SHORT COMMUNICATIONS

CORNER CONDITIONS FOR WEAK SHOCK DIFFRACTION BY A WEDGE*

By N. J. de Mestre†

In deriving the first-order approximate solutions to the problem of the diffraction of a propagating pressure discontinuity by a rigid wedge in a non-viscous, non-thermally conducting, polytropic gas, Miles (1952) and Friedlander (1958) have taken as the corner conditions that the pressure remains finite and that the velocity may have an integrable singularity, whereas Keller and Blank (1951) gave no discussion of the corner conditions at all. Friedlander imposed the above-mentioned conditions in order to ensure the validity of the uniqueness theorem of the initial value problem for the wave equation, whereas Miles invoked them through physical requirements but gave no details.

Ordinarily finite force (integrable pressure) would seem to be the condition needed and this led the author to consider the possibility of singularities at the corner.

Consider the problem of a plane weak shock wave of strength

$$(p_1-p_0)/\gamma p_0 \equiv \epsilon$$

being diffracted by a rigid convex-angled wedge. (Here p_1 and p_0 are respectively the pressures behind and in front of the incident shock, while γ is the adiabatic index of the gas.) The essential features of the investigation are shown by choosing the direction of incidence of the shock such that the problem can be treated as two dimensional and unsteady, with the added simplification that there are no reflected shocks (see Fig. 1). Cartesian axes $0x$ and $0y$ are chosen with the corner as the origin 0, the y axis being in the direction of propagation of the incident shock and the x axis such that its positive direction points into the area corresponding to the shadow region of geometrical acoustics. Let the angle of the wedge be denoted by $\frac{1}{2}\pi+\omega$.

If the assumption of "conical" flow is made, the first-order approximate problem reduces to finding a solution (inside the boundaries) of the equation

$$(1-r^2)P^{(1)}_{rr}+(r^{-1}-2r)P^{(1)}_{r}+r^{-2}P^{(1)}_{\theta\theta} = 0, \tag{1}$$

subject to the boundary conditions

$$P^{(1)} = 0 \qquad (r = 1, \quad \omega \leqslant \theta < \tfrac{1}{2}\pi), \tag{2}$$

$$P^{(1)} = 1 \qquad (r = 1, \quad \tfrac{1}{2}\pi < \theta \leqslant \tfrac{3}{2}\pi), \tag{3}$$

$$P^{(1)}_{\theta} = 0 \qquad (0 \leqslant r \leqslant 1, \quad \theta = \omega \text{ and } \tfrac{3}{2}\pi), \tag{4}$$

where

$$(p-p_0)/\gamma p_0 = \epsilon P^{(1)}+\epsilon^2 P^{(2)} + \dots$$

* Manuscript received August 28, 1967.

† Royal Military College, Duntroon, A.C.T. 2600.

and (r, θ) are the conical polar coordinates corresponding to the Cartesian coordinates $(x/a_0 t, y/a_0 t)$, the sound speed in the gas ahead of the shock being denoted by a_0.

The conditions of finite pressure and integrable singularity for the velocity yield a unique solution of equations (1)–(4), which, when expanded near $r = 0$, gives

$$(p-p_0)/\gamma p_0 = -\epsilon(2^{1-\lambda}/\pi)\sin\{\lambda(\tfrac{1}{2}\pi-\omega)\}\cos\{\lambda(\theta-\omega)\}r^{\lambda}+o(\epsilon r^{\lambda}), \tag{5}$$

$$u = \epsilon\{\lambda 2^{1-\lambda}/\pi(1-\lambda)\}a_0\sin\{\lambda(\tfrac{1}{2}\pi-\omega)\}\cos\{(1-\lambda)\theta+\lambda\omega\}r^{\lambda-1}+o(\epsilon r^{\lambda-1}), \tag{6}$$

and

$$v = \epsilon\{\lambda 2^{1-\lambda}/\pi(1-\lambda)\}a_0\sin\{\lambda(\tfrac{1}{2}\pi-\omega)\}\sin\{(1-\lambda)\theta+\lambda\omega\}r^{\lambda-1}+o(\epsilon r^{\lambda-1}), \tag{7}$$

where

$$\lambda = 2\pi/(3\pi-2\omega).$$

These results are the dominant terms of the velocity components (u, v) and the pressure p near the corner, and hold as long as the linearized approximations to the problem are valid.

Suppose instead that a pressure singularity is allowed in the first-order approximate solution. Then an expression

$$\epsilon A\cos\{n\lambda(\theta-\omega)\}(r^{-n\lambda}-r^{n\lambda}) \qquad (n = 1, 2, 3, \ldots),$$

where A is a constant, must be added to the local solution (5) for $(p-p_0)/\gamma p_0$. However, the requirement that any pressure singularity be integrable eliminates all $n \geqslant 2$. Thus for $n = 1$ the additional contribution is

$$\epsilon A\cos\{\lambda(\theta-\omega)\}(r^{-\lambda}-r^{\lambda}),$$

and so as $r \to 0$ the term

$$\epsilon A\cos\{\lambda(\theta-\omega)\}/r^{\lambda}$$

is now the leading one for $\epsilon P^{(1)}$, that is, the pressure has a new local asymptotic expansion

$$(p-p_0)/\gamma p_0 = \epsilon A\cos\{\lambda(\theta-\omega)\}/r^{\lambda}+O(\epsilon r^{\lambda}). \tag{8}$$

It is then easily deduced from the linearized momentum equations that the leading terms of the velocity components are

$$u = \frac{\epsilon\lambda A a_0\cos\{(1-\lambda)\theta+\lambda\omega\}}{(1+\lambda)r^{\lambda+1}}+O(\epsilon r^{\lambda-1}) \tag{9}$$

and

$$v = \frac{\epsilon\lambda A a_0\sin\{(1-\lambda)\theta+\lambda\omega\}}{(1+\lambda)r^{\lambda+1}}+O(\epsilon r^{\lambda-1}). \tag{10}$$

With p, u, and v given by equations (8)–(10) the linearized approximations to the full nonlinear equations of the problem are seen to break down when r is $O(\epsilon^{1/(2+\lambda)})$, for which value the velocity components are $O(\epsilon^{1/(2+\lambda)})$ while $(p-p_0)/\gamma p_0$ is $O(\epsilon^{2/(2+\lambda)})$. Thus the energy flux across the transition boundary where the linearized approximations first break down is $O(\epsilon^{4/(2+\lambda)})$, this being calculated from the product of the orders of $(p-p_0)/\gamma p_0$, r, and the normal velocity component.

Now a nonlinear solution of this weak shock diffraction problem (using the techniques developed by Lighthill (1949) and Whitham (1957)) reveals that the diffracted shock front consists of segment I lying entirely in the shadow region, for which $(p-p_0)/\gamma p_0$ and the velocity components are $O(\epsilon^2)$, and segment II lying inside a region of angular width $O(\epsilon^{\frac{1}{2}})$ about the geometrical shadow boundary, for which $(p-p_0)/\gamma p_0$ and the velocity components are $O(\epsilon)$. The remainder of the diffracted front lies behind the shock front and is a boundary separating the fluid into a uniform-state region and a disturbed region. This boundary is divided into segments III, IV, and V, as shown in Figure 1, the value of $(p-p_0)/\gamma p_0$ and the velocity components being $O(\epsilon)$ across each segment.

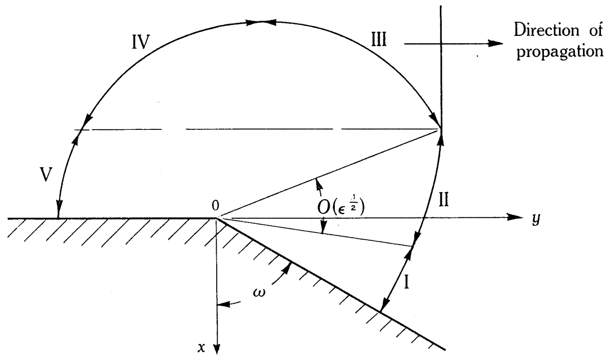

Fig. 1.—Segments I–V of the diffracted fronts.

When account is taken of the lengths of the segments, the flux across I is $O(\epsilon^4)$, across II is $O(\epsilon^{5/2})$, across V is $O(\epsilon^{5/2})$, while that across III and IV combined is only $O(\epsilon^3)$, since the contributions $O(\epsilon^2)$ from each part cancel. Therefore the overall rate of surplus energy entering the disturbed region is $O(\epsilon^{5/2})$.

For convex-angled corners $4/(2+\lambda) < \frac{5}{2}$, since the range of λ is $\frac{1}{2} \leqslant \lambda < 1$, and so the assumed occurrence at the corner of the pressure singularity shows that there is an increase in energy flux in the disturbed region between the diffracted fronts and the transition boundary. However, for the present problem no mechanism exists for such an energy increase in this region and hence the assumption of a pressure singularity at the corner is false, showing that the only corner conditions physically possible for the first-order approximate problem are those taken by Miles (1952) and Friedlander (1958).

The author acknowledges with gratitude useful discussions with Professor J. J. Mahony and the late Professor H. C. Levey. Thanks are also due to Mr. C. W. Thomas for his assistance with the diagram.

References

Friedlander, F. G. (1958).—"Sound Pulses." p. 110. (Cambridge Univ. Press.)
Keller, J. B., and Blank, A. (1951).—*Communs pure appl. Math.* **4**, 75.
Lighthill, M. J. (1949).—*Phil. Mag.* **40**, 1202.
Miles, J. W. (1952).—*Proc. R. Soc.* A **212**, 543.
Whitham, G. B. (1957).—*J. Fluid Mech.* **2**, 145.