\right); \frac{4ar}{1+ar^2}\frac{\partial r}{\partial x^1}; \frac{4ar}{1+ar^2}\frac{\partial r}{\partial x^2}; \frac{4ar}{1+ar^2}\frac{\partial r}{\partial x^3}\right\}, \tag{12}$$

where

$$r^2 = (x^1)^2+(x^2)^2+(x^3)^2.$$

The form of the vector C_{ν} agrees with our assumption, as the time component C_0 corresponds to Hubble's expansion $(1/y)(\partial y/\partial x^0)$ and C_1, C_2, C_3 are components of $4a\mathbf{r}/(1+ar^2)$, that is,

$$4ax^1/(1+ar^2);\; 4ax^2/(1+ar^2);\; 4ax^3/(1+ar^2); \tag{13}$$

corresponding to the isotropy assumed.

This method can safely be extended to other cosmological models.

IV. Gravitational Force in Terms of $\bar{C}_1$

Approximate gravitational force in terms of $\bar{C}_1$ (the gravitating mass contribution to C_1) has been found for Schwarzschild's exterior and interior fields. Hence, an approximation for force can be obtained from $\bar{C}_1$ for the following fields.

(a) *Schwarzschild's Interior Field in Isotropic Coordinates*

Choosing the metric (Wyman 1946)

$$ds^2 = e^{\sigma}\,dt^2 - e^{\mu}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2),$$

where

$$e^{\sigma} = \{2a-2m+m(4a-m)r^2/2a^3\}\{(2a+m)(1+mr^2/2a^3)\}^{-1}$$

and

$$e^{\mu} = (1+m/2a)^6(1+mr^2/2a^3)^{-2},$$

we obtain

$$C_{\nu} = \left\{0; -\frac{2}{r}-\frac{mr}{a^3(1+mr^2/2a^3)}\left[2-\frac{2a-m}{2a-2m+m(4a-m)r^2/2a^3}\right]; -\cot\theta; 0\right\}. \tag{14}$$

Now the value of C_{ν} in Schwarzschild's external field (Singh and Verma 1962) is

$$C_{\nu} = \left\{0; -\frac{2}{r}-\frac{M}{r^2}(1-2M/r)^{-1};\; -\cot\theta;\; 0\right\}.$$

$$\bar{C}_1 = -\frac{M}{r^2}(1-2M/r)^{-1},$$

where $\bar{C}_1$ is the contribution to C_1 due to the gravitating mass. The other contributions depend only on the choice of the coordinates. For Schwarzschild's internal field, to the first approximation (Singh and Verma 1962),

$$\bar{C}_1 \simeq -4\pi\rho_0 r/3.$$

We see that $\bar{C}_1$ corresponds to the gravitational force. Hence in Schwarzschild's internal field in isotropic coordinates, the gravitational force must be given by (from equation (14))

$$\bar{C}_1 = -\frac{mr}{a^3(1+mr^2/2a^3)}\left[2-\frac{2a-m}{2a-2m+m(4a-m)r^2/2a^3}\right],$$

as the gravitating matter contributes only to the $\bar{C}_1$ term.

(b) *Force in Schwarzschild's External Field for Charged Particles*

The metric is (Eddington 1924)

$$ds^2 = \left(1-\frac{2m}{r}+\frac{4\pi e^2}{r^2}\right)dt^2-\left(1-\frac{2m}{r}+\frac{4\pi e^2}{r^2}\right)^{-1}dr^2-r^2\,d\theta^2-r^2\sin^2\theta\,d\phi^2.$$

The contribution of charge and mass, that is, $\bar{C}_1$ comes out as

$$\bar{C}_1 = -\left(\frac{m}{r^2}-\frac{4\pi e^2}{r^3}\right)\left(1-\frac{2m}{r}+\frac{4\pi e^2}{r^2}\right)^{-1},$$

which must correspond to the force in this field.

(c) *Force in a Spatially Isotropic Universe*

The metric (Weber 1961)

$$ds^2 = (dx^0)^2-y^2(1+ar^2)^{-2}[dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2]$$

gives the value of C_ν as

$$C_\nu = \left\{-3\left(\frac{1}{y}\frac{\partial y}{\partial x^0}\right);\ -\frac{2}{r}\cdot\frac{1-ar^2}{1+ar^2};\ \cot\theta;0\right\}.$$

$$C_1 = -\frac{2}{r}+\frac{4ar}{1+ar^2},$$

or

$$\bar{C}_1 = 4ar/(1+ar^2).$$

Hence the force in a spatially isotropic universe is represented by $4ar/(1+ar^2)$. We see that $\bar{C}_1$ here is the vector whose components are given by C_1, C_2, and C_3 of equation (13).

V. Discussion

We are led to a clear understanding of vector tetrad in a spatially isotropic universe. We can always choose a vector in a tetrad which gives

(i) the Hubble expansion of cosmological red shift in terms of its time component, and

(ii) the gravitational field force.

In other words, we can also say that the x^0 component of the gravitational field force corresponds to Hubble's expansion, and components along x^1, x^2, x^3 are components along space axes.

VI. References

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