AVERAGE ELECTROMAGNETIC FORCES AND ENERGY*

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Recent results (Smith 1961, 1965*a*) expressing average generalized forces and average electromagnetic energy in terms of the observable external phenomenological behaviour of the system may be combined to give simple relationships between average forces and total stored energy. The relationships are similar to well-known energy theorems in electrostatics or magnetostatics but are different in detail. Universal results are obtained only for loss-free (energy-conserving) systems.

Initially, we consider loss-free systems for which the total average energy $\overline{W}$ may be written (Montgomery, Dicke, and Purcell 1948; Smith 1965*a*)

$$\overline{W} = \tfrac{1}{2}\mathbf{I}\dagger(\partial\mathbf{X}/\partial\omega)\mathbf{I} \tag{1}$$

$$= \tfrac{1}{2}\mathbf{V}\dagger(\partial\mathbf{B}/\partial\omega)\mathbf{V}\,, \tag{2}$$

where $\mathbf{I}$ and $\mathbf{V}$ are complex r.m.s. source "current" or "voltage" column vectors specifying the external excitation (angular frequency ω) of the system and $\mathbf{X}$ and $\mathbf{B}$ are generalized reactance and susceptance matrices describing the observable exterior behaviour. The average generalized force $\overline{F}_x$ corresponding to a mechanical parameter x of the system may also be expressed, using the same variables as equations (1) and (2) (Smith 1961), by

$$\omega\overline{F}_x = \tfrac{1}{2}\mathbf{I}\dagger(\partial\mathbf{X}/\partial x)\mathbf{I} \tag{3}$$

$$= \tfrac{1}{2}\mathbf{V}\dagger(\partial\mathbf{B}/\partial x)\mathbf{V}\,. \tag{4}$$

By differentiating equations (1) and (3) with $\mathbf{I}$ constant and (2) and (4) with $\mathbf{V}$ constant, $\mathbf{X}$ and $\mathbf{B}$ may be eliminated to give

$$\overline{F}_x+\omega(\partial\overline{F}_x/\partial\omega)_{\mathbf{I}} = (\partial\overline{W}/\partial x)_{\mathbf{I}}\,, \tag{5}$$

$$\overline{F}_x+\omega(\partial\overline{F}_x/\partial\omega)_{\mathbf{V}} = (\partial\overline{W}/\partial x)_{\mathbf{V}}\,. \tag{6}$$

It must be emphasized that the vectors $\mathbf{V}$ and $\mathbf{I}$ refer only to the external excitation of the system. Constancy of $\mathbf{I}$, for instance, does not imply that internal currents of the system are maintained constant throughout the differentiations in equation (5). Equations (5) and (6), applicable to any linear loss-free system at any frequency of excitation, may be regarded as average-energy conservation equations generalizing familiar results in magnetostatics or electrostatics (Jackson 1962). Although true for arbitrary loss-free systems, equations (5) and (6) are not usually valid for energy-dissipating systems, as is easily demonstrated by particular counter-examples.

The production of average electromagnetic levitation forces has been investigated anew recently (Smith 1965*b*, 1965*c*, 1965*d*; Vigoureux 1965; Graneau 1966). In this case also, a simple relationship between force and energy may be established. If the

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excitation coils and the levitated material are perfectly conducting and radiation effects are negligible, the system is loss-free and equation (5) applies. Further, within the quasi-stationary approximation, the levitation force for constant excitation current is independent of frequency, so that

$$\bar{F}_x = (\partial \bar{W}/\partial x)_{\mathbf{I}}. \tag{7}$$

Usually for levitation only a single excitation current I is involved, and (7) simplifies to

$$\bar{F}_x = (\partial \bar{W}/\partial x)_I. \tag{8}$$

Vigoureux takes equation (8) as the starting point for his discussion of levitation forces when the materials are not perfectly conducting, claiming that it is a universal result. There is no valid justification for this assertion. Equation (5) shows that it is not always true even for loss-free systems. Careful investigation of electromagnetic levitation forces (Smith 1965*b*, 1965*c*, 1965*d*) reveals that equation (8) does actually apply, but, far from being universal, it is specific to the particular system under consideration. It is important for equation (8) not to be confused with formulae (Maxwell 1881) that require *all* currents (internal currents included), and not just *terminal* currents, to be maintained constant during the differentiation (White and Woodson 1959). Graneau's discussion confuses this point.

Another variant of equations (5) and (6) for loss-free systems is of interest. Alternative forms for $\bar{W}$ and $\bar{F}_x$ are (Smith 1964, 1965*a*)

$$\mathrm{i}\bar{W} = -\tfrac{1}{2}\mathbf{b}\dagger(\partial \mathbf{S}/\partial \omega)\mathbf{a}, \tag{9}$$

$$\mathrm{i}\omega\bar{F}_x = -\tfrac{1}{2}\mathbf{b}\dagger(\partial \mathbf{S}/\partial x)\mathbf{a}, \tag{10}$$

where $\mathbf{a}$, $\mathbf{b}$ are the complex r.m.s. incoming and outgoing partial-wave column vectors and the scattering matrix $\mathbf{S}$ is defined by $\mathbf{b} = \mathbf{S}\mathbf{a}$. Differentiation of (9) and (10), maintaining the incoming wave vector constant, gives, in analogy with (5) and (6),

$$\bar{F}_x + \omega(\partial \bar{F}_x/\partial \omega)_{\mathbf{a}} = (\partial \bar{W}/\partial x)_{\mathbf{a}} + \tfrac{1}{2}\mathrm{i}\mathbf{a}\dagger\{(\partial \mathbf{S}\dagger/\partial \omega)(\partial \mathbf{S}/\partial x) - (\partial \mathbf{S}\dagger/\partial x)(\partial \mathbf{S}/\partial \omega)\}\mathbf{a}. \tag{11}$$

If a single partial wave is sufficient to describe the system (i.e. a one-port), the term involving $\mathbf{a}$ in the right-hand side of (11) vanishes, giving

$$\bar{F}_x + \omega(\partial \bar{F}_x/\partial \omega)_a = (\partial \bar{W}/\partial x)_a, \tag{12}$$

which is of similar form to (5) and (6). Vigoureux also discusses the radiation pressure of an unbounded plane wave, using

$$\bar{F}_x = (\partial \bar{W}/\partial x)_a \tag{13}$$

in place of equation (12). A correct result is obtained nevertheless, since in this particular case $(\partial \bar{F}_x/\partial \omega)_a = 0$. However, this agreement is fortuitous, and equation (13) must be modified when the force production process is dispersive.

Another way of writing equations (3) and (4) for loss-free systems uses, instead of the total energy, the average Lagrangian $\bar{L}$, which may be written

$$\omega\bar{L} = \tfrac{1}{2}\mathbf{I}\dagger\mathbf{X}\mathbf{I} \tag{14}$$

$$= -\tfrac{1}{2}\mathbf{V}\dagger\mathbf{B}\mathbf{V}. \tag{15}$$

Then,

$$\bar{F}_x = (\partial \bar{L}/\partial x)_{\mathrm{I}} \tag{16}$$

$$= -(\partial \bar{L}/\partial x)_{\mathrm{V}}, \tag{17}$$

where, as before, the differentiations are to be performed with constant complex current or voltage sources at the excitation "terminals". These results are not obtained by trivial time-averaging of Lagrange's equations of a network, because of this constraint in the differentiations. Internal coordinates of the system are not held constant by maintaining the excitation vectors constant. It is, of course, this involvement of the excitation vectors alone that makes the present results useful practically. For lossy systems neither (16) nor (17) is usually true.

References

GRANEAU, P. (1966).—*Br. J. appl. Phys.* **17**, 139.

JACKSON, J. D. (1962).—"Classical Electrodynamics." pp. 127, 177. (Wiley: New York.)

MAXWELL, J. C. (1881).—"A Treatise on Electricity and Magnetism." (2 vols.) Section 590. (Clarendon Press: Oxford.)

MONTGOMERY, C. G., DICKE, R. H., and PURCELL, E. M. (1948).—"Principles of Microwave Circuits." (McGraw-Hill: New York.)

SMITH, W. E. (1961).—*Aust. J. Phys.* **14**, 152.

SMITH, W. E. (1964).—*Aust. J. appl. Sci.* **15**, 65.

SMITH, W. E. (1965*a*).—*Aust. J. Phys.* **18**, 287.

SMITH, W. E. (1965*b*).—*Br. J. appl. Phys.* **16**, 377.

SMITH, W. E. (1965*c*).—*Aust. J. Phys.* **18**, 195.

SMITH, W. E. (1965*d*).—*J. Proc. R. Soc. N.S.W.* **98**, 151.

VIGOUREUX, P. (1965).—*Br. J. appl. Phys.* **16**, 757.

WHITE, D. C., and WOODSON, H. H. (1959).—"Electromechanical Energy Conversion." p. 22. (Wiley: New York.)