THE DETECTION OF A NON-STATIONARY SIGNAL IN NOISE

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Summary

Keen *et al.* (1965) have described the use of seismometer arrays for detecting earthquakes and underground atomic explosions. The purpose of the present paper is to investigate the merits of various correlation statistics based on the records from such an array. Assuming the probabilities of incorrect detection to be fixed in advance, a good statistic is one that (1) depends upon the observations only, (2) can detect a disturbance with small signal-to-noise ratio, and (3) is easy to compute. All the statistics considered possess property (1). Property (2) can be made more precise by defining the "efficiency" of a statistic. The amount of computation may be reduced (as suggested by Watts 1962) by digitalizing the records from the array, an extreme case of which is the situation when only the sign of each observation is recorded.

The results of the present paper suggest that considerable gain in efficiency may be achieved by using a correlator that sums all possible products of pairs of synchronized records, rather than one that simply multiplies the average disturbances from two arrays. While this involves a large number of multiplications, the computation may be simplified either by a coarse digitalization of both terms in each product, or by multiplying the first term in each product by the sign of the second. In both cases it is shown that the efficiency is not greatly reduced.

I. Introduction

The problem of detecting a stationary signal in the presence of stationary random noise has received much attention in recent years (see, for example, Wainstein and Zubakov 1962). The stationarity of the signal enables one to estimate its power consistently, that is, to find estimates that tend in probability to the true value of the power as the length of the record tends to infinity. A very good method for determining both the power and direction of the signal is the so-called crosscorrelation technique, in which the records from two receiving stations, sufficiently distant in location for the outputs to be statistically independent, are synchronized, multiplied, and averaged over time. The crosscorrelation method is improved by averaging the records from several receiving stations (see Jacobson 1957).

The case when the signal is not stationary, but dies out after a finite time (as would be the case for an earthquake or atomic explosion), does not seem to have received much attention in the literature. In this situation it is clearly impossible to find consistent statistics for estimating the power or detecting the presence of the signal with only two receiving stations. However, if we have a number of stations, it is possible to construct statistics that tend in probability to the true value as the number of stations tends to infinity, and are thus consistent.

In the present paper we investigate the problem of testing for the presence of a signal of the latter type. Various statistics, consistent in the above sense, will be

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defined, and judged according to two criteria, ease of computation and efficiency. A natural way to compare the efficiencies of several statistics is to compute the signal-to-noise ratio detectable by each statistic with pre-assigned errors of the first and second kind. Clearly the smaller the signal-to-noise ratio required, the more efficient will be the statistic. Since quantization of the values of a record (which includes digitalizing and rounding-off) considerably reduces the amount of computation and, as we shall see, hardly reduces the efficiency of the test statistic, we will consider this aspect in detail.

II. The Statistical Model

Let $S(t)$, with $-\infty < t < \infty$, denote the signal. We wish to test the hypothesis

$$H_0 : S(t) \equiv 0 \qquad \text{for} \quad t' \leqslant t \leqslant t'+T$$

against the alternative

$$H_1 : S(t) \equiv S^*(t) \qquad \text{for} \quad t' \leqslant t \leqslant t'+T\,,$$

for particular values of t', T, where $S^*(t)$ is of known form but may involve one or more unknown parameters. The data consist of the set of $2n$ records

$$X_i(t) = S(t+\tau_i)+N_i(t+\tau_i)\,, \qquad i = 1,2, \ldots 2n\,, \tag{1}$$

recorded at receiving stations sufficiently different in location for $X_i(t)$ and $X_j(t+\tau)$, $i \neq j$, to be statistically independent for all values of τ. The stations are arranged in the form of a right-angled cross and there are n stations on each arm. Let each $N_i(t)$ have zero mean value and finite variance σ^2, and common covariance function

$$E[N(t)\,N(t-\tau)] = R(\tau)\,. \tag{2}$$

Finally we assume that the "lags" τ_i, $i = 1,2, \ldots 2n$, are known. This is true if the direction of the signal is known.

Since the signal may be expected to undergo some distortion in passing from one station to another, it would be more realistic to replace equation (1) by

$$X_i(t) = S_i(t+\tau_i)+N_i(t+\tau_i)\,, \qquad i = 1,2, \ldots 2n\,. \tag{3}$$

However, if the functions $S_i(.)$ are monotonic functions of S, the above refinement will make little difference to the results we are about to prove, and for mathematical convenience equation (1) will be assumed.

III. Definitions of the Statistics

It is fruitless to look for fully efficient statistics (see Appendix) for testing H_0 against H_1. However, it is natural, and mathematically convenient, to use statistics based on quadratic forms in the observations. Let

$$C_1(t',T) = (n^2T)^{-1} \int_{t'}^{t'+T} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} X_i(t-\tau_i)\,X_j(t-\tau_j)\,dt\,. \tag{4}$$

$C_1(t',T)$ is the statistic used by Jacobson (1957) to estimate the direction of a stationary signal.

Define

$$C_2(t', T) = \{n(2n-1)T\}^{-1} \int_{t'}^{t'+T} \sum_{\substack{i=1,j=2 \\ i<j}}^{2n} X_i(t-\tau_i)\, X_j(t-\tau_j)\, \mathrm{d}t\,. \tag{5}$$

Clearly $C_1(t',T)$ is simpler to compute than $C_2(t',T)$, since $C_2(t',T)$ involves $n(2n-1)$ multiplications before averaging whereas $C_1(t',T)$ requires only one. However, it will be shown that $C_2(t',T)$ is more efficient than $C_1(t',T)$. Put

$$Q_1(t', T) = (n^2 T)^{-1} \int_{t'}^{t'+T} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} g(X_i(t-\tau_i))\, g(X_j(t-\tau_j))\, \mathrm{d}t\,, \tag{6}$$

where $g(.)$ is a single-valued function. If $g(.)$ is a step function, $g(X_j)$ may be called the quantized or, in the case of a step function with uniform jumps and intervals, digitalized value of X_j. An example of very coarse quantization occurs when $g(x) = \operatorname{sgn} x$. In this case the record takes one of two values at each instant, and is said to be "infinitely clipped" (see, for example, McFadden 1958).

In Section VI we shall discuss statistics of the form

$$Q_1'(t', T) = (n^2 T)^{-1} \int_{t'}^{t'+T} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} g_1(X_i(t-\tau_i))\, g_2(X_j(t-\tau_j))\, \mathrm{d}t\,, \tag{6a}$$

the functions $g_1(.)$ and $g_2(.)$ being different. As a special case of this, statistics in which $g_1(x) = x$ will be considered. This corresponds to quantizing only one of the factors in each product $X_i X_j$, and the merit of this was discussed by Watts (1962).

An alternative to $Q_1(t',T)$ is the statistic

$$Q_2(t', T) = \{n(2n-1)T\}^{-1} \int_{t'}^{t'+T} \sum_{\substack{i=1,j=2 \\ i<j}}^{2n} g(X_i(t-\tau_i))\, g(X_j(t-\tau_j))\, \mathrm{d}t\,, \tag{7}$$

which may be regarded as the quantized analogue to $C_2(t',T)$. Using equations (1) and (2), it follows from equations (4), (5), (6), and (7) that the mean values of the above statistics are given by

$$E[C_r(t', T)] = T^{-1} \int_{t'}^{t'+T} S^2(t)\, \mathrm{d}t, \qquad r = 1, 2\,, \tag{8}$$

and

$$E[Q_r(t', T)] = T^{-1} \int_{t'}^{t'+T} E^2[g(S(t)+N(t))]\, \mathrm{d}t, \qquad r = 1, 2\,. \tag{9}$$

In view of equation (8), $C_1(t',T)$ and $C_2(t',T)$ are unbiased estimators of the mean square of the signal over the interval $(t', t'+T)$, while, from equation (9), $Q_1(t',T)$ and $Q_2(t',T)$ are biased estimators of this quantity. Let us determine the magnitude of this bias in the case when $g(S(t)+N(t))$ represents a digitalized record. Put

$$g(x) = pd + \epsilon \qquad \text{for} \quad (p-\tfrac{1}{2})d + \epsilon \leqslant x < (p+\tfrac{1}{2})d + \epsilon\,, \tag{10}$$

where p ranges over all integers and $d > 0$ and ϵ are constants representing the size and position of the digitalizing intervals. It follows that

$$E[g(S(t)+N(t))] = \sum_{p=-\infty}^{\infty} (pd+\epsilon) \int_{(p-\frac{1}{2})d+\varepsilon}^{(p+\frac{1}{2})d+\varepsilon} f(x)\, \mathrm{d}x\,, \tag{11}$$

where $f(x)$ is the probability density function of $S(t)+N(t)$ for fixed t. Expanding the right-hand side of equation (11) in a Fourier series, after Fisher (1922), we have

$$E[g(S(t)+N(t))] = S(t)+\frac{d}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p}\phi(-2\pi p/d)\sin[2\pi p\{S(t)-\epsilon\}/d], \tag{12}$$

where $\phi(u)$ is the characteristic function of $N(t)$, that is,

$$\phi(u) = E[\exp\{\mathrm{i}uN(t)\}] = \int_{-\infty}^{\infty}\exp(\mathrm{i}ux)\, f(x+S(t))\,\mathrm{d}x. \tag{13}$$

If the noise is Gaussian, $\phi(u) = \exp(-\frac{1}{2}\sigma^2u^2)$ and, substituting equation (12) in (9), we obtain

$$E[Q_r(t',T)] = T^{-1}\int_{t'}^{t'+T}\left(S(t)+\frac{d}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^p}{p}\exp(-2\pi^2\sigma^2p^2/d^2)\sin[2\pi p\{S(t)-\epsilon\}/d]\right)^2\mathrm{d}t,$$

$$r = 1,2. \tag{14}$$

Observe that the bias due to digitalizing, which is given by the right-hand side of equation (14), is very small if d is smaller than σ, the standard deviation of the noise.

The bias due to infinite clipping, that is, when $g(x) = \operatorname{sgn} x$, may be obtained from equation (12) by the following argument. It is clear that infinite clipping corresponds to taking the limiting case of digitalization as $d\to\infty$ when $\epsilon = \frac{1}{2}d$. Thus from equation (12)

$$\begin{aligned} E[\operatorname{sgn}(S(t)+N(t))] &= \lim_{d\to\infty}\frac{2}{d}\left(S(t)+\frac{d}{\pi}\sum_{p=1}^{\infty}\frac{1}{p}\phi(-2\pi p/d)\sin\{2\pi pS(t)/d\}\right) \\ &= \frac{2}{\pi}\lim_{d\to\infty}\sum_{p=1}^{\infty}\frac{1}{p}\phi(-2\pi p/d)\sin\{2\pi pS(t)/d\}. \end{aligned} \tag{15}$$

The above infinite series may be replaced by the equivalent integral in the limit as $d\to\infty$, and we find

$$E[\operatorname{sgn}(S(t)+N(t))] = \int_0^{\infty}(2/\pi x)\sin\{2\pi xS(t)\}\,\phi(-2\pi x)\,\mathrm{d}x. \tag{16}$$

For Gaussian noise, equation (16) becomes

$$\begin{aligned} E[\operatorname{sgn}(S(t)+N(t))] &= \int_0^{\infty}(2/\pi x)\sin\{2\pi xS(t)\}\exp(-2\pi^2\sigma^2x^2)\,\mathrm{d}x \\ &= \operatorname{erf}\{S(t)/(2\sigma^2)^{\frac{1}{2}}\}. \end{aligned} \tag{17}$$

It follows that when $g(x) = \operatorname{sgn} x$

$$E[Q_r(t',T)] = T^{-1}\int_{t'}^{t'+T}[\operatorname{erf}\{S(t)/(2\sigma^2)^{\frac{1}{2}}\}]^2\,\mathrm{d}t, \qquad r = 1,2. \tag{18}$$

Observe that if $S(t)/(2\sigma^2)^{\frac{1}{2}}$ is small

$$E[Q_r(t',T)] = (\tfrac{1}{2}\pi\sigma^2T)^{-1}\int_{t'}^{t'+T}S^2(t)\,\mathrm{d}t + O(\{S(t)/\sigma\}^4). \tag{19}$$

IV. A Comparison of the Efficiencies of $C_1(t',T)$ and $C_2(t',T)$

In order to compare the efficiencies of the four statistics defined above, it is necessary to obtain expressions for their variances. Consider first the unquantized statistics. From equation (4)

$$E[C_1^2(t',T)] = (n^4T^2)^{-1} \iint_{t'}^{t'+T} E\left[\sum_{i=1}^{n}\sum_{k=1}^{n} X_i(t-\tau_i)\, X_k(u-\tau_k)\right] \times E\left[\sum_{j=n+1}^{2n}\sum_{l=n+1}^{2n} X_j(t-\tau_j)\, X_l(u-\tau_l)\right] \mathrm{d}t\,\mathrm{d}u$$

$$= (n^4T^2)^{-1} \iint_{t'}^{t'+T} \{n^2S(t)\,S(u) + nR(t-u)\}^2\,\mathrm{d}t\,\mathrm{d}u\,. \qquad (20)$$

Thus, using equation (8),

$$\operatorname{var}[C_1(t',T)] = \frac{2}{nT^2}\iint_{t'}^{t'+T} S(t)\,S(u)\,R(t-u)\,\mathrm{d}t\,\mathrm{d}u + \frac{1}{n^2T^2}\iint_{t'}^{t'+T} R^2(t-u)\,\mathrm{d}t\,\mathrm{d}u\,. \qquad (21)$$

Making the change of variables $t-u = x$, $u = y$, equation (21) may be written

$$\operatorname{var}[C_1(t',T)] = \frac{4}{nT^2}\int_0^T R(x)\,\mathrm{d}x \int_{t'}^{t'+T-x} S(y+x)\,S(y)\,\mathrm{d}y + \frac{2}{n^2T}\int_0^T (1-x/T)R^2(x)\,\mathrm{d}x. \qquad (22)$$

In a similar manner we obtain from equations (5) and (8)

$$\operatorname{var}[C_2(t',T)] = \frac{4(4n-1)}{3n(2n-1)T^2}\int_0^T R(x)\,\mathrm{d}x \int_{t'}^{t'+T-x} S(y+x)\,S(y)\,\mathrm{d}y + \frac{2}{n(2n-1)T}\int_0^T (1-x/T)R^2(x)\,\mathrm{d}x\,. \qquad (23)$$

When H_0 is true, the first terms in the right-hand sides of equations (22) and (23) vanish, and

$$\operatorname{var}[C_r(t',T)] = O(n^{-2}), \qquad r = 1,2\,. \qquad (24)$$

When H_1 is true, however,

$$\operatorname{var}[C_r(t',T)] = O(n^{-1}), \qquad r = 1,2\,. \qquad (25)$$

In each case it may be shown, using the central limit theorem, that the distributions of $C_r(t',T)-E[C_r(t',T)]$, $r = 1,2$, suitably scaled, are asymptotically normal as $n \to \infty$. It follows from equations (24) and (25) that $C_1(t',T)$ and $C_2(t',T)$ are consistent statistics for testing H_0 against H_1.

Let α and β be the probabilities of error of the first and second kind respectively, that is,

$$\alpha = \text{prob. (rejecting } H_0 \text{ when } H_0 \text{ is true)}, \qquad (26)$$

$$\beta = \text{prob. (accepting } H_0 \text{ when } H_1 \text{ is true)}. \qquad (27)$$

In order to form a basis for comparison of the statistics we will assume that α and β are fixed in advance. This imposes restrictions on the function $S^*(t)$, which specifies the alternative hypothesis. For example, if $S^*(t)$ is of known form, but involves one unknown parameter, its amplitude, say, then there is a value of this parameter corresponding to each pair of values of α and β. For this value of $S^*(t)$ we define the quantity

$$\mu = (\sigma^2 T)^{-1} \int_{t'}^{t'+T} \{S^*(t)\}^2 \, \mathrm{d}t \tag{28}$$

as the signal-to-noise ratio corresponding to a test of strength (α, β). The value of μ depends on the test statistic employed, and clearly the smaller the value of μ, the more efficient is the statistic. Let us compare the efficiencies of $C_1(t', T)$ and $C_2(t', T)$.

Assume n is sufficiently large for $C_1(t', T)$ and $C_2(t', T)$ to be approximately normally distributed, the discrepancy being negligible. Let the functions $S_1(t)$ and $S_2(t)$ specify alternative hypotheses that give a test of strength (α, β) for the statistics $C_1(t', T)$ and $C_2(t', T)$ respectively, so that from equation (28)

$$\mu_r = (\sigma^2 T)^{-1} \int_{t'}^{t'+T} S_r^2(t) \, \mathrm{d}t, \qquad r = 1, 2. \tag{29}$$

Write

$$S_r^2 = 2T^{-2} \int_0^T R(t) \int_{t'}^{t'+T-t} S_r(u+t)\, S_r(u) \, \mathrm{d}u \, \mathrm{d}t, \qquad r = 1, 2, \tag{30}$$

and

$$R^2 = 2T^{-1} \int_0^T (1 - t/T) R^2(t) \, \mathrm{d}t. \tag{31}$$

Because of the normality assumption, it follows from equations (8), (22), (23), and (31) that the distributions of $nC_1(t', T)/R$ and $\{n(2n-1)\}^{\frac{1}{2}} C_2(t', T)/R$ are standardized normal when H_0 is true. Employing, in addition, equations (29) and (30), we find that when H_1 is true

$$n\{C_1(t', T) - \sigma^2 \mu_1\}/(R^2 + 2nS_1^2)^{\frac{1}{2}} \quad \text{and} \quad \{n(2n-1)\}^{\frac{1}{2}}\{C_2(t', T) - \sigma^2 \mu_2\}/(R^2 + \tfrac{2}{3}(4n-1)S_2^2)^{\frac{1}{2}}$$

are also distributed as standardized normal variables. Consequently, for α and β to be the same for each statistic, the critical values λ_1 and λ_2 must satisfy

$$\left(\frac{n^2}{2\pi R^2}\right)^{\frac{1}{2}} \int_{\lambda_1}^{\infty} \exp(-\tfrac{1}{2} n^2 x^2 / R^2) \, \mathrm{d}x = \left(\frac{n(2n-1)}{2\pi R^2}\right)^{\frac{1}{2}} \int_{\lambda_2}^{\infty} \exp\{-\tfrac{1}{2} n(2n-1) x^2 / R^2\} \, \mathrm{d}x = \alpha, \tag{32}$$

and

$$\begin{aligned} &\left(\frac{n^2}{2\pi(R^2 + 2nS_1^2)}\right)^{\frac{1}{2}} \int_{\lambda_1}^{\infty} \exp\{-\tfrac{1}{2} n^2 (x - \sigma^2 \mu_1)^2 / (R^2 + 2nS_1^2)\} \, \mathrm{d}x \\ &\quad = \left(\frac{n(2n-1)}{2\pi\{R^2 + \frac{2}{3}(4n-1)S_2^2\}}\right)^{\frac{1}{2}} \int_{\lambda_2}^{\infty} \exp[-\tfrac{1}{2} n(2n-1)(x - \sigma^2 \mu_2)^2 / \{R^2 + \tfrac{2}{3}(4n-1)S_2^2\}] \, \mathrm{d}x \\ &\quad = 1 - \beta. \end{aligned} \tag{33}$$

Equations (32) reduce to

$$\lambda_1 n / R = \lambda_2 \{n(2n-1)\}^{\frac{1}{2}} / R = c(\alpha), \tag{34}$$

where $c(\alpha)$ is given by

$$\alpha = \int_{c(\alpha)}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2)\, \mathrm{d}x. \tag{35}$$

Substituting for λ_1 and λ_2 from equations (34), equations (33) reduce to

$$\frac{c(\alpha)\, R - n\sigma^2\mu_1}{(R^2+2nS_1^2)^{\frac{1}{2}}} = \frac{c(\alpha)\, R - \{n(2n-1)\}^{\frac{1}{2}}\sigma^2\mu_2}{\{R^2+\frac{2}{3}(4n-1)S_2^2\}^{\frac{1}{2}}} = c(1-\beta). \tag{36}$$

From equations (36) the values of μ_1 and μ_2 may be found, provided the signal is specified except for a single unknown parameter. Suppose this parameter is the amplitude, and let us investigate the effect of the form of the signal upon μ_1 and μ_2.

Define

$$\lambda = \frac{S_1^2}{\sigma^4\mu_1} = \frac{2\int_0^T R(t)\, \mathrm{d}t \int_{t'}^{t'+T-t} S_1(u+t)\, S_1(u)\, \mathrm{d}u}{\sigma^2 T \int_0^T S_1^2(t)\, \mathrm{d}t}. \tag{37}$$

The form of the function $S_1(t)$ determines the value of λ. Using equation (37), equations (36) imply that

$$\mu_1^2 - \frac{2}{n}\left\{\frac{Rc(\alpha)}{\sigma^2} + \lambda c^2(1-\beta)\right\}\mu_1 + \frac{R^2}{\sigma^4 n^2}\left\{c^2(\alpha) - c^2(1-\beta)\right\} = 0, \tag{38}$$

and, consequently, that the maximum (minimum) value of μ_1, and similarly μ_2, occurs when λ takes its maximum (minimum) value. It seems difficult to find the precise forms of the function $S_1(t)$ that respectively maximize and minimize the right-hand side of equation (37), but some indication of the wide range of variation of λ is demonstrated by the following two examples.

Put $R(t) = \sigma^2 \exp\{-(t/T)^2\}$. Then

(1) if $S_1(t) = A$, where A is a constant,

$$\lambda = \left(2\int_0^T (1-t/T)\exp\{-(t/T)^2\}\, \mathrm{d}t\right) \Big/ \int_0^T \mathrm{d}t = 0{\cdot}8615\,;$$

(2) if $S_1(t) = A\sin\{2\pi(t-t')/T\}$, where A is a constant,

$$\lambda = \frac{\int_0^T \{\cos(2\pi t/T) + (2\pi)^{-1}\sin(2\pi t/T)\}\exp(-t^2/T^2)\, \mathrm{d}t}{\int_0^T \sin^2(2\pi t/T)\, \mathrm{d}t} = 0{\cdot}0500.$$

From equation (31), $R = 0{\cdot}8741\,\sigma^2$. Suppose now $n = 20$ and $\alpha = \beta = 0{\cdot}025$. Then, from equation (38), $\mu_1 = 0{\cdot}502$ for case (1), and $\mu_1 = 0{\cdot}191$ for case (2). It would thus appear that the "square wave" is much more difficult to detect than the sine wave.

To compare μ_1 and μ_2 for various values of α, β, and n, let us put

$$R(t) = \sigma^2 \exp\{-(t/T)^2\} \quad \text{and} \quad S_r(t) = A_r \sin\{2\pi(t-t')/T\}, \qquad r = 1, 2.$$

Equation (38) becomes

$$\mu_1^2 - n^{-1}\{1\cdot 7482\, c(\alpha) + 0\cdot 1000\, c^2(1-\beta)\}\mu_1 + 0\cdot 7640\, n^{-2}\{c^2(\alpha) - c^2(1-\beta)\} = 0, \tag{39}$$

and the corresponding equation for μ_2, obtained from equations (36) in a similar manner, is

$$\mu_2^2 - \{n(2n-1)\}^{-\frac{1}{2}}\left(1\cdot 7482\, c(\alpha) + 0\cdot 0333\frac{(4n-1)c^2(1-\beta)}{\{n(2n-1)\}^{\frac{1}{2}}}\right)\mu_2 + \frac{0\cdot 7640}{n(2n-1)}\left(c^2(\alpha) - c^2(1-\beta)\right) = 0. \tag{40}$$

TABLE 1

VALUES OF SIGNAL-TO-NOISE RATIOS FOR VARIOUS VALUES OF α AND β

The values are for the statistics $C_1(t',T)$ (upper value) and $C_2(t',T)$ (lower value) for tests of strength (α,β), when the signal is sinusoidal and 20 records are used

β	α 0·01	0·05	0·10	0·20	0·50
0·01	0·461 0·332	0·379 0·284	0·348 0·258	0·290 0·227	0·207 0·167
0·05	0·394 0·274	0·315 0·227	0·274 0·202	0·228 0·172	0·144 0·114
0·10	0·358 0·244	0·280 0·198	0·241 0·174	0·194 0·144	0·113 0·087
0·20	0·315 0·210	0·238 0·165	0·199 0·141	0·154 0·112	0·074 0·056
0·50	0·232 0·150	0·158 0·106	0·121 0·083	0·077 0·054	— —

Note that μ_1 and μ_2 are each approximately inversely proportional to n. Values of μ_1 and μ_2 for $n = 10$ and various values of α and β are shown in Table 1, while Table 2 gives values of μ_1 and μ_2 for various values of n when $\alpha = \beta = 0\cdot 05$. Clearly, for all values, $C_2(t',T)$ is more efficient than $C_1(t',T)$.

V. THE EFFECT OF QUANTIZATION

We now investigate the effect of quantization on the efficiency of each test statistic. For the statistic $Q_1(t',T)$ we have from equation (5)

$$E[Q_1(t',T)] = (n^4T^2)^{-1}\iint_{t'}^{t'+T}\sum_{i=1}^{n}\sum_{k=1}^{n}\Big(E[g(X_i(t-\tau_i))\,g(X_k(u-\tau_k))] \times \sum_{j=n+1}^{2n}\sum_{l=n+1}^{2n} E[g(X_j(t-\tau_j))\,g(X_l(u-\tau_l))]\Big)\,\mathrm{d}t\,\mathrm{d}u$$

$$= (n^2T^2)^{-1}\iint_{t'}^{t'+T}\{(n-1)E[g(S(t)+N(t))]\,E[g(S(u)+N(u))] + E[g(S(t)+N(t))\,g(S(u)+N(u))]\}^2\,\mathrm{d}t\,\mathrm{d}u.$$

Now using equation (9) we obtain

$$\operatorname{var}[Q_1(t',T)] = 2n^{-1}U^2 + n^{-2}V^2, \tag{41}$$

where

$$U^2 = T^{-2}\iint_{t'}^{t'+T} E[g(S(t)+N(t))]\,E[g(S(u)+N(u))] \times \operatorname{cov}[g(S(t)+N(t))\,;\, g(S(u)+N(u))]\,\mathrm{d}t\,\mathrm{d}u \tag{42}$$

and

$$V^2 = T^{-2}\iint_{t'}^{t'+T} \{\operatorname{cov}[g(S(t)+N(t))\,;\, g(S(u)+N(u))]\}^2\,\mathrm{d}t\,\mathrm{d}u. \tag{43}$$

TABLE 2

VALUES OF SIGNAL-TO-NOISE RATIOS FOR VARIOUS VALUES OF n

The values are for a test of strength $(0\cdot05, 0\cdot05)$ when the signal is sinusoidal

$\frac{1}{2}$(No. of Records) n	μ_1 (for $C_1(t',T)$)	μ_2 (for $C_2(t',T)$)
5	0·629	0·467
10	0·315	0·227
15	0·210	0·150
20	0·157	0·112
25	0·126	0·090
50	0·063	0·045
100	0·031	0·022

Similarly it may be shown that

$$\operatorname{var}[Q_2(t',T)] = \frac{2(4n-1)}{3n(2n-1)}U^2 + \frac{1}{(2n-1)n}V^2. \tag{44}$$

It follows from equations (41) and (44) that $Q_1(t',T)$ and $Q_2(t',T)$ are consistent statistics for testing H_0 against H_1. Let $S_r(t)$ and $U_r(t)$, $r = 1,2$, specify alternative hypotheses that correspond to tests of strength (α,β) for $C_r(t',T)$ and $Q_r(t',T)$ respectively. Write

$$M_r = E[Q_r(t',T)], \qquad r = 1,2, \tag{45}$$

and denote by U_r^2 and V_r^2 the right-hand sides of equations (42) and (43) respectively with $S(.)$ replaced by $U_r(.)$, $r = 1,2$. From equation (9)

$$M_r = T^{-1}\int_{t'}^{t'+T} E[g(U_r(t)+N(t))]^2\,\mathrm{d}t, \tag{46}$$

and, using equation (28), signal-to-noise ratios for $Q_1(t',T)$ and $Q_2(t',T)$ are given by

$$\nu_r = (\sigma^2 T)^{-1}\int_{t'}^{t'+T} U_r^2\,(t)\,\mathrm{d}t, \qquad r = 1,2. \tag{47}$$

Proceeding as for equations (36), the signal-to-noise ratios for the four statistics for tests of strength (α, β) are given by equations (29) and (47), where

$$\frac{c(\alpha)\,R-n\sigma^2\mu_1}{(R^2+2nS_1^2)^{\frac{1}{2}}}=\frac{c(\alpha)\,V_1-nM_1}{(V_1+2nU_1^2)^{\frac{1}{2}}}=c(1-\beta) \tag{48}$$

and

$$\frac{c(\alpha)R-\{n(2n-1)\}^{\frac{1}{2}}\sigma^2\mu_2}{\{R^2+\frac{2}{3}(4n-1)S_2^2\}^{\frac{1}{2}}}=\frac{c(\alpha)\,V_2-\{n(2n-1)\}^{\frac{1}{2}}M_2}{\{V_2^2+\frac{2}{3}(4n-1)U_2^2\}^{\frac{1}{2}}}=c(1-\beta)\,. \tag{49}$$

Let us obtain expressions for V^2 and U^2 in the case when the signal is digitalized, that is, where $g(x)$ is given by equation (10). From equation (10)

$$\operatorname{cov}[g(S(t)+N(t)),g(S(u)+N(u))]=\sum_{p_1}\sum_{p_2}{}_{-\infty}^{\infty}(p_1d+\epsilon)(p_2d+\epsilon)\iint_{(p-\frac{1}{2})d+\varepsilon}^{(p+\frac{1}{2})d+\varepsilon}f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$-E[g(S(t)+N(t))]\,E[g(S(u)+N(u))], \tag{50}$$

where $f(x,y)$ is the joint probability density function of $S(t)+N(t)$, $S(u)+N(u)$. Using the Fourier series representation, as in equation (12), and assuming the noise to be Gaussian, we obtain

$$\operatorname{cov}[g(S(t)+N(t)),g(S(u)+N(u))]$$
$$=R(t-u)\left[1+2\sum_{p=1}^{\infty}(-1)^p\exp(-2\pi^2\sigma^2p^2/d^2)\left\{\cos\left(\frac{2\pi p(S(t)+\epsilon)}{d}\right)+\cos\left(\frac{2\pi p(S(u)+\epsilon)}{d}\right)\right\}\right]$$
$$-\frac{d^2}{4\pi^2}\sum_p{}'\sum_q{}'\frac{(-1)^{p+q}}{pq}\exp\{-2\pi^2\sigma^2(p^2+q^2)/d^2\}\Big(\exp\{-4\pi^2R(t-u)pq/d^2\}$$
$$\times\cos[2\pi\{p(S(t)+\epsilon)+q(S(u)+\epsilon)\}/d]+\sin\{2\pi p(S(t)+\epsilon)/d\}\sin\{2\pi p(S(u)+\epsilon)/d\}\Big), \tag{51}$$

where Σ' means summation over all integers except zero. V^2 and U^2 are now found by substituting equations (12) and (51) in the right-hand sides of equations (43) and (42) respectively. In particular, if d/σ is sufficiently small for terms involving $\exp(-2\pi^2p^2\sigma^2/d^2)$ to be neglected for all $p>0$ (that is, $d/\sigma \leqslant 1$), we find

$$V^2=T^{-2}\iint_{t'}^{t'+T}\left\{R(t-u)+\frac{d^2}{2\pi^2}\sum_{p=1}^{\infty}\left(\frac{1}{p}\right)^2\exp[-4\pi^2p^2\sigma^2\{1-\rho(t-u)\}/d^2]\right.$$
$$\left.\times\cos\{2\pi p(S(t)-S(u))/d\}\right\}^2\mathrm{d}t\,\mathrm{d}u \tag{52}$$

and

$$U^2=T^{-2}\iint_{t'}^{t'+T}S(t)\,S(u)\left\{R(t-u)+\frac{d^2}{2\pi^2}\sum_{p=1}^{\infty}\left(\frac{1}{p}\right)^2\exp[-4\pi^2p^2\sigma^2\{1-\rho(t-u)\}/d^2]\right.$$
$$\left.\times\cos\{2\pi pS(t)-S((u))/d\}\right\}\mathrm{d}t\,\mathrm{d}u\,, \tag{53}$$

where $\rho(t) = R(t)/\sigma^2$ and is supposed non-negative for $t\epsilon(-T,T)$. To this approximation equation (46) becomes

$$M_r = T^{-1} \int_{t'}^{t'+T} U_r^2(t)\,\mathrm{d}t, \qquad r = 1,2. \tag{54}$$

For a numerical comparison of μ_r and ν_r, $r = 1,2$, put $S_r(t) = A_r$, $U_r(t) = B_r$, $r = 1,2$, where A_r and B_r are constants and $R(t) = \sigma^2 \exp(-t^2/T^2)$ as before. From equations (52) and (53) we obtain

$$V_r^2 = 2\sigma^4 \int_0^1 (1-t)\left(\exp(-t^2) + \frac{d^2}{2\pi^2\sigma^2} \sum_{p=1}^{\infty} p^{-2} \exp[-4\pi^2 p^2 (\sigma/d)^2 \{1-\exp(-t^2)\}]\right)^2 \mathrm{d}t, \tag{55}$$

and

$$U_r^2 = 2\sigma^2 B_r^2 \int_0^1 (1-t)\left(\exp(-t^2) + \frac{d^2}{2\pi^2\sigma^2} \sum_{p=1}^{\infty} p^{-2} \exp[-4\pi^2 p^2 (\sigma/d)^2 \times \{1-\exp(-t^2)\}]\right) \mathrm{d}t, \qquad r = 1,2. \tag{56}$$

TABLE 3

VALUES OF SIGNAL-TO-NOISE RATIOS WHEN THE QUANTIZED RECORD IS DIGITALIZED

The values are for the statistics $C_r(t',T)$ and $Q_r(t',T)$, $r = 1,2$, for various values of the test strength (α,β), when the signal is a square wave and 20 records are used

(α,β)	$(0\cdot01, 0\cdot01)$	$(0\cdot05, 0\cdot05)$	$(0\cdot1, 0\cdot1)$	$(0\cdot2, 0\cdot2)$	$(0\cdot01, 0\cdot5)$	$(0\cdot5, 0\cdot01)$
μ_2	0·932	0·527	0·356	0·190	0·147	0·670
ν_2	0·950	0·537	0·363	0·194	0·150	0·682
μ_1	1·339	0·753	0·507	0·269	0·203	0·975
ν_1	1·365	0·768	0·517	0·274	0·208	0·986

Also equations (47) and (54) imply that

$$M_r = B_r^2 = \sigma^2 \nu_r, \qquad r = 1,2, \tag{57}$$

and from equations (30) and (31) we find

$$R^2 = 0\cdot7640\,\sigma^4 \text{ and } S_r^2 = 0\cdot8615\,A_r^2\sigma^2, \quad r = 1,2. \tag{58}$$

If $d = \sigma$, equations (55) and (56) become

$$V_r^2 = 0\cdot7961\,\sigma^4 \text{ and } U_r^2 = 0\cdot8775\,B_r^2\sigma^2, \quad r = 1,2. \tag{59}$$

Now using equations (57), (58), and (59), and putting $n = 10$, equations (48) and (49) reduce to

$$\frac{0\cdot8741\,c(\alpha) - 10\,\mu_1}{(0\cdot7640 + 17\cdot23\,\mu_1)^{\frac{1}{2}}} = \frac{0\cdot8922\,c(\alpha) - 10\,\nu_1}{(0\cdot7961 + 17\cdot55\,\nu_1)^{\frac{1}{2}}} = c(1-\beta), \tag{60}$$

and

$$\frac{0\cdot8741\,c(\alpha) - 13\cdot79\,\mu_2}{(0\cdot7640 + 22\cdot40\,\mu_2)^{\frac{1}{2}}} = \frac{0\cdot8922\,c(\alpha) - 13\cdot79\,\nu_2}{(0\cdot7961 + 22\cdot81\,\nu_2)^{\frac{1}{2}}} = c(1-\beta). \tag{61}$$

Values of μ_1, μ_2, ν_1, and ν_2 are given in Table 3 for various values of α and β. Observe the trifling effect of digitalization on the efficiency of the test.

When the quantizing function is $g(x) = \operatorname{sgn} x$, that is, when the record is completely clipped, we would expect a much greater loss in efficiency than in the above case of digitalization. As for $E[\operatorname{sgn}(S(t)+N(t))]$, given by equation (15), an expression for $\operatorname{cov}[\operatorname{sgn}(S(t)+N(t)),\ \operatorname{sgn}(S(u)+N(u))]$ is obtainable by taking the limiting case as $d \to \infty$ of digitalization with $\epsilon = \frac{1}{2}d$. For Gaussian noise we obtain from equations (17) and (51)

$$\begin{aligned}
&\operatorname{cov}[\operatorname{sgn}(S(t)+N(t)),\ \operatorname{sgn}(S(u)+N(u))] \\
&\quad = \lim_{d\to\infty} \frac{4}{d^2}\Bigg(-\frac{d^2}{4\pi^2} \sum_p{}' \sum_q{}' (pq)^{-1} \exp\{-2\pi^2\sigma^2(p^2+2pq\rho(t-u)+q^2)/d^2\} \\
&\qquad\qquad \times \cos\{2\pi(pS(t)+qS(u))/d\}\Bigg) - \operatorname{erf}\{S(t)/(2\sigma^2)^{\frac{1}{2}}\} \operatorname{erf}\{S(u)/(2\sigma^2)^{\frac{1}{2}}\}, \\
&\quad = \frac{1}{\pi^2} \iint_{-\infty}^{\infty} \frac{1}{xy} \exp[-\tfrac{1}{2}\{x^2+2xy\rho(t-u)+y^2\}] \cos\{(xS(t)+yS(u))/\sigma\}\, \mathrm{d}x\, \mathrm{d}y \\
&\qquad - \operatorname{erf}\{S(t)/(2\sigma^2)^{\frac{1}{2}}\} \operatorname{erf}\{S(u)/(2\sigma^2)^{\frac{1}{2}}\}. \qquad (62)
\end{aligned}$$

The integral on the right-hand side of equation (62) is not expressible in closed form. However, if $S(.)/\sigma$ is sufficiently small, we may neglect terms involving powers of $S^2(t)$, $S^2(u)$, and $S(t)S(u)$ higher than the first, and write for equation (62)

$$\begin{aligned}
&\operatorname{cov}[\operatorname{sgn}(S(t)+N(t)), \operatorname{sgn}(S(u)+N(u))] \\
&\quad = \frac{2}{\pi}\left(\arcsin\{\rho(t-u)\} - \frac{(S^2(t)+S^2(u))\rho(t-u)-2S(t)\,S(u)}{2\sigma^2[1-\rho^2(t-u)]^{\frac{1}{2}}} - \frac{S(t)\,S(u)}{\sigma^2}\right). \qquad (63)
\end{aligned}$$

Thus, using equations (19) and (63), equations (42) and (43) become

$$U^2 = \left(\frac{2}{\sigma\pi T}\right)^2 \iint_{t'}^{t'+T} S(t)\,S(u) \arcsin\{\rho(t-u)\}\, \mathrm{d}t\, \mathrm{d}u, \qquad (64)$$

and

$$\begin{aligned}
V^2 = \left(\frac{2}{\pi T}\right)^2 \iint_{t'}^{t'+T} &\arcsin\{\rho(t-u)\}\Big(\arcsin\{\rho(t-u)\} - 2S(t)\,S(u)/\sigma^2 \\
&-[(S^2(t)+S^2(u))\rho(t-u)-2S(t)\,S(u)]/[\sigma^2\{1-\rho^2(t-u)\}^{\frac{1}{2}}]\Big)\, \mathrm{d}t\, \mathrm{d}u. \qquad (65)
\end{aligned}$$

Similarly, equations (19) and (46) imply that

$$M_r = \frac{2}{\sigma^2 \pi T} \int_{t'}^{t'+T} U_r^2(t)\, \mathrm{d}t = 0{\cdot}6366\, \nu_r. \qquad (66)$$

For a numerical comparison of the signal-to-noise ratios, let us put

$$R(t) = \sigma^2 \exp(-t^2/T^2) \quad \text{and} \quad U_r(t) = B_r \sin\{2\pi(t-t')/T\}.$$

From equations (64), (65), and (66) we find

$$B_r^2 = 2\sigma^2 \nu_r ,$$

$$U_r^2 = \left(\frac{2B_r}{\sigma\pi}\right)^2 \int_0^1 \{(1-t)\cos 2\pi t + (\sin 2\pi t)/2\pi\} \arcsin\{\exp(-t^2)\}\, dt ,$$

$$= 0 \cdot 0619\, \nu_r ,$$

and

$$V_r^2 = \frac{8}{\pi^2} \int_0^1 \Big\{ (1-t)\arcsin^2\{\exp(-t^2)\} - \frac{B_r}{\sigma} \arcsin\{\exp(-t^2)\} \Big((1-t)\cos(2\pi t) + (\sin 2\pi t)/2\pi$$

$$+ \{1-\exp(-2t^2)\}^{-\frac{1}{2}} [\exp(-t^2)\{1-t+(\sin 4\pi t)/4\pi\} - (1-t)\cos(2\pi t)$$

$$+ (\sin 2\pi t)/2\pi] \Big) \Big\}\, dt$$

$$= 0 \cdot 6751 - 2 \cdot 2543\, \nu_r .$$

TABLE 4

VALUES OF SIGNAL-TO-NOISE RATIOS FOR INFINITELY CLIPPED DATA

The values are for the statistics $C_1(t', T)$, $C_2(t', T)$, and their analogues for various values of the test strength (α, β), when the signal is sinusoidal and 200 records are used

(α, β)	$(0 \cdot 05, 0 \cdot 05)$	$(0 \cdot 1, 0 \cdot 1)$	$(0 \cdot 2, 0 \cdot 2)$	$(0 \cdot 3, 0 \cdot 3)$	$(0 \cdot 4, 0 \cdot 4)$
μ_2	0·0222	0·0170	0·0109	0·0067	0·0032
ν_2	0·0346	0·0261	0·0165	0·0095	0·0047
μ_1	0·0315	0·0240	0·0154	0·0094	0·0045
ν_1	0·0472	0·0360	0·0230	0·0135	0·0067

Therefore equations (48) and (49) become

$$\frac{c(\alpha)(0 \cdot 6751 - 2 \cdot 2543\, \nu_1)^{\frac{1}{2}} - 0 \cdot 6366\, n\nu_1}{\{0 \cdot 6751 + (0 \cdot 1238\, n - 2 \cdot 2543)\nu_1\}^{\frac{1}{2}}} = \frac{c(\alpha)(0 \cdot 6751 - 2 \cdot 2543\, \nu_2)^{\frac{1}{2}} - 0 \cdot 6366\{n(2n-1)\}^{\frac{1}{2}}\nu_2}{\{0 \cdot 6751 + (0 \cdot 2051\, n - 2 \cdot 2956)\nu_2\}^{\frac{1}{2}}}$$

$$= c(1-\beta) , \qquad (67)$$

together with equations (39) and (40). Values of μ_1, μ_2, ν_1, and ν_2, for $n = 100$ and various values of (α, β), are given in Table 4. In this case the loss in efficiency due to quantizing the data is approximately 50%. Since infinite clipping is the most extreme form of quantization, this gives us a rough upper bound for the efficiency loss due to quantization.

VI. Partial Quantization

In Section III it was mentioned that the statistics $Q_1(t', T)$ and $Q_2(t', T)$ could be generalized to the case when each of the factors in the product $X_i X_j$ is quantized differently, and the generalization of $Q_1(t', T)$ is given by equation (6a). This gives rise to no mathematical difficulties, and it is easily shown that all the results of Section V remain valid, if equations (42), (43), and (46) are replaced by

$$U^2 = \tfrac{1}{2}T^{-2} \iint_{t'}^{t'+T} \{E[g_1(S(t)+N(t))]\,E[g_1(S(u)+N(u))] \times \operatorname{cov}[g_2(S(t)+N(t))\,;\, g_2(S(u)+N(u))] + E[g_2(S(t)+N(t))] \times E[g_2(S(u)+N(u))]\operatorname{cov}[g_1(S(t)+N(t))\,;\, g_1(S(u)+N(u))]\}\,\mathrm{d}t\,\mathrm{d}u\,, \tag{42a}$$

$$V^2 = T^{-2} \iint_{t'}^{t'+T} \operatorname{cov}[g_1(S(t)+N(t))\,;\, g_1(S(u)+N(u))] \times \operatorname{cov}[g_2(S(t)+N(t))\,;\, g_2(S(u)+N(u))]\,\mathrm{d}t\,\mathrm{d}u\,, \tag{43a}$$

and

$$M_r = T^{-1} \int_{t'}^{t'+T} E[g_1(U_r(t)+N(t))]\,E[g_2(U_r(t)+N(t))]\,\mathrm{d}t, \qquad r = 1,2. \tag{46a}$$

A special case of the above situation arises when $g_1(x) = x$, in which case the above equations become

$$U^2 = \tfrac{1}{2}T^{-2} \iint_{t'}^{t'+T} \{S(t)\,S(u)\operatorname{cov}[g(S(t)+N(t))\,;\, g(S(u)+N(u))] + R(t-u)\,E[g(S(t)+N(t))]\,E[g(S(u)+N(u))]\}\,\mathrm{d}t\,\mathrm{d}u\,, \tag{42b}$$

$$V^2 = T^{-2} \iint_{t'}^{t'+T} R(t-u)\operatorname{cov}[g(S(t)+N(t))\,;\, g(S(u)+N(u))]\,\mathrm{d}t\,\mathrm{d}u\,, \tag{43b}$$

and

$$M_r = T^{-1} \int_{t'}^{t'+T} U_r(t)\,E[g(U_r(t)+N(t))]\,\mathrm{d}t, \qquad r = 1,2. \tag{46b}$$

Let us consider the case when the data are "partially quantized" (that is, $g_1(x) = x$) and the second factor in the product $X_i X_j$ is infinitely clipped, or $g_2(x) = \operatorname{sgn} x$. Then proceeding as in Section V, and making the same assumptions concerning the magnitude of the signal compared with the standard deviation of the Gaussian noise, equations (42b), (43b), and (46b) become

$$U^2 = (1/\pi T^2) \iint_{t'}^{t'+T} S(t)\,S(u)[\arcsin\{\rho(t-u)\} + \rho(t-u)]\,\mathrm{d}t\,\mathrm{d}u\,, \tag{68}$$

$$V^2 = \frac{2}{\pi T^2} \iint_{t'}^{t'+T} \left\{R(t-u)\left(\arcsin\{\rho(t-u)\} - \frac{2S(t)\,S(u)}{\sigma^2} - \frac{(S^2(t)+S^2(u))\,\rho(t-u) - 2S(t)\,S(u)}{\sigma^2\{1-\rho^2(t-u)\}^{\frac{1}{2}}}\right)\right\}\mathrm{d}t\,\mathrm{d}u\,, \tag{69}$$

and

$$M_r = \frac{(2/\pi)^{\frac{1}{2}}}{\sigma T} \int_{t'}^{t'+T} U_r^2(t)\,\mathrm{d}t, \qquad r = 1,2. \tag{70}$$

Now putting $R(t) = \sigma^2 \exp(-t^2/T^2)$ and $U_r(t) = B_r \sin\{2\pi(t-t')/T\}$, we find

$$M_r = 0\cdot3989\, B_r^2/\sigma = 0\cdot7978\,\sigma\nu_r\,,$$

$$V_r^2 = (0\cdot6396-1\cdot6256\,\nu_r)\sigma^2, \qquad U_r^2 = 0\cdot0947\,\sigma^2\nu_r\,, \qquad r = 1,2\,,$$

and equations (48) and (49) give

$$\frac{c(\alpha)(0\cdot6396-1\cdot6256\,\nu_1)^{\frac{1}{2}}-0\cdot7978\,n\nu_1}{\{0\cdot6396+(0\cdot1894\,n-1\cdot6256)\nu_1\}^{\frac{1}{2}}} = \frac{c(\alpha)(0\cdot6396-1\cdot6256\,\nu_2)^{\frac{1}{2}}-0\cdot7978\{n(2n-1)\}^{\frac{1}{2}}\nu_2}{[0\cdot6396+\{0\cdot0631(4n-1)-1\cdot6256\}\nu_2]^{\frac{1}{2}}}$$

$$= c(1-\beta)\,, \tag{71}$$

together with equations (39) and (40). Values of ν_1 and ν_2 for $n = 100$ and various values of (α,β) are given in Table 5. These may be compared with the values in Table 4. Note that the loss in efficiency in this case is approximately 17%, whereas when both factors were infinitely clipped the loss was 50%.

Table 5

VALUES OF SIGNAL-TO-NOISE RATIOS FOR PARTIALLY INFINITELY CLIPPED DATA

The values are for the statistics $C_1'(t',T)$ and $C_2'(t',T)$ for various values of the test strength (α,β), when the signal is sinusoidal and 200 records are used

(α,β)	$(0\cdot05, 0\cdot05)$	$(0\cdot1, 0\cdot1)$	$(0\cdot2, 0\cdot2)$	$(0\cdot3, 0\cdot3)$	$(0\cdot4, 0\cdot4)$
ν_1	0·0393	0·0284	0·0186	0·0114	0·0052
ν_2	0·0279	0·0210	0·0132	0·0079	0·0037

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APPENDIX

In the case when the noise is Gaussian it may be shown, using the maximum likelihood method, that the optimum statistic for testing H_1 is of the form

$$C(t', T) = \frac{1}{2nT} \int_{t'}^{t'+T} H(t-t') \sum_{i=1}^{2n} X_i(t-\tau_i) \, dt,$$

where $H(t)$ depends upon $S^*(t)$ and $R(t)$. However, $C(t',T)$ suffers from the fact that usually $H(t)$ is not known beforehand. Consequently we are concerned exclusively with statistics that are functions of the observations only.