SHORT COMMUNICATIONS

A NEW KINETIC ENERGY CORRECTION FOR CAPILLARY TUBE VISCOMETERS*

By R. I. Tanner†

Source of the New Correction

The work of Caw and Wylie (1961) on viscometers with special capillary tubes of slowly varying radius has enabled the usual kinetic energy corrections to be reduced to a very small fraction of those with abrupt-ended capillaries. Thus one of the uncertainties in the use of such tubes is eliminated; also, the remaining small effect is rendered amenable to calculation (Tanner and Linnett 1965). It has been shown theoretically that the relation between viscosity and flow time is expected to be of the form

$$\nu = At + Ct^{-3} + O(t^{-7}), \tag{1}$$

where A is a positive constant, and C may be positive or negative. Relation (1) is valid up to Reynolds numbers (based on capillary diameter and mean velocity) of 100 or more with typical flared viscometers (Caw and Wylie 1961). Agreement between experiment and equation (1) is satisfactory up to such Reynolds numbers.

However, the value of the Ct^{-3} term now makes it necessary to consider another correction, of order t^{-1}, which can be completely ignored in the usual abrupt-ended tube because of the relatively large uncertainty in the usual correction term, which is also of order t^{-1} (Merrington 1943). This apparently new correction is due to the unsteadiness of the flow in typical viscometers caused by the variable head in the instrument. The calculation of such a correction is straightforward, and is given below for a capillary tube of constant radius R.

Calculation

To a first approximation the head of liquid (H) varies exponentially with time —the viscometer is equivalent to a capacitor discharging through a linear resistor. Hence if H_0 is the initial head,

$$H = H_0 \exp(-kt), \tag{2}$$

where k is a constant, which may be evaluated from a single test on the viscometer. Thus, to a first approximation, the velocity components in the capillary vary in a similar manner with time, and we seek an axi-symmetric solution of the Navier–Stokes equations in cylindrical polar coordinates (r, θ, z) (Rosenhead 1963) in which the velocity components vary exponentially with time. Assume the velocity vector $\mathbf{V}$ is given by

$$\mathbf{V} = \{0, 0, u(r)\exp(-kt)\}, \tag{3}$$

* Manuscript received February 10, 1965.

† Mechanical Engineering Department, University of Sydney.

then the Navier–Stokes equations reduce to

$$\frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-\rho\frac{\partial u}{\partial t}, \tag{4}$$

where the pressure $p = p(z,t)$, the viscosity of the fluid is μ, and ρ is the fluid density. Explicitly,

$$\frac{\partial p}{\partial z} = -\frac{\rho g H_0}{l_c}\exp(-kt), \tag{5}$$

where l_c is the capillary length and g is the acceleration of gravity. Hence, substituting in (4) one obtains

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}+\frac{\rho k}{\mu}u = -\frac{\rho g H_0}{\mu l_c}. \tag{6}$$

The solution for (6) finite on the axis and zero at $r = R$ is given by

$$u = \left(\frac{g H_0}{k l_c}\right)\left\{1-\frac{\mathrm{J}_0(\alpha r)}{\mathrm{J}_0(\alpha R)}\right\}, \tag{7}$$

where $\alpha = (\rho k/\mu)^{\frac{1}{2}}$. For small αR, which is the case of interest, the Bessel function J_0 may be represented by a few terms of the series. Integration to find the total volume discharge $\mathscr{V}$ in a time of flow t gives, to the second approximation,

$$\mathscr{V} = \frac{\pi\rho g H_0 R^4 t}{8\mu l_c}\left\{1+\frac{\rho k R^2}{6\mu}\right\}\left\{1-\tfrac{1}{2}kt\right\}. \tag{8}$$

Hence, since $k = t^{-1}\ln(H_0/H)$, (8) may be rearranged to give the relation

$$\nu = \frac{\pi g H_0 R^4 t}{8\mathscr{V} l_c}\left\{1-\ln(H_0/H)\right\}\left\{1+\frac{R^2}{6\nu t}\ln(H_0/H)\right\}, \tag{9}$$

which, substituting for ν in the right-hand side of (9), is seen to be of the form

$$\nu = At+B/t, \tag{10}$$

where

$$B = \tfrac{1}{6}R^2\ln(H_0/H). \tag{11}$$

B is thus positive, in contrast to the usual kinetic energy correction.

Discussion

The new correction is easily calculated on the assumption that the parallel capillary is the only source of flow resistance. For flaring capillaries, which are the cases of interest, no difficulty will arise provided the parallel portion is the main source of resistance. If this is not the case, then at low Reynolds numbers one may apply equation (5) in differential form, and then define the equivalent radius (R_e^2) as the ratio

$$R_e^2 = \int R^{-2}\,\mathrm{d}z \Big/ \int R^{-4}\,\mathrm{d}z. \tag{12}$$

The integrals are taken over the length of the viscometer. R_e^2 should then be used in equation (11).

For the viscometers used by Caw and Wylie (1961) the new correction is negligible. However, as an example of the orders of magnitude that may be involved, consider viscometers of the BS/U types G and H (British Standards Institution 1957) used at Reynolds numbers of 50; the steady flow kinetic energy errors from a flared version of these viscometers would be at most of the order of 0·1% while the calculation presented here would give an error of the order of 1%. Thus, particularly with flared capillaries of larger bore, the two effects should be considered together.

References

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