MULTIPARTICLE COLLISIONS

II.‡ APPLICATION OF UNITARITY

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Summary

The application of unitarity to multiparticle production processes is studied and relationships between production and scattering amplitudes are derived.

I. Introduction

In Part I of this work (Cook 1967) the possible structures of many-particle partial wave amplitudes were reviewed. In the present paper, the application of unitary properties of the scattering matrix is investigated to ascertain how much information is provided by the principle of unitarity. From standard tests such as those of Blatt and Weisskopf (1952) as applied by Kibble (1960), the unitary condition may be written

$$S^\dagger S = I,$$

from which one gets

$$(2i)^{-1}(A_{fi} - A_{fi}^*) = \sum_n \int d\Omega_n \, A_{fn}^* \, A_{ni}, \tag{1}$$

where $A_{fi} = \langle f|T|i \rangle$ is the transition amplitude between states of i and f particles respectively, $T = (2i)^{-1}(I - S)$ is the transition matrix, where S is the scattering matrix, and $d\Omega_n$ is the volume element of all degrees of freedom in intermediate states of n particles.

The theorem of reciprocity states that

$$T_{fi} = T_{-i-f}. \tag{2}$$

Only cases where the interacting particles have no spin are dealt with, and then A_{fi} is simply a scalar complex number and both sides of equation (1) are real.

II. Integration over Intermediate States

If the partial wave expansion of the general vertex describing transitions from a state with i particles to a state with f particles were known, it would prove

possible to carry out the integrations on the right-hand side of equation (1) and so obtain algebraic relationships between the partial wave amplitudes for production and scattering. As in Part I, we write

$$A_{fi} = (2\pi)^{-\frac{1}{2}} \sum_L (2L+1)\, \psi_f(LM', \boldsymbol{\Omega}')\, a_{fi}(L, W)\, D^L_{M'M}(W_{fi})\, \psi_i(LM, \boldsymbol{\Omega})\,, \qquad (3)$$

where ψ_f, ψ_i are multiparticle states of orbital angular momentum L and z component M, for f and i particles respectively, $D^L_{M'M}(W_{fi})$ is the rotation group operator, A_{fi} is the partial wave amplitude, and $\boldsymbol{\Omega}', \boldsymbol{\Omega}$ are the additional degrees of freedom required to specify final and initial configurations respectively.

The integrated product in (1) is evaluated as follows. After elimination of the kinematical constraints we obtain

$$\int \mathrm{d}\boldsymbol{\Omega}_n\, A^{\bullet}_{fn}\, A_{ni} = \int \mathrm{d}\boldsymbol{\Omega}''_n \sum_{L'} \sum_{L''} \psi^{\bullet}_f(L'M'; \boldsymbol{\Omega})\, a^{\bullet}_{fn}(L'; W) \times \psi_n(L'M''; \boldsymbol{\Omega}'')$$

$$D^{L'}_{M'M''}(W_{fn})\, D^L_{M'M}(W_{ni})\, \psi_n(LM; \boldsymbol{\Omega}'')\, a_{ni}(L; W) \times \psi_i(LM; \boldsymbol{\Omega})\,. \qquad (4)$$

The phase space factors J_n incorporated into the ψ_n and $\mathrm{d}\boldsymbol{\Omega}''_n$ will cancel and, following from the orthogonality of the ψ_n,

$$\int \mathrm{d}\boldsymbol{\Omega}''_n\, \psi_n(L'M'; \boldsymbol{\Omega}'')\, \psi_n(LM; \boldsymbol{\Omega}'') = \delta(L', L)\, \delta(M'M)\,,$$

as well as from the addition theorem for the rotation group operators (Edmunds 1957), namely,

$$\sum_{M''} D^L_{M'M''}(W_{fn})\, D^L_{M''M}(W_{ni}) = D^L_{M'M}(W_{fi})\,,$$

one obtains

$$\int \mathrm{d}\boldsymbol{\Omega}_n\, A^{\bullet}_{fn}\, A_{ni} = \sum_{L\,M'\,M} \psi_f(LM'; \boldsymbol{\Omega}')\, a^{\bullet}_{fn}(L, W)\, a_{ni}(L, W)\, D^L_{M'M}(W_{fi}) \times \psi_i(LM; \boldsymbol{\Omega})\,. \quad (5)$$

Now we select coordinates in initial and final states such that the z axis lies in a plane perpendicular to L in each case; hence $M' = M = 0$. The rotation group operators obey the property (Rose 1957)

$$\int \mathrm{d}w\, D^{L_1}_{M_1M_1'}(w)\, D^{L_2}_{M_2M_2'}(w) = \frac{8\pi^2}{2L_1+1}\, \delta(M_1, M_2)\, \delta(M_1', M_2')\, \delta(L_1, L_2)\,, \qquad (6)$$

while it is assumed that $\psi_n(L0; \boldsymbol{\Omega})$ is a real function. Using these rules, we can project out the $M = M' = 0$ states in (5), and the left-hand side of (1), to obtain

$$(2\mathrm{i})^{-1}\{a_{fi}(L, W) - a^{\bullet}_{fi}(L, W)\} = \sum_n a^{\bullet}_{fn}(L, W)\, a_{ni}(L, W)\,. \qquad (7)$$

Any scalar amplitude may be written

$$a_{fi} = \rho_{fi}\, \exp \mathrm{i}\, \delta_{fi}\,,$$

and substituting this form into (7), we obtain

$$\operatorname{Im} a_{fi}(L, W) = \sum_n \rho_{fn}(L, W)\, \rho_{ni}(L, W)\, \exp[\mathrm{i}\{\delta_{ni}(L, W) - \delta_{fn}(L, W)\}]\,. \qquad (8)$$

The left-hand side of (8) is a real number, and the right-hand side must therefore satisfy

$$\sum_n \rho_{fn} \rho_{ni} \sin(\delta_{ni} - \delta_{fn}) = 0, \tag{9}$$

where the (L, W) have been dropped for convenience.

One simple way to satisfy the stringent condition (9) is to introduce an equiphase principle. We assume that

$$\delta_{fn} = \delta_{ni} \qquad \text{for all } (f, n, i), \tag{10}$$

and explore the consequences.

The validity of the equiphase principle depends upon the dynamics of the intermediate processes. It is rigorously correct in the Breit–Wigner (1936) single-level approximation, where the branching ratios become the partial widths of the resonant state. If one assumes that resonant amplitudes are additive (Feshbach, Porter, and Weisskopf 1954), and that all phase shifts can be explained as the contribution from distant levels, it is approximately true if the partial widths for particular processes are roughly equal. Using the multilevel formalism of Reich and Moore (1958) it is apparently approximately true if the level separations are much greater than their widths. The exact principle has also been applied to photoproduction of pions in relation to scattering (Bethe and de Hoffman 1955). There exists no proof of the principle or the general conditions under which it holds, although Hamilton (1959) relates a proof of its validity near thresholds.

III. Partial Wave Amplitudes in the Equiphase Assumption

If ρ_{fi} is regarded as an $n \times n$ matrix, where up to n initial, final, or intermediate particles are kinematically possible, then in the equiphase assumption, equation (8) states

$$\boldsymbol{\rho} \cdot \boldsymbol{\rho} = \boldsymbol{\rho} \sin \delta. \tag{11}$$

The partial wave projections from the $\boldsymbol{T}$ matrix elements can be written

$$\boldsymbol{T} = \boldsymbol{\rho} \exp i \delta. \tag{12}$$

Since $\boldsymbol{\rho}$ is a real symmetric matrix, it follows from (11) that

$$\boldsymbol{\rho}^n = (\sin \delta)^{n-1} \boldsymbol{\rho} = (\sin \delta) \boldsymbol{\rho}^{n-1}.$$

When the determinant of both sides of (11) is taken, one finds

$$(\det \boldsymbol{\rho})^2 = (\sin \delta)^n \det \boldsymbol{\rho},$$

that is, $\det \boldsymbol{\rho}$ equals either 0 or $(\sin \delta)^n$.

The second result corresponds to the trivial solution

$$\boldsymbol{\rho} = (\sin \delta) \boldsymbol{I},$$

but for the first solution, the physically interesting one, $\boldsymbol{\rho}$ is a singular matrix. Its characteristic equation

$$\det |\boldsymbol{\rho} - \lambda I| = 0,$$

has $n - 1$ zero roots with one root equal to $\sin \delta$. Therefore $\boldsymbol{\rho}$ is of unit rank and

all principal minors of order greater than unity vanish. It follows from the expansion of the characteristic equation and the Cayley–Hamilton theorem (Mirsky 1955) that

$$(-\sin\delta)^n + (-\sin\delta)^{n-1}\,\text{trace}\,\boldsymbol{\rho} = 0\,,$$

so that

$$\text{trace}\,\boldsymbol{\rho} = \sin\delta\,, \qquad \text{trace}\,\boldsymbol{T} = \sin\delta\exp i\,\delta\,, \qquad \text{and} \quad \det\boldsymbol{T} = 0. \qquad (13)$$

Thus $\boldsymbol{\rho}$ and $\boldsymbol{T}$ possess no inverse; on the other hand, the matrix

$$\boldsymbol{S} = \boldsymbol{I} - 2i\,\boldsymbol{\rho}\exp i\,\delta$$

fulfils the conditions

$$\boldsymbol{S}^\dagger\boldsymbol{S} = \boldsymbol{I}\,,$$

$$\det\boldsymbol{S} = \exp 2i\,\delta\,,$$

$$\det\boldsymbol{S}\det\boldsymbol{S}^\dagger = 1\,,$$

$$\boldsymbol{S}^\dagger = \boldsymbol{S}^{-1}\,.$$

$\boldsymbol{\rho}$ also has the curious property that

$$(\text{trace}\,\boldsymbol{\rho})^n = \text{trace}\,\boldsymbol{\rho}^n\,.$$

IV. Branching Ratios

The condition that every minor of $\boldsymbol{\rho}$ of order greater than unity should vanish leads to the condition that all 2×2 minors should vanish. Hence

$$\rho_{fi}\rho_{kl} = \rho_{fl}\rho_{ki}\,. \qquad (14)$$

For example

$$\rho_{23}^2 = \rho_{22}\rho_{33}\,.$$

We define branching ratios Γ_{fi} by

$$\Gamma_{fi} = \Gamma\rho_{fi}/\sin\delta\,, \qquad (15)$$

and from (13) we find

$$\Gamma = \sum_i \Gamma_{ii}\,. \qquad (16)$$

Also, equation (14) relates all off-diagonal elements to diagonal ones by the relation

$$\Gamma_{fi} = \pm(\Gamma_{ff}\Gamma_{ii})^{\frac{1}{2}}\,. \qquad (17)$$

Since Γ_{fi} appears in total cross sections as a factor of proportionality, we choose the positive roots of (17). The matrix

$$B_{fi} = \Gamma_{fi}/\Gamma = \rho_{fi}/\sin\delta \qquad (18a)$$

satisfies

$$\boldsymbol{B}^2 = \boldsymbol{B}\,, \qquad (18b)$$

$$\text{trace}\,\boldsymbol{B} = 1\,, \qquad (18c)$$

and

$$\det\boldsymbol{B} = 0. \qquad (18d)$$

V. CROSS SECTIONS AND PHASE SHIFTS

The total cross section for a particular reaction is defined by

$$\sigma_{fi}(W, \mathbf{\Omega}) = \int d\mathbf{\Omega}_f \, |A_{fi}|^2 \,. \tag{19}$$

This expression can be evaluated by substituting (3) into (19) to obtain

$$\sigma_{fi}(W, \mathbf{\Omega}) = 4\pi \sum_L (2L+1) \rho_{fi}^2(L,W) \, |\psi_i(L0;\mathbf{\Omega})|^2 \,. \tag{20}$$

With the equiphase assumption we get:

$$\sigma_{fi}(W, \mathbf{\Omega}) = 4\pi \sum_L (2L+1) \frac{\Gamma_{fi}^2(L,W)}{\Gamma^2(L,W)} \sin^2\delta(L,W) \, |\psi_i(L0;\mathbf{\Omega})|^2 \,,$$

$$= 4\pi \sum_L (2L+1) \frac{\Gamma_{ff}(L,W)\,\Gamma_{ii}(L,W)}{\Gamma^2(L,W)} \sin^2\delta(L,W) \, |\psi_i(L0;\mathbf{\Omega})|^2 \,. \tag{21}$$

For a two-particle initial state

$$\psi_2(L0;\mathbf{\Omega}) = J_2^{-\frac{1}{2}} \,, \tag{22}$$

where J_2 is the phase space factor for the state. The inelastic cross sections obtained from a two-particle state are found by substituting (22) into (21) to obtain

$$\sigma_{f2}(W) = (4\pi/J_2) \sum_L (2L+1) \frac{\Gamma_{ff}(L,W)\,\Gamma_{22}(L,W)}{\Gamma^2(L,W)} \sin^2\delta(L,W) \,. \tag{23}$$

Now the scattering amplitude is usually represented in terms of a complex phase shift $(\alpha + i\beta)$ such that in each eigenstate of L

$$a_n(L,W) = \sin(\alpha + i\beta) \exp i(\alpha + i\beta) \,. \tag{24}$$

By equating (24) to the polar form, one finds:

$$\rho_{22} = \tfrac{1}{2}\{(1 + \exp(-4\beta)) - 2\cos\alpha \exp(-2\beta)\}^{\frac{1}{2}}$$

and $\qquad \delta = \tan^{-1}\{(1 - \exp(-2\beta))\cos 2\alpha / \exp(-\beta)\sin 2\alpha\} \,.$ $\qquad$ (25)

The inverse transformations are

$$\alpha = \tfrac{1}{2}\tan^{-1}\{(2\rho_{22}\cos\delta)/(1 - 2\rho_{22}\sin\delta)\}$$

and $\qquad \beta = -\tfrac{1}{4}\ln\{1 + 4\rho_{22}(\rho_{22} - \sin\delta)\} \,.$ $\qquad$ (26)

The absorption coefficient

$$\eta(L,W) = \exp\{-2\beta(L,W)\}$$

is such that in each eigenstate

$$\rho_{22}(\sin\delta - \rho_{22}) = \tfrac{1}{4}\{1 - \exp(-4\beta)\} \,,$$

that is,

$$(\Gamma_{22}/\Gamma)\,(1 - \Gamma_{22}/\Gamma)\sin^2\delta = \tfrac{1}{4}(1 - \eta^2) \,.$$

The total production cross section from an initial state of two particles becomes

$$\sigma_{\mathrm{prod},2}(W) = \sum_{f=3}^{n} \sigma_{f2}(W) = (4\pi/J_2) \sum_{f=3}^{n} \sum_{L} (2L+1)\rho_{f2}^2(L,W)$$

$$= (4\pi/J_2) \sum_{f=3}^{n} \sum_{L} (2L+1)\rho_{22}(L,W)\{\rho_{33}(L,W)+\rho_{44}(L,W)+\ldots\}$$

$$= (4\pi/J_2) \sum_{L} (2L+1)\frac{\Gamma_{22}(L,W)}{\Gamma(L,W)}\left(1-\frac{\Gamma_{22}(L,W)}{\Gamma(L,W)}\sin^2\delta(L,W)\right)$$

$$= (\pi/J_2) \sum_{L} (2L+1)\{1-\eta^2(L,W)\}. \tag{27}$$

For single-level approximations, one may put

$$\cot\delta = 2(E_r-E)/\Gamma,$$

where E_r is the energy at resonance and E the total energy of the initial particles. This equation yields the Breit–Wigner (19) form in (23) for the partial cross sections.

VI. A Simple Coupling Scheme

In the matrix $\boldsymbol{B}$ of the branching ratios, given by equation (18a), the diagonal elements are unrelated. The reciprocity theorem (2) leads to the result

$$\rho_{fi} = \rho_{if}.$$

If we assume that this assumption may be generalized in such a way that

$$\rho_{fi} = \rho_{f-r,i+r} = \rho_{f+r,i-r}, \qquad |i-r|, |f-r| \geqslant 2, \tag{28}$$

the diagonal elements become related. The consequence of the postulate (28) is that the total cross sections (21) integrated over the initial configuration become invariant under the complex Lorentz transformations that change a particle from an initial state incoming to a final state outgoing configuration. That is, if

$$\sigma_{fi}(W) = \int d\boldsymbol{\Omega}_i \, \sigma_{fi}(W,\boldsymbol{\Omega})$$

$$= 4\pi \sum_{L} (2L+1)\rho_{fi}^2(L,W),$$

then
$$\sigma_{fi}(W) = \sigma_{f+r,\,i-r}(W). \tag{29}$$

The diagonal elements of $\boldsymbol{B}$ become related by

$$\xi = \Gamma_{33}/\Gamma_{22} = \Gamma_{44}/\Gamma_{33} = \Gamma_{ii}/\Gamma_{i-1,i-1}, \tag{30}$$

in which case

$$\boldsymbol{B} = \frac{\Gamma_{22}}{\Gamma}\begin{pmatrix} 1 & \xi & \ldots \\ \xi & \xi^2 & \ldots \end{pmatrix},$$

$$\mathrm{trace}\,\boldsymbol{B} = \frac{\Gamma_{22}}{\Gamma}\frac{1-\xi^{2(n-1)}}{1-\xi^2} = 1, \tag{31}$$

and
$$\rho_{fi} = (\xi)^{i+f-4}\rho_{22}. \tag{32}$$

In this way all production amplitudes are related to scattering and the entire set of n^2 reactions is specified by two parameters such as (ρ_{22}, δ) or (Γ_{22}, δ) per eigenstate of L.

Using the above theory, similar results are derived for different types of $(2 \to 2)$ or $(i \to f)$ reactions. In these cases we simply subdivide Γ_{22} into subsets

$$\Gamma_{22} = \Gamma_{22}(a) + \Gamma_{22}(b) + \ldots$$

$$\Gamma_{fi} = \Gamma_{fi}(a) + \Gamma_{fi}(b) + \ldots$$

to obtain branching ratios for reactions (a), (b), ... respectively.

VII. REFERENCES

BETHE, H. A., and DE HOFFMAN, F. (1955).—"Mesons and Fields. Volume II, Mesons." p. 184. (Row, Peterson, & Co.: New York.)

BLATT, J. M., and WEISSKOPF, V. F. (1952).—"Theoretical Nuclear Physics." (Wiley: New York.)

BREIT, G., and WIGNER, E. (1936).—*Phys. Rev.* **49**, 519.

COOK, J. L. (1967).—*Aust. J. Phys.* **20**, 347.

EDMUNDS, A. R. (1957).—"Angular Momentum in Quantum Mechanics." (Princeton Univ. Press.)

FESHBACH, H., PORTER, C. E., and WEISSKOPF, V. F. (1954).—*Phys. Rev.* **96**, 448.

HAMILTON, J. (1959).—"The Theory of Elementary Particles." p. 360. (Oxford Univ. Press.)

KIBBLE, T. W. B. (1960).—*Phys. Rev.* **117**, 1159.

MIRSKY, L. (1955).—"Introduction to Linear Algebra." p. 206. (Oxford Univ. Press.)

REICH, C. W., and MOORE, M. S. (1958).—*Phys. Rev.* **111**, 929.

ROSE, M. E. (1957).—"Elementary Theory of Angular Momentum." (Wiley: New York.)