QUARK–ANTIQUARK MODELS OF MESONS WITH RELATIVISTIC KINEMATICS

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Summary

Mesons are treated as bound states of a quark and an antiquark. Several models are given in which the interaction between quark and antiquark is described by a potential, and the motion of the quarks is described by the Klein–Gordon equation. Different relations among the masses of mesons within a multiplet are obtained for different models. The same relation among masses obtained from non-relativistic Schrödinger equation models can be obtained for particular Klein–Gordon equation models either among masses or among squares of masses.

I. Introduction

A number of authors (Dalitz 1965; Morpurgo 1965; Sinanoğlu 1966) have considered a model in which mesons are bound states of a quark and antiquark (Gell-Mann 1964; Zweig 1964). Treating the motion of the quark and antiquark as non-relativistic, these authors have obtained relations among the masses of the mesons, and, in comparing the relations with experiment, they have arbitrarily replaced masses by squares of masses in order to obtain better agreement with the experimental results.

Macfarlane and Socolow (1966) have expressed doubt whether experiment really favours the use of the squares of masses in meson mass formulae. However, although the experimental evidence is inconclusive, we believe that the tentative evidence favours relations among the squares of meson masses rather than relations that are linear in the masses.

One possible defect of the non-relativistic model of Morpurgo, Dalitz, and others has been pointed out by Greenberg (personal communication 1965) and by Domokos and Palmer (personal communication 1966). This is that if two particles are deeply bound in a potential that is singular‡ at the origin, such as a Yukawa potential, the motion of the quarks is essentially relativistic. In this case, not only should relativistic kinematics be used, but perhaps the very concept of a potential becomes meaningless.

It is the purpose of the present paper simply to explore the consequences of using relativistic kinematics in a description of a meson as a bound state of a quark and antiquark. We find that such a model can yield the Schwinger (1964) relation

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‡ *Note added in proof.* Professor Morpurgo has kindly pointed out that he stated in his original paper that his model did not apply to potentials that are too singular. In that paper he also stated that in his scheme it is difficult to understand a quadratic mass formula.

between either masses or squares of masses, depending on the nature of the quark–antiquark interaction.

It is assumed that the interaction between a quark and an antiquark can be described by a static potential of simple form and that the motion of the quarks is described by the Klein–Gordon equation. Thus, this model is a combination of relativistic kinematics and non-relativistic dynamics. The spins of the quarks are not treated correctly in the Klein–Gordon equation, but, since the potential is allowed to depend on the spin states of the quarks, this should not be a serious defect of the model.

We ignore the question of the meaningfulness of a description by means of a static potential. Although perhaps some argument can be made that a Yukawa potential is more suitable than a square well, we choose the latter to make the calculations simpler. Thus, we do not explore in detail the question of how mass formulae depend on the shape of the potential as a function of the separation between quark and antiquark. Rather, we treat the question of how the results are affected by the way in which the potential is inserted into the Klein–Gordon equation.

Domokos and Palmer have treated this problem relativistically by means of dispersion relations, and Horwitz (personal communication 1966) has considered a model in which the Dirac equation is used to describe the quark-antiquark motion.

II. Models of Mesons

With no interaction, the Hamiltonian in the centre-of-mass system is

$$H_0 = (p^2+m_1^2)^{\frac{1}{2}}+(p^2+m_2^2)^{\frac{1}{2}}. \tag{1}$$

We shall use small m for the quark masses and capital M for the meson masses.

All potentials are taken as very deep square wells of radius a. Using the approximation that the well is of infinite depth, the eigenfunctions of the Hamiltonian H are also eigenfunctions of p^2. For, an infinite potential is equivalent to a condition on the eigenfunction

$$\psi(r) = 0 \qquad r \geqslant a. \tag{2}$$

Thus,

$$\psi(r) = j_l(pr) \qquad r < a, \tag{3}$$

where l is the orbital angular momentum. For instance, for the lowest s state,

$$p = \pi/a. \tag{4}$$

As in the work of Dalitz (1965), the interaction is assumed invariant under SU(3), and SU(3) is then broken by one quark having a mass $m+\Delta$, heavier than the mass m of the other two quarks.

An interaction between quark and antiquark that is invariant under SU(3) is specified by an octet potential and a singlet potential. The mesons are arranged in nonets, and it is convenient to use a representation of SU(3)×SU(3), since this correctly describes the meson states when the singlet and octet potentials are the same.

There is no unique way of introducing a potential into the Hamiltonian (1). Several ways of doing this have been investigated, and we now give some examples.

Model I

The potential is added to the free-particle Hamiltonian (1), giving

$$H = H_0 + V. \tag{5}$$

The masses of the mesons are given by the eigenvalues of H. m is the mass of the two quarks with $Y = +1$; $m+\Delta$ is the mass of the quark with $Y = 0, I = 0$; M_I is the mass of the meson with isotopic spin I. Then,

$$M_{\frac{1}{2}} = (p^2+m^2)^{\frac{1}{2}} + \{p^2+(m+\Delta)^2\}^{\frac{1}{2}} + V_8, \tag{6}$$

$$M_1 = 2(p^2+m^2)^{\frac{1}{2}} + V_8, \tag{7}$$

where V_8 is the octet potential.

If the singlet and octet potential were the same ($V_1 = V_8 = V$), then, for the two mesons with $I = 0$,

$$M_0^{(1)} = 2(p^2+m^2)^{\frac{1}{2}} + V, \tag{8}$$

$$M_0^{(2)} = 2\{p^2+(m+\Delta)^2\}^{\frac{1}{2}} + V. \tag{9}$$

The difference between V_8 and V_1 causes mixing of these two mesons, so that we now identify the masses of the two mesons having $I = 0$ as M_0' and M_0'', the two eigenvalues of the matrix

$$\begin{pmatrix} 2(p^2+m^2)^{\frac{1}{2}} + \frac{1}{3}V_8 + \frac{2}{3}V_1 & \frac{1}{3}\sqrt{2}(V_1 - V_8)\alpha \\ \frac{1}{3}\sqrt{2}(V_1 - V_8)\alpha^* & 2\{p^2+(m+\Delta)^2\}^{\frac{1}{2}} + \frac{2}{3}V_8 + \frac{1}{3}V_1 \end{pmatrix}. \tag{10}$$

Here α is the overlap integral between the wavefunction of two quarks of mass m and the wavefunction of two quarks of mass $m+\Delta$, when $V_1 = V_8$.

We then obtain (see Appendix)

$$\begin{aligned}(M_0' - M_1)(M_0'' - M_1) - \tfrac{4}{3}(M_{\frac{1}{2}} - M_1)(M_0' + M_0'' - 2M_{\frac{1}{2}}) \\ = \tfrac{2}{9}(M_0' + M_0'' - 2M_{\frac{1}{2}})^2(1 - |\alpha|^2) \geqslant 0.\end{aligned} \tag{11}$$

The mass relation obtained by Dalitz (1965) from a non-relativistic model differs from equation (11) only because of the use of a different overlap integral. Dalitz uses the overlap integral between octet and singlet states of SU(3) with all quark masses equal.

From

$$M_{\frac{1}{2}} - M_1 = \{p^2+(m+\Delta)^2\}^{\frac{1}{2}} - (p^2+m^2)^{\frac{1}{2}} \tag{12}$$

it is seen that, unless p^2 is large and varies from multiplet to multiplet, $d = M_{\frac{1}{2}} - M_1$ should be the same for different multiplets. Experimentally, this is not the case, as is shown in Table 1, where we have used the masses from Rosenfeld *et al.* (1965).

Model II

The interaction between quark and antiquark is described by a potential ϕ (of dimension of the square of an energy), which is added to H_0^2 rather than to H_0, so

$$H^2 = H_0^2 + \phi. \tag{13}$$

Following the procedure used for Model I, we obtain

$$M_{\frac{1}{2}}^2 = [(p^2+m^2)^{\frac{1}{2}}+\{p^2+(m+\Delta)^2\}^{\frac{1}{2}}]^2+\phi_8\,, \tag{14}$$

$$M_1^2 = 4(p^2+m^2)+\phi_8\,, \tag{15}$$

and $M_0'^2$ and $M_0''^2$ are the two eigenvalues of the matrix

$$\begin{pmatrix} 4(p^2+m^2)+\frac{1}{3}\phi_8+\frac{2}{3}\phi_1 & \frac{1}{3}\sqrt{2}(\phi_1-\phi_8)\alpha \\ \frac{1}{3}\sqrt{2}(\phi_1-\phi_8)\alpha^* & 4\{p^2+(m+\Delta)^2\}+\frac{2}{3}\phi_8+\frac{1}{3}\phi_1 \end{pmatrix}. \tag{16}$$

Table 1

EXPERIMENTAL† VALUES OF THE PARAMETERS d, δ, γ, AND $M_{\frac{1}{2}}/M_1$

Meson Nonet	$d = M_{\frac{1}{2}}-M_1$ (BeV)	$\delta = M_{\frac{1}{2}}^2-M_1^2$ (BeV^2)	$\gamma = M_{\frac{1}{2}}(M_{\frac{1}{2}}-M_1)$ (BeV^2)	$M_{\frac{1}{2}}/M_1$
0^-	0·358	0·227	0·178	3·59
1^-	0·126	0·208	0·112	1·16
2^+	0·081	0·221	0·114	1·06

† Model I predicts that the values of d should be equal, Model II that the values of δ should be equal, Model III that the values of γ should be equal, and Model IV that the ratios $M_{\frac{1}{2}}/M_1$ should be equal for the 0^-, 1^-, and 2^+ nonets.

From equations (14) and (15), it is seen that the difference in the squares of masses within a multiplet are of order $m\Delta$, and, since the quark mass is large compared to all other masses, Δ is small. Neglecting terms of order Δ^3/m^3, the matrix (16) can be written as

$$\begin{pmatrix} M_1^2+\frac{2}{3}(\phi_1-\phi_8) & \frac{1}{3}\sqrt{2}(\phi_1-\phi_8)\alpha \\ \frac{1}{3}\sqrt{2}(\phi_1-\phi_8)\alpha^* & 2M_{\frac{1}{2}}^2-M_1^2+\frac{1}{3}(\phi_1-\phi_8) \end{pmatrix}. \tag{17}$$

We then obtain

$$\begin{aligned}(M_0'^2-M_1^2)(M_0''^2-M_1^2)-\tfrac{4}{3}(M_{\frac{1}{2}}^2-M_1^2)(M_0'^2+M_0''^2-2M_{\frac{1}{2}}^2)& \\ = \tfrac{2}{9}(M_0'^2+M_0''^2-2M_{\frac{1}{2}}^2)^2(1-|\alpha|^2)\geqslant 0\,.&\end{aligned} \tag{18}$$

For $\alpha = 1$, this is the Schwinger relation (Schwinger 1964) for the squares of meson masses.

Unless p^2 is large and varies from multiplet to multiplet, the quantity $\delta = M_{\frac{1}{2}}^2-M_1^2$ should be the same for all multiplets, according to this model. Experimentally, this is very closely the case for 0^-, 1^-, and 2^+ meson nonets, as can be seen from Table 1. The constancy of δ has been remarked by many authors (Coleman and Glashow 1964; Schwinger 1964; Dalitz 1965).

In this model, a very small difference in quark masses can cause a large difference in meson masses.

Model III

Another way to insert ϕ into the Klein–Gordon equation is to write

$$H = (p^2+m_1^2+\phi)^{\frac{1}{2}}+(p^2+m_2^2+\phi)^{\frac{1}{2}}. \tag{19}$$

Then,

$$M_{\frac{1}{2}} = (p^2+m^2+\phi_8)^{\frac{1}{2}}+\{p^2+(m+\Delta)^2+\phi_8\}^{\frac{1}{2}}, \tag{20}$$

$$M_1 = 2(p^2+m^2+\phi_8)^{\frac{1}{2}}, \tag{21}$$

and $M_0'^2$ and $M_0''^2$ are the eigenvalues of the matrix

$$4\times\begin{pmatrix} p^2+m^2+\frac{1}{3}\phi_8+\frac{2}{3}\phi_1 & \frac{1}{3}\sqrt{2}(\phi_1-\phi_8)\alpha \\ \frac{1}{3}\sqrt{2}(\phi_1-\phi_8)\alpha^* & p^2+(m+\Delta)^2+\frac{2}{3}\phi_8+\frac{1}{3}\phi_1 \end{pmatrix}$$
$$= \begin{pmatrix} M_1^2+\frac{8}{3}(\phi_1-\phi_8) & \frac{4}{3}\sqrt{2}(\phi_1-\phi_8)\alpha \\ \frac{4}{3}\sqrt{2}(\phi_1-\phi_8)\alpha^* & (2M_{\frac{1}{2}}-M_1)^2+\frac{4}{3}(\phi_1-\phi_8) \end{pmatrix}. \tag{22}$$

We get the mass formula

$$(M_0'^2-M_1^2)(M_0''^2-M_1^2)-\tfrac{8}{3}\gamma(M_0'^2+M_0''^2-4\gamma-2M_1^2)$$
$$= \tfrac{2}{9}(M_0'^2+M_0''^2-4\gamma-2M_1^2)^2(1-|\alpha|^2)\geqslant 0\ , \tag{23}$$

where

$$\gamma = M_{\frac{1}{2}}(M_{\frac{1}{2}}-M_1). \tag{24}$$

This model gives a mass formula that is intermediate between the linear and quadratic mass formulae of equations (11) and (18). The quantity γ can be expressed in terms of quark masses as

$$\gamma = 2m\Delta+\Delta^2. \tag{25}$$

According to this model, γ should be the same for different multiplets. In Table 1, γ is shown for 0^-, 1^-, and 2^+ mesons.

Model IV

We assume H^2 is given by

$$H^2 = H_0^2+2HV. \tag{26}$$

For $m_1 = m_2$, this model has the same non-relativistic limit, $p^2/m^2 \to 0$, $V/m \to 0$, as Model III.

In the limit of large quark mass, $m \gg M_1$, we obtain

$$M_{\frac{1}{2}}/M_1 = 1+\Delta/m. \tag{27}$$

The ratio $M_{\frac{1}{2}}/M_1$ is predicted to be the same for all multiplets, but this is not in agreement with the experimental masses as shown in Table 1.

III. Quark Velocity

To investigate whether the quarks are moving non-relativistically, consider the state with $m_1 = m_2 = m$. An estimate of the quark velocity is obtained by treating the Hamiltonian classically. Then,

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{28}$$

where $\dot{q}$ is twice the quark velocity.

For Models I and IV, we have

$$\dot{q} = 4p/H_0 \to 2p/m \quad \text{as } p \to 0, \tag{29}$$

so that it is possible to have deep binding and non-relativistic motion, and there is some justification for describing the interaction by an unretarded potential.

For Models II and III, we have

$$\dot{q} = 4p/H = 4p/M, \tag{30}$$

and $\dot{q} \ll 2$ only if $a \gg 2\pi/M$. The motion can be non-relativistic only if the radius of the meson is large compared with its Compton wavelength. Otherwise, for deep binding, the motion in Models II and III is essentially relativistic.

IV. Discussion

It should be emphasized that the four models discussed here do not exhaust the possibilities, even with the restriction to extremely deep square wells.

The result of Model II can be obtained by replacing masses by squares of masses in the expression obtained by Dalitz (1965), but this is not so for other models.

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Appendix

Derivation of Equation (11)

Writing the matrix (10) as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{31}$$

we have, from equations (6) and (7),

$$A = M_1 + \tfrac{2}{3}W, \tag{32}$$

$$B = \tfrac{1}{3}\sqrt{2}W\alpha, \tag{33}$$

$$C = \tfrac{1}{3}\sqrt{2}W\alpha^*, \tag{34}$$

$$D = 2M_{\frac{1}{2}} - M_1 + \tfrac{1}{3}W, \tag{35}$$

where $W = V_1 - V_8$. Now, M_0' and M_0'' are the eigenvalues of (10), so that

$$M_0' + M_0'' = A + D = 2M_{\frac{1}{2}} + W.$$

Therefore,

$$W = M_0' + M_0'' - 2M_{\frac{1}{2}}. \tag{36}$$

We also have

$$(A - M_0')(D - M_0') = BC.$$

Substituting for A, B, C, and D from (32), (33), (34), and (35) respectively, and substituting for W from (36), we obtain

$$\left(M_1 - \tfrac{4}{3}M_{\frac{1}{2}} - M_0' + \tfrac{2}{3}M_0''\right)\left(\tfrac{4}{3}M_{\frac{1}{2}} - M_1 - \tfrac{2}{3}M_0' + M_0''\right) = \tfrac{2}{9}\left(M_0' + M_0'' - 2M_{\frac{1}{2}}\right)^2 |\alpha|^2.$$

This can be rearranged to yield equation (11). Since α is an overlap integral, $|\alpha|^2 \leqslant 1$, and so the right-hand side of equation (11) is positive. Incidentally, since $|\alpha|^2 > 0$, we have altogether that

$$0 \leqslant (M_0' - M_1)(M_0'' - M_1) - \tfrac{4}{3}(M_{\frac{1}{2}} - M_1)(M_0' + M_0'' - 2M_{\frac{1}{2}}) \leqslant \tfrac{2}{9}(M_0' + M_0'' - 2M_{\frac{1}{2}})^2.$$