DIFFUSION OF HEAT FROM A SPHERE TO A SURROUNDING MEDIUM*

By A. Brown†

In a recent paper, Philip (1964) discussed a heat flow problem in which, as an idealization, spherical symmetry is assumed and the medium is regarded as composite, with a spherical core of one material embedded in a larger mass of a second material. One application in mind was to laccoliths, where an intrusion of igneous rock in sedimentary material can produce dome-shaped bulging at the surface. The idea of treating this as a problem involving spherical symmetry was put forward by Lovering (1935), who gave a solution for the case where the core and its surroundings are taken as uniform. In a subsequent paper, Lovering (1936) considered in more detail what variations in the thermal conductivity and diffusivity are to be expected. From the figures he quotes, a composite model should give a better approximation, though remaining a considerable simplification of the physical problem.

In Philip's model, the outer region is taken to be infinite in extent, and at time $t = 0$ the core and the outer region are each at a constant temperature, with the core at a higher temperature than its surroundings. Philip showed that solutions for the temperature and the heat flux at the interface between the two regions could be developed as series expansions, and gave the first few terms of these expansions both for small and large values of t.

The present paper gives some additional results for the same problem. In particular, the temperature at any point is given as an infinite integral. Although this integral converges for any time t greater than zero, the convergence is more rapid when t is large. For small values of t, a more convenient solution is obtained in the special case where the thermal conductivity of the two materials is the same although their diffusivities differ. It is possible to extend this solution to a number of related problems, and these extensions are outlined in the final section of the present paper.

Notation and Method of Solution

We take the spherical core as the region $0 \leqslant r \leqslant a$, where r is the distance from the centre, and use K_1, k_1, T_1 for the conductivity, diffusivity, and temperature in the core, with K_2, k_2, T_2 as the corresponding quantities in the outer region. The relevant equations for T_1 and T_2 are then:

$$\frac{\partial T_1}{\partial t} = k_1\left(\frac{\partial^2 T_1}{\partial r^2} + \frac{2}{r}\frac{\partial T_1}{\partial r}\right) \qquad 0 < r < a; \tag{1}$$

$$\frac{\partial T_2}{\partial t} = k_2\left(\frac{\partial^2 T_2}{\partial r^2} + \frac{2}{r}\frac{\partial T_2}{\partial r}\right) \qquad r > a; \tag{2}$$

$$T_1 = T_2, \qquad K_1(\partial T_1/\partial r) = K_2(\partial T_2/\partial r), \qquad \text{at } r = a;$$

$$T_1 = T_0, \qquad T_2 = 0, \qquad \text{at } t = 0.$$

* Manuscript received May 24, 1965.

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For convenience, the initial temperature in the outer region has been taken as the zero level for T_1 and T_2, and we shall take T_0 to be a positive constant. Additional restrictions which affect the form of T_1 and T_2 are that T_1 must be finite at $r = 0$, and T_2 must tend to zero as r approaches infinity.

If $\bar{T}_1$ and $\bar{T}_2$ are the Laplace transforms of T_1 and T_2 with respect to t, that is,

$$\bar{T}_i = \int_0^\infty T_i(r,t)\exp(-pt)\,\mathrm{d}t \qquad (i = 1,2),$$

then it can be shown that

$$\bar{T}_1 = \frac{T_0}{p}\left[1 - \frac{K_2(1+q_2 a)\sinh q_1 r}{r\{K_1 q_1 \cosh q_1 a + (K_2 q_2 + K)\sinh q_1 a\}}\right], \tag{3}$$

$$\bar{T}_2 = \frac{T_0}{p}\left[\frac{K_1(q_1 a \cosh q_1 a - \sinh q_1 a)\exp\{-q_2(r-a)\}}{r\{K_1 q_1 \cosh q_1 a + (K_2 q_2 + K)\sinh q_1 a\}}\right], \tag{4}$$

where

$$q_1 = (p/k_1)^{\frac{1}{2}}, \qquad q_2 = (p/k_2)^{\frac{1}{2}}, \tag{5}$$

$$K = (K_2 - K_1)/a. \tag{6}$$

Equations (3) and (4) are equivalent to equations (3.6) and (3.7) of Philip's paper (Philip 1964). However, the notation follows more closely that of Carslaw and Jaeger (1939), who considered a somewhat similar problem of heat transfer in a composite system with spherical symmetry.

Carslaw and Jaeger (1939) also used the Laplace transform method, and obtained transforms with essentially the same denominator as in $\bar{T}_1$ and $\bar{T}_2$ above. To invert these transforms, they used the standard inversion integral (Churchill 1958, p. 176), and integrated around a suitable contour in the (complex) p-plane. In this way, they obtained solutions for the temperature in a form involving real integrals. In the present problem, their procedure can be taken over with only minor variations, and this gives solutions for T_1 and T_2 in the form

$$\frac{T_1}{T_0} = \frac{2Qa}{\pi r}\int_0^\infty \frac{(\sin u - u\cos u)\sin(ur/a)\exp(-k_1 u^2 t/a^2)}{(u\cos u + L\sin u)^2 + (Qu\sin u)^2}\,\mathrm{d}u, \tag{7}$$

$$\frac{T_2}{T_0} = \frac{2a}{\pi r}\int_0^\infty \frac{(\sin u - u\cos u)\,F(u)\exp(-k_1 u^2 t/a^2)}{(u\cos u + L\sin u)^2 + (Qu\sin u)^2}\,\frac{\mathrm{d}u}{u}, \tag{8}$$

where

$$F(u) = (u\cos u + L\sin u)\sin\{u(r-a)/\sigma a\} + Qu\sin u\cos\{u(r-a)/\sigma a\},$$

$$Q = K_2/K_1\sigma, \qquad L = (K_2 - K_1)/K_1, \qquad \sigma = (k_2/k_1)^{\frac{1}{2}}.$$

It will be seen that the integrals involved are of Fourier type, in that separable solutions of equations (1) and (2) are combined in suitable proportions to satisfy the boundary conditions and initial conditions. For example, equation (7) is of the form

$$T_1 = \int_0^\infty G(u)\,\{(1/r)\sin(ur/a)\exp(-k_1 u^2 t/a^2)\}\,\mathrm{d}u,$$

where the expression in braces is a separable solution of equation (1) for all values of u, and $G(u)$ can be regarded as an amplitude factor.

Temperature and Heat Flow at Interface

The temperature at the interface, which we denote by T^*, can be obtained by putting $r = a$ in either equation (7) or equation (8). This gives

$$T^* = (2QT_0/\pi)\int_0^\infty (1/D^2)(\sin u - u\cos u)\sin u \exp(-k_1 u^2 t/a^2)\, du, \qquad (9)$$

where $\quad D^2 = (u\cos u + L\sin u)^2 + (Qu\sin u)^2.$

Similarly, the heat flux from the core to the exterior region can be obtained from either of equations (7) and (8). If we use $H'(t)$ for this heat flux, then

$$\begin{aligned} H'(t) &= -4\pi K_1 a^2(\partial T_1/\partial r)_{r=a} = -4\pi K_2 a^2(\partial T_2/\partial r)_{r=a} \\ &= 8T_0 a K_1 Q \int_0^\infty \{(\sin u - u\cos u)^2/D^2\}\exp(-k_1 u^2 t/a^2)\, du. \qquad (10)\end{aligned}$$

By integrating from 0 to t, we get the heat loss from the core after time t as

$$H(t) = 8T_0 a^3 K_1 Q/k_1 \int_0^\infty \{(\sin u - u\cos u)^2/u^2D^2\}\{1-\exp(-k_1 u^2 t/a^2)\}\, du.$$

As $t \to \infty$, the temperature in the core will tend to zero, and $H(t)$ must approach H_0, where H_0 is the total amount of energy which the core can lose by heat flow to the surrounding material. Now

$$\begin{aligned} H_0 &= (\text{core volume})\times(\text{density})\times(\text{specific heat})\times(\text{initial temperature difference}) \\ &= (4\pi/3)a^3(K_1/k_1)T_0. \qquad (11)\end{aligned}$$

The density and specific heat here are those appropriate to the core and, since

$$\text{diffusivity} = (\text{thermal conductivity})/(\text{density}\times\text{specific heat}),$$

we can use K_1/k_1 instead of (density$\times$specific heat) in equation (11). It follows that

$$\int_0^\infty \{(\sin u - u\cos u)^2/u^2D^2\}\, du = \pi/6Q,$$

and

$$\begin{aligned} (H_0-H)/H_0 &= \text{fraction of total heat flux from core that has yet to occur at time } t \\ &= (6Q/\pi)\int_0^\infty \{(\sin u - u\cos u)^2/u^2D^2\}\exp(-k_1 u^2 t/a^2)\, du. \qquad (12)\end{aligned}$$

For large values of t, the integrals in equations (9), (10), and (12) can be approximated by an asymptotic expansion in descending powers of t. In the case of equation (9), for example, the asymptotic expansion is obtained by expressing $(1/D^2)(\sin u - u\cos u)\sin u$ as a power series in u, and integrating the new form of the integrand term by term. The resulting expansion for T^* agrees exactly with the asymptotic expansion given by Philip (1964, equation (6.4)). Similarly, the asymptotic expansion for $H'(t)$ from equation (10) agrees with Philip's equation (6.5), and the asymptotic expansion for $(H_0-H)/H_0$ confirms Philip's conclusions regarding the equilibration time of the composite system.

Temperature Distribution for $K_1 = K_2$

The expressions for T_1 and T_2 given by equations (7) and (8) are valid for all positive values of t, but it is clear that they converge more rapidly when t is large than when t is small. In the case where $K_1 = K_2$, equations (7) and (8) can be supplemented by series solutions which converge more rapidly for small values of t.

If we put $K_1 = K_2$ in equations (3)–(6), then $K = 0$ and

$$\bar{T}_1 = \frac{T_0}{p}\left[1 - \frac{(1+q_2 a)\sinh q_1 r}{rq_1\{\cosh q_1 a + (1/\sigma)\sinh q_1 a\}}\right], \tag{13}$$

$$\bar{T}_2 = \frac{T_0}{p}\left[\frac{(q_1 a\cosh q_1 a - \sinh q_1 a)\exp\{-q_2(r-a)\}}{rq_1\{\cosh q_1 a + (1/\sigma)\sinh q_1 a\}}\right], \tag{14}$$

since $\sigma = (k_2/k_1)^{\frac{1}{2}} = q_1/q_2$. These Laplace transforms can be inverted to give solutions for T_1 and T_2 in terms of the error function and its integral. One way of inverting $\bar{T}_1$ and $\bar{T}_2$ is to express the hyperbolic functions in terms of exponentials and replace $\{\cosh q_1 a + (1/\sigma)\sinh q_1 a\}^{-1}$ by a series expansion in powers of $\exp(-2q_1 a)$. An example of this procedure is given by Carslaw and Jaeger (1959, Chap. XII, § 12.5). An alternative method, which leads into some interesting by-ways, is to use the result that, if

$$\bar{G}(p) = (\pi/p)^{\frac{1}{2}}\,\bar{F}(p^{\frac{1}{2}}),$$

then

$$G(t) = t^{-\frac{1}{2}}\int_0^\infty F(u)\exp(-u^2/4t)\,\mathrm{d}u \qquad (t > 0),$$

where $\bar{F}(p)$ is the Laplace transform of $F(t)$. However, the first method is simpler and more direct.

To write the solutions for T_1 and T_2 compactly, we introduce the notation

$$E_0(x) = \operatorname{erfc} x, \qquad E_1(x) = \operatorname{ierfc} x = \int_x^\infty E_0(u)\,\mathrm{d}u,$$
$$P_N = \{Na + \tfrac{1}{2}(a-r)\}/(k_1 t)^{\frac{1}{2}}, \qquad Q_N = \{Na + \tfrac{1}{2}(a+r)\}/(k_1 t)^{\frac{1}{2}},$$
$$R_N = \{N\sigma a + \tfrac{1}{2}(r-a)\}/(k_2 t)^{\frac{1}{2}}.$$

Then

$$\frac{T_1}{T_0} = 1 - \frac{(a/r)}{1+\sigma}\sum_{N=0}^{\infty}\left(\frac{1-\sigma}{1+\sigma}\right)^N\left[E_0(P_N) - E_0(Q_N) + \frac{2(k_2 t)^{\frac{1}{2}}}{a}\{E_1(P_N) - E_1(Q_N)\}\right], \tag{15}$$

$$\frac{T_2}{T_0} = \frac{\sigma(a/r)}{1+\sigma}\left\{E_0(R_0) - \frac{2(k_1 t)^{\frac{1}{2}}}{a}E_1(R_0)\right\}$$
$$+ \frac{2\sigma(a/r)}{(1+\sigma)^2}\sum_{N=1}^{\infty}\left(\frac{1-\sigma}{1+\sigma}\right)^{N-1}\left\{E_0(R_N) + \frac{2(k_2 t)^{\frac{1}{2}}}{a}E_1(R_N)\right\}. \tag{16}$$

Equations (15) and (16) agree with the solution given by Lovering (1935). As mentioned earlier, he treated the core and its surroundings as if they were of the same material. (In his numerical work, he used an average value of the thermal conductivity and of the diffusivity.) With this assumption, $k_1 = k_2$ and hence $\sigma = 1$, so the series for T_1 and T_2 terminate, giving

$$T_1/T_0 = 1-(a/2r)\{E_0(P_0)-E_0(Q_0)\}-(1/r)(k_1 t)^{\frac{1}{2}}\{E_1(P_0)-E_1(Q_0)\},$$

$$T_2/T_0 = (a/2r)\{E_0(R_0)+E_0(R_1)\}-(1/r)(k_1 t)^{\frac{1}{2}}\{E_1(R_0)-E_1(R_1)\},$$

with

$$P_0 = -R_0 = \tfrac{1}{2}(a-r)/(k_1 t)^{\frac{1}{2}}, \qquad Q_0 = R_1 = \tfrac{1}{2}(a+r)/(k_1 t)^{\frac{1}{2}}.$$

Using the relations

$$E_0(-x) = 2-E_0(x),$$

$$E_1(-x) = 2x+E_1(x),$$

$$E_1(x) = -x\,E_0(x)+\pi^{-\frac{1}{2}}\exp(-x^2),$$

it is easy to verify that

$$T_1 = T_2 = \tfrac{1}{2}T_0(\operatorname{erf} P_0 + \operatorname{erf} Q_0)-(T_0/r)(k_1 t/\pi)^{\frac{1}{2}}\{\exp(-P_0^2)-\exp(-Q_0^2)\},$$

which agrees with Lovering's solution.

When $K_1 = K_2$, the integral in equation (7) simplifies slightly, since $L = 0$ and $Q = 1/\sigma$. Hence, for $K_1 = K_2$,

$$\frac{T_1}{T_0} = \frac{2\sigma a}{\pi r}\int_0^\infty \frac{(\sin u - u\cos u)\sin(ur/a)\exp(-k_1 u^2 t/a^2)}{u^2(\sigma^2\cos^2 u+\sin^2 u)}\,du. \tag{17}$$

Equations (15) and (17) now give two strikingly different expressions for the ratio T_1/T_0. Similarly, equation (8) may be used to give an infinite integral for T_2/T_0 as an alternative to the series on the right-hand side of equation (16).

Temperature and Heat Flow at Interface for $\mathrm{K}_1 = \mathrm{K}_2$

The temperature at the interface can be obtained by putting $r = a$ in either equation (15) or (16). Since $E_0(0) = 1$ and $E_1(0) = \pi^{-\frac{1}{2}}$, we get

$$T^* = \frac{\sigma T_0}{1+\sigma}\left\{1-\left(\frac{2}{a}\right)\left(\frac{k_1 t}{\pi}\right)^{\frac{1}{2}}\right\}+\frac{2\sigma T_0}{(1+\sigma)^2}\sum_{N=1}^{\infty}\left(\frac{1-\sigma}{1+\sigma}\right)^{N-1}\left\{E_0(P_N^*)+\frac{2(k_2 t)^{\frac{1}{2}}}{a}E_1(P_N^*)\right\},$$

where $\quad P_N^* = \text{value of } P_N \text{ at interface} = Na/(k_1 t)^{\frac{1}{2}}.$

Similarly, the heat flux from the core can be obtained from either equation (15) or equation (16). This gives

$$H'(t) = \frac{4\pi a K_1 T_0}{1+\sigma}\left[(\sigma-1)+\frac{a^2-2\sigma k_1 t}{a(\pi k_1 t)^{\frac{1}{2}}}\right.$$
$$\left.+\sum_{N=1}^{\infty}\frac{2(1-\sigma)^{N-1}}{(1+\sigma)^N}\left\{\frac{Na^2+2\sigma k_1 t}{k_1 t}E_0(P_N^*)+\frac{a^2+2k_2 t}{a(k_1 t)^{\frac{1}{2}}}E_1(P_N^*)\right\}\right].$$

These expressions for T^* and $H'(t)$ can be compared with equations (7.2) and (7.3) of Philip's paper (Philip 1964), noting that Philip's symbols B, τ, $\vartheta(1)$, and Φ

are equal, respectively, to $\sigma/(1+\sigma)$, $k_1 t/a^2$, T^*/T_0, and $(4\pi a K_1 T_0)^{-1} H'(t)$ of the present paper. The comparison shows, in the higher order terms, discrepancies which arise because of the omission of a bracket in equation (4.1) of Philip's paper. Dr. Philip (personal communication; see also Philip 1965) has kindly supplied me with amended forms of the equations affected by the missing bracket, and our results are now in agreement.

Related Problems for $K_1 = K_2$

A well-known device for dealing with heat conduction problems where there is spherical symmetry is to use variables $U_1 = rT_1$ and $U_2 = rT_2$, in place of T_1 and T_2. Then U_1 and U_2 satisfy the standard diffusion equation for linear flow of heat, that is, instead of equations (1) and (2) we get

$$\frac{\partial U_1}{\partial t} = k_1 \frac{\partial^2 U_1}{\partial r^2}, \qquad \frac{\partial U_2}{\partial t} = k_2 \frac{\partial^2 U_2}{\partial r^2}.$$

In general, linear flow solutions cannot be taken over unaltered, because the boundary conditions at the interface are

$$U_1 = U_2, \qquad K_1 \frac{\partial}{\partial r}\left(\frac{U_1}{r}\right) = K_2 \frac{\partial}{\partial r}\left(\frac{U_2}{r}\right), \tag{18}$$

whereas in a linear flow problem the boundary conditions are

$$U_1 = U_2, \qquad K_1(\partial U_1/\partial r) = K_2(\partial U_2/\partial r). \tag{19}$$

Carslaw (1921) illustrates this point by quoting a Cambridge tripos question in which the answer supplied was wrong because (19) had been used when (18) was the appropriate pair of equations. But in the special case where $K_1 = K_2$, the distinction between (18) and (19) disappears and the corresponding linear flow solution can be taken over unchanged, if a linear flow solution is available. Where a solution of the corresponding linear flow problem is not immediately available, there may be standard techniques which give a solution readily. Thus, in any heat conduction problem involving a composite material with spherical symmetry, it may be possible to obtain a detailed solution for the case $K_1 = K_2$, and thus obtain some preliminary information about the heat flow, even where it is difficult to find a solution in the general case.

In the problem with which we are concerned, the initial conditions are that $U_1 = rT_0$, for $0 \leqslant r < a$, and $U_2 = 0$, for $r > a$. In addition, U_1 must be zero at $r = 0$, and U_2 must approach zero as $r \to \infty$, for $t > 0$. The justification for the latter requirement is that, for large values of r, T_2 should behave like the solution for a heat source at the origin at $t = 0$, with subsequent diffusion into an infinite medium. Hence, we can expect to have $T_2 \sim t^{-3/2} \exp(-r^2/4k_2 t)$ for large values of r and t greater than zero. It follows that U_2 must tend to zero as r approaches infinity.

An equivalent problem is to find the temperature in an infinite rod with a central portion, $-a < x < a$, which is of different material from the remainder, given that the initial temperature is xT_0 in the central portion and zero outside the central portion. To correspond to the restriction $K_1 = K_2$, the conductivity must be unchanged along the rod. Then the solution of the linear flow problem for $0 < x < a$ corresponds to U_1, and the solution for $x > a$ corresponds to U_2. In the linear flow problem, taking the conductivity to be different in the two media adds no essential difficulty; if the Laplace transform method of solution is used, the transforms that have to be inverted are of the same type as those in equations (13) and (14).

The heat flow problem where the core is initially at a uniform temperature can be related to problems where the core contains a heat source. For example, if heat is produced at a constant rate in the core, and if the core and its surroundings are initially at zero temperature, the subsequent temperature distribution can be obtained by integrating T_1 and T_2 with respect to t, and multiplying by a suitable factor to allow for the strength of the source. This solution in turn can be used to solve the case where the strength of the source decays exponentially with time. Thus, although the model considered is restrictive in its assumptions, it can be regarded as fundamental to a wider range of problems.

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