COMPRESSIONAL ALFVÉN WAVE PROPAGATION BELOW ION CYCLOTRON FREQUENCY

By R. C. Cross* and J. A. Lehane*

[*Manuscript received November* 20, 1967]

Summary

When the compressional wave cutoff frequency is below ion cyclotron frequency both compressional and torsional Alfvén waves may be present simultaneously. Compressional wave measurements in this regime are particularly important because of their relevance to certain ion cyclotron resonance heating experiments. This paper extends the work of others in this important regime.

The measurements deal mainly with the dispersion of the azimuthally symmetric wave and the radial variation of its magnetic field components. The observed radial variation agrees with that expected theoretically, as does the change in the relative amplitudes of the field components with frequency. The dispersion relation measurements, made in a decaying plasma, yield values of the plasma parameters that change with time after breakdown in a way consistent with spectroscopic observations. These measurements bring out the importance of ion–neutral coupling and yield a value of the ion–neutral momentum transfer cross section of $7\cdot 5(\pm 2\cdot 5)\times 10^{-15}\ cm^2$, which is consistent with estimates made from torsional wave studies.

I. Introduction

The dispersion relation for hydromagnetic waves propagating parallel to a magnetic field in a cylindrical plasma has two branches, one corresponding to what is generally known as the torsional or slow wave and the other to the compressional or fast wave. The torsional wave, whose electric vector rotates in the same direction as the ions, propagates only for frequencies below the ion cyclotron frequency where there is a resonance. The compressional wave propagates for frequencies below the electron cyclotron frequency and has a wave-guide cutoff, the frequency of which is proportional to the Alfvén velocity and to the radial wave number. The cutoff may thus be either above or below the ion cyclotron frequency depending on the plasma parameters.

The case where the cutoff frequency is above the ion cyclotron frequency and the torsional wave is not propagating has been extensively studied by Kovan *et al.* (1961), Hooke *et al.* (1962), and Jephcott and Malein (1964).

When the cutoff frequency is below the ion cyclotron frequency both waves may be present; there has been considerable interest in this laboratory and elsewhere in the use of the torsional wave resonance at the ion cyclotron frequency for plasma

* School of Physics, University of Sydney, N.S.W. 2006.

heating, where any energy fed into the compressional wave is lost unproductively in wave rather than particle energy. It has been necessary therefore to extend the work of Gould, Swanson, and Hertel (1964) and Malein (1965) in this important regime of compressional wave propagation.

II. Theory

The theory of the propagation of magnetohydrodynamic waves in a uniform partly ionized plasma has been fully developed by a number of workers (e.g. Woods 1962; Brown 1965; Sodek and Willett 1967). In each of these treatments the equations of motion for ions, electrons, and neutral particles are obtained by linearizing the Boltzmann equation and then applying the conservation and Maxwell's equations to obtain the dispersion relation. Brown (1965) considers the propagation of azimuthally symmetric ($m = 0$) waves of angular frequency ω that are travelling in the z direction with a propagation constant p such that all wave variables vary as $\exp \mathrm{i}(pz-\omega t)$ and obtains the dispersion relation

$$ap^4+bp^2+c = 0\,, \tag{1}$$

where a, b, and c, are algebraic functions of the plasma parameters and are given in the Appendix to his paper. The real and imaginary parts of p are respectively the wave number k and the attenuation constant ϵ, that is,

$$p = k+\mathrm{i}\epsilon\,. \tag{2}$$

The attenuation constant ϵ is just the reciprocal of the attenuation length L.

The numerical evaluation of equation (1), which is quadratic in p^2, gives the velocity and attenuation length of both the torsional and compressional waves as a function of frequency. These depend on the axial magnetic field B_0, total particle density n, percentage ionization P, electron temperature T_{e}, and cross section for ion–neutral momentum transfer σ, as well as the radial wave number ν, which is determined by the boundary conditions.

The radial variations of the wave magnetic field components are also obtained. These are given by

$$b_\theta = A\,\mathrm{J}_1(\nu r)\,, \qquad b_r = B\,\mathrm{J}_1(\nu r)\,, \qquad b_z = C\,\mathrm{J}_0(\nu r)\,,$$

where J_0 and J_1 are Bessel functions of the first kind of order zero and one respectively. The amplitude constants A, B, and C depend on the frequency, the wave type, and the radial mode in addition to the plasma characteristics listed above. Their relative amplitudes are given by (see Spillman 1963)

$$C/B = \mathrm{i}\nu/p\,,$$

$$B/A = -\mathrm{i}\Omega U^2/\{\beta^2-(1-\alpha_\perp)(1+U^2)\}\,,$$

where $U = p/\nu$, $\beta = \omega/V_{\mathrm{A}}\nu$, $\Omega = (\rho_1/\rho_{\mathrm{i}})(\omega/\omega_{\mathrm{ci}})$ and $\alpha_\perp = \omega\eta_\perp/\mu_0 V_{\mathrm{A}}^2$. Here ω_{ci} is the ion cyclotron frequency, $\eta_\perp$ is the plasma resistivity perpendicular to the axial magnetic field B_0, and V_A is the Alfvén velocity given by $V_A = B_0/(\mu_0 \rho_1)^{\frac{1}{2}}$. In these expressions ρ_1 is a complex mass density, the modulus of which is the effective mass density determining the wave velocity. It is given by

$$\rho_1 = \rho_i + \rho_n(1+i\tau)/(1+\tau^2),$$

where the subscripts i and n refer to ions and neutral particles respectively and $\tau = \omega/\nu_{ni}$, ν_{ni} being the ion–neutral collision frequency. In the work to be described ν_{ni} is approximately 11 MHz and thus, at frequencies appreciably less than 1 MHz, ρ_1 is the total density (the sum of the ion and neutral mass densities) whereas, at frequencies comparable with 11 MHz, it is approximately the ion mass density.

III. Experimental Method

(a) *Plasma Source*

The measurements to be discussed were made in the Supper II machine, which is described in detail by Brennan *et al.* (1963). This machine consists basically of a stainless steel vessel 170 cm long and 21·5 cm in diameter which is immersed in an axial magnetic field that is variable up to 13 kG. One end of the machine is sealed with a quartz plate to which is attached a central molybdenum electrode of diameter 7·6 cm and length 7·6 cm. The other end is sealed with a Pyrex glass plate to which is attached on the inside an earthed stainless steel wire mesh. Hydrogen, purified by a palladium filter, is continuously circulated through the machine.

The plasma was prepared by the hybrid method discussed by Brown *et al.* (1967), whereby an initial ionizing front is followed by an axial discharge of peak current 45 kA decaying to zero in 200 μsec. Most of the measurements were made for the same conditions used by the above authors, namely a magnetic field of 7 kG and a filling pressure of 70 mtorr. These authors give the radial and axial distributions of both ion density and electron temperature in the decaying plasma. They find that these parameters have no axial variation but do have some variation with radius. Near the vessel wall there is a region of low ion density that extends radially inwards as the plasma decays, and for times up to 350 μsec there is also a region of low ion density at the centre of the plasma column. For comparison with the wave results to be presented here, average values of ion density and electron temperature were calculated from these distributions. As typical values, the average ion density and electron temperature 200 μsec after breakdown are $2 \cdot 4 \times 10^{15}$ cm^{-3} and 12 000°K respectively.

(b) *Wave Excitation and Detection*

Azimuthally symmetric waves were excited by discharging appropriate capacitors through a quartz-sheathed 11·6 cm diameter loop, which was placed in the plasma 30 cm from the plasma preparation end of the vessel with its axis along the vessel axis. The diameter was chosen to minimize the second radial mode. Frequencies used ranged from 2 to 8 MHz and the quality factor Q in the presence of the plasma ranged from 20 at 2 MHz to 10 at 8 MHz, the wave launched being thus essentially monochromatic.

The waves were detected via their perturbation magnetic fields by using small calibrated magnetic probes. These consisted of 30-turn coils wound on 0·2 cm diameter formers. The coils were centre-tapped, the signals from the two sides being subtracted using a differential amplifier to remove capacitive pickup. The probes were inserted into the plasma through re-entrant quartz tubes. The signals from the probes were transmitted via double-screened coaxial cable to oscilloscopes that were operated in a screened room.

The radial distributions of the field components were measured by a probe that could be moved along a diameter of the vessel. Two probes were used, one to measure either b_θ or b_z and the other to measure b_r. Measurements were made at a distance of 30 cm from the launcher. The field profiles were determined by making a series of six shots for each component at several radial positions and averaging.

The velocity and attenuation of the wave were determined using a probe that was moved longitudinally through the plasma. Measurements were made over a distance of 50 cm starting at the wave exciter. Six shots were taken every 5 cm. The wave component used in these measurements was the b_z component and it was detected on the vessel axis. This component and radial position were chosen to minimize interference from the torsional wave.

IV. Results

The results fall into three groups. Measurements were made of (1) the cutoff frequency as a function of both magnetic field and original neutral particle density, (2) the wave magnetic field components as a function of radius for two different frequencies, and (3) the velocity and attenuation length as a function of frequency for several different times after breakdown.

Those in the first and second groups are important in showing that it is the compressional wave that is being studied and, to a good approximation, that it is only the $m = 0$ mode with the lowest radial wave number.

(a) *Variation of Cutoff Frequency*

The compressional wave cutoff frequency ω_c is given by the expression

$$\omega_c = \nu V_A ,$$

where ν is the radial wave number and V_A the Alfvén velocity. It thus depends on the plasma conditions and, providing all other parameters are held fixed, should be proportional to the magnetic field B_0 and inversely proportional to the square root of the particle density.

The cutoff frequencies for various conditions were measured using the method suggested by Gould, Swanson, and Hertel (1964). A pulse of width 0·3 μsec was launched by critically damping the wave excitation circuitry. This pulse contains a spectrum of frequencies and so a group of waves is excited. Of these, the high frequency waves have the largest group velocity and reach the magnetic probe first. The last frequency to appear is just above the cutoff frequency, which is hence determined. This method enables the cutoff frequency to be determined rapidly and in those cases where both measurements were made it was found that the frequency obtained by this method agreed well with that obtained from the complete dispersion relation.

The variation of cutoff frequency with (1) magnetic field for three different filling pressures and (2) filling pressure for two different magnetic fields are given in Figures 1 and 2 respectively. All the measurements were made 200 μsec after breakdown of the gas. As indicated by the straight lines that have been fitted to them, the

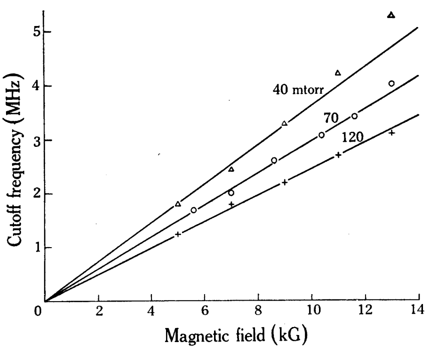

Fig. 1.—The $m = 0$ compressional wave cutoff frequency as a function of magnetic field for gas-filling pressures of 40, 70, and 120 mtorr.

results in Figure 1 show in general that the cutoff frequency varies with magnetic field in the expected way. There is, however, some tendency for the cutoff frequency to increase more rapidly than expected, particularly at low pressures. This is probably due to an increased radial wave number arising from the more peaked radial density distribution that occurs at high magnetic fields. The effect, however, is also to be expected since decoupling of the neutral particles from the ions is taking place.

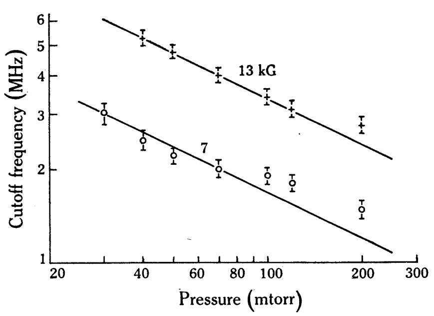

Fig. 2.—The $m = 0$ compressional wave cutoff frequency as a function of gas-filling pressure for magnetic fields of 7 and 13 kG.

The results in Figure 1 also indicate that the cutoff frequency decreases with increasing pressure. A comparison of the slopes of the fitted lines shows, however, that the variation is not as the inverse of the square root of the pressure. This is shown further by the results presented in Figure 2. The implication is that the percentage ionization decreases as the neutral pressure increases, an effect independently observed by Millar and Watson-Munro (1965) of this laboratory.

Finally it should be noted that the absolute values of the cutoff frequencies imply that the densities determining the Alfvén velocities are significantly less than the original filling densities down to the lowest pressures used. Impurities are thus not important in determining the wave propagation.

(b) *Radial Variation of Wave Magnetic Field Components*

To check that basically only one azimuthal mode (the $m = 0$ mode) is being propagated and that radial modes other than the first are insignificant, it is important to make measurements of the radial variation of the wave field components. Such measurements were made at 2·8 and 6·1 MHz using a field of 7 kG and a filling pressure of 70 mtorr, for a time after gas breakdown of 200 μsec. For such conditions the cutoff frequency is 2 MHz. The radial variation at 2·8 MHz is presented in Figure 3 for the three components b_r, b_θ, and b_z. The theoretical curves given are for the $m = 0$ mode with the lowest radial wave number and for plasma parameters determined from the 200 μsec dispersion results, as discussed in the next section. The fit was made by normalizing to the b_z component and retaining the relative amplitudes. The agreement is quite good.

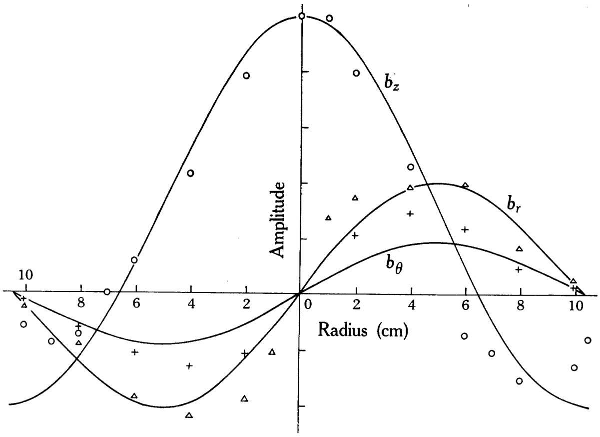

Fig. 3.—Radial variation of the $m = 0$ compressional wave magnetic field components at a frequency of 2·8 MHz. The theoretical curves are for a plasma with axial magnetic field 7 kG, total density $1\cdot8\times10^{15}$ cm^{-3}, percentage ionization 71%, and electron temperature 10 500°K. For such conditions the compressional wave cutoff frequency is about 2 MHz.

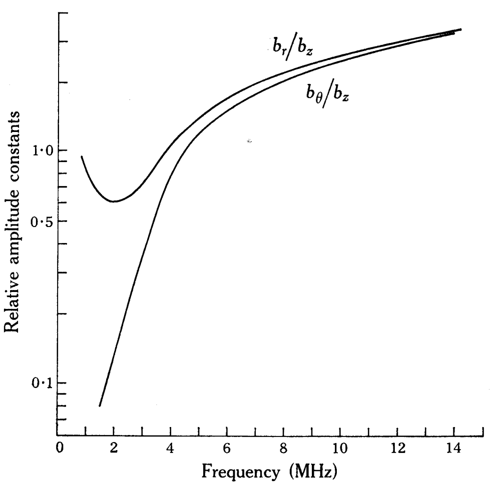

Fig. 4.—Theoretical relative amplitudes of the $m = 0$ compressional wave magnetic field components as a function of frequency for the following values of the relevant parameters: total particle density $1\cdot7\times10^{15}$ cm^{-3}, percentage ionization 70%, electron temperature 10 000°K, axial magnetic field 7 kG, radial wave number $0\cdot365$ cm^{-1}, ion–neutral momentum transfer cross section 5×10^{-15} cm^2. For such conditions the cutoff frequency is about 2 MHz.

As indicated in Section II, an interesting feature of the wave field components is that the relative amplitudes of the components should change with frequency. This is shown in Figure 4, where the theoretical ratios b_r/b_z and b_θ/b_z are plotted as a function of frequency, 200 μsec after breakdown, for the plasma produced at 7 kG with a filling pressure of 70 mtorr. In Table 1 the maximum values of the three components at 6·1 MHz together with those at 2·8 MHz obtained from Figure 3 are compared with the theoretical values obtained from Figure 4. The amplitudes are given relative to the b_z component.

It is clear that the relative amplitudes change in the expected way. For the b_r and b_z components there is good numerical agreement between the observed and calculated relative amplitudes. The disagreement for the b_θ component is due to interference from the azimuthal magnetic field of the torsional wave, which was observed to be propagating with small amplitude, by making measurements of b_θ below cutoff frequency.

TABLE 1

COMPARISON BETWEEN EXPERIMENTAL AND THEORETICAL VALUES OF WAVE MAGNETIC FIELD COMPONENTS AT TWO FREQUENCIES

Wave Field Component	2·8 MHz		6·1 MHz	
	Experimental Value	Theoretical Value	Experimental Value	Theoretical Value
b_z	1·00	1·00	1·00	1·00
b_r	0·42	0·41	1·02	1·05
b_θ	0·30	0·19	0·64	1·04

(c) *Wave Velocity and Attenuation*

The velocity and attenuation length of the wave were determined as a function of frequency for a field of 7 kG and a filling pressure of 70 mtorr for four different times after breakdown (200, 250, 300, and 350 μsec). The results are given in Figure 5 together with theoretical curves that have been fitted to them. For each time the velocity and attenuation results were simultaneously fitted. In making the fit the magnetic field was taken as known but all the other parameters (total density, percentage ionization, electron temperature, ion–neutral momentum transfer cross section, and radial wave number) were taken as variables, although ν was not allowed to be less than the value 0·365 cm^{-1}, which is expected for a plasma of diameter 21·5 cm bounded by a conducting wall. The amount of information contained in the velocity and attenuation results is such that all of the parameters are well determined by the fitting process. The curves giving velocity and attenuation length as a function of frequency depend quite differently on the different parameters, as shown in Figure 6. It will be noted for example, that, whereas the total particle density n affects both the cutoff frequency and the value of velocity at the higher frequencies, the percentage ionization P has an appreciable effect only on the latter. The values of the parameters obtained by the fitting process are given in Table 2. The curves shown in Figure 5 correspond to these best-fit values.

The results in Table 2 show well the changes that take place in the plasma as it decays. Whereas the total density remains approximately constant, the temperature and ion density decrease and do so at a rate that is consistent with a three-body recombination model. Further, the radial wave number increases as the plasma decays. This is in accord with spectroscopic and framing camera observations, which show that the radius of the plasma column decreases as the plasma decays.

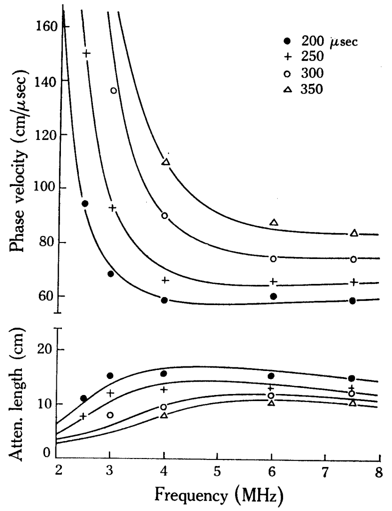

Fig. 5.—Velocity and attenuation length of the $m = 0$ compressional wave as a function of frequency for four different times after plasma preparation. The curves shown are the best-fit theoretical curves.

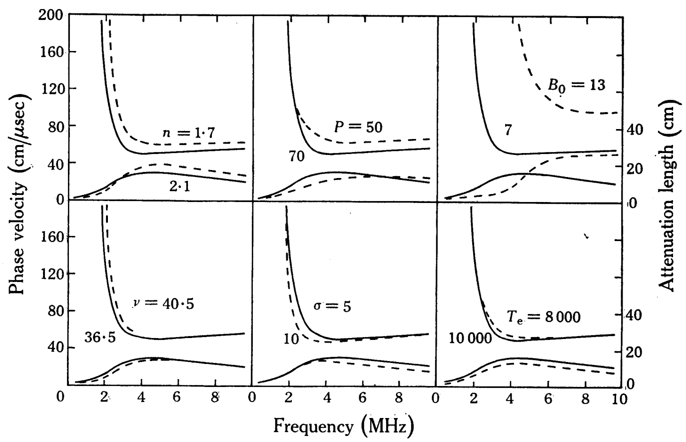

Fig. 6.—Dependence of the velocity versus frequency and attenuation length versus frequency curves on the various plasma parameters. The solid curves are for plasma parameter values of total particle density $n = 2 \cdot 1 \times 10^{15}$ cm^{-3}, percentage ionization $P = 70\%$, magnetic field $B_0 = 7$ kG, radial wave number $\nu = 0 \cdot 365$ cm^{-1}, ion–neutral momentum transfer cross section $\sigma = 5 \cdot 0 \times 10^{-15}$ cm^2, and electron temperature $T_e = 10^4$ °K. The dashed curves show, in turn, the effect of changing each of these parameters to the new values indicated on the figures.

It is of interest to compare the values of the various parameters deduced here with the corresponding values deduced from spectroscopic observations. As discussed in Section III, average values of ion density and temperature were determined from the distributions given by Brown *et al.* (1967). These are given in Table 3.

Comparison of the values given in Tables 2 and 3 shows that there is good agreement at all times between the spectroscopically and wave determined temperatures. There is, however, disagreement between the two estimates of ion density, though the relative values agree well. This discrepancy is due in part to the difficulty of obtaining reliable averages from the spectroscopically determined distributions. These have to be extrapolated to the wall and there is hence significant error since the ion densities are weighted proportionately to their distances from the centre.

TABLE 2

VALUES OF PLASMA PARAMETERS DEDUCED FROM COMPRESSIONAL WAVE STUDIES

Time (μsec)	n (10^{15} cm^{-3})	P (%)	T (°K)	ν (cm^{-1})	σ (10^{-15} cm^2)	Ion Density (10^{15} cm^{-3})
200	$1\cdot8\pm0\cdot1$	71 ± 5	$10\,500\pm1500$	$0\cdot365\pm0\cdot01$	$7\cdot5\pm2\cdot5$ (all times)	$1\cdot25\pm0\cdot1$
250	$1\cdot6\pm0\cdot1$	68 ± 5	$7\,000\pm1000$	$0\cdot375\pm0\cdot01$		$1\cdot1\pm0\cdot1$
300	$1\cdot6\pm0\cdot1$	55 ± 5	$6\,000\pm1000$	$0\cdot395\pm0\cdot01$		$0\cdot8\pm0\cdot1$
350	$1\cdot6\pm0\cdot1$	45 ± 5	$4\,500\pm500$	$0\cdot395\pm0\cdot01$		$0\cdot7\pm0\cdot1$

The dispersion results are of further interest as they show that a considerable change in ion–neutral coupling is taking place over the frequency range used. Figure 5 shows that at the highest frequencies used the velocity increases as the time after breakdown increases. Since the total particle density is expected to remain constant with time, this change must be associated with a decrease in ion density and hence in

TABLE 3

VALUES OF PLASMA PARAMETERS DEDUCED FROM SPECTROSCOPIC OBSERVATIONS

Time (μsec)	Average Ion Density (10^{15} cm^{-3})	Average Electron Temperature (°K)
200	2·4	12 000
250	1·9	7 500
300	1·5	6 000
350	1·2	4 500

percentage ionization (see Fig. 6). It is thus apparent that at these frequencies the velocity is determined essentially by the ion density. Indeed the theoretical calculations show, for example, that for a time of 200 μsec after breakdown the effective number density determining the wave propagation (the absolute value of ρ_1 divided by the ion mass) is $1\cdot28\times10^{15}$ cm^{-3} at $7\cdot5$ MHz, which is approximately equal to the ion density $1\cdot25\times10^{15}$ cm^{-3}. At the cutoff frequency (2 MHz), however, this effective number density is $1\cdot60\times10^{15}$ cm^{-3}, which is closer to the total particle number density $1\cdot80\times10^{15}$ cm^{-3}.

V. Conclusions

Measurements dealing with the azimuthally symmetric compressional wave have been presented. These deal in the main with the dispersion of the wave and the radial variation of the wave magnetic field components. The observed radial variation agrees quite well with that expected theoretically, as does the change in the relative amplitudes of the field components with frequency.

The dispersion relation measurements, made at various times in a decaying plasma, result in estimates of the plasma parameters at these times. They have the expected trend in that the total density stays approximately constant whereas the ion density and electron temperature decrease with time after breakdown. Though there is some disagreement in absolute values, the relative changes with time of the parameters are in agreement with those determined spectroscopically. Some evidence for the contraction of the plasma as it decays is also given.

The dispersion relation measurements are also of importance in that they clearly show the effect of ion–neutral coupling on the wave propagation. The value of the ion–neutral momentum transfer cross section that is obtained ($7\cdot5\pm2\cdot5\times10^{-15}$ cm^2) is the same as the value determined by Brown (1965) from torsional wave studies.

VI. Acknowledgments

The authors are grateful to the Australian Institute of Nuclear Science and Engineering, the Australian Research Grants Committee, and the University of Sydney Research Committee for financial support, and to Professor H. Messel for provision of facilities in the School of Physics. Thanks are due to Mr. N. Lowe for maintaining the apparatus and for his assistance in taking the measurements.

VII. References

Brennan, M. H., Lehane, J. A., Millar, D. D., and Watson-Munro, C. N. (1963).—*Aust. J. Phys.* **16**, 340.

Brown, I. G. (1965).—*Aust. J. Phys.* **18**, 437.

Brown, I. G., Lehane, J. A., Potter, I. C., and Watson-Munro, C. N. (1967).—*Aust. J. Phys.* **20**, 47.

Gould, R. W., Swanson, D. G., and Hertel, R. H. (1964).—*Phys. Fluids* **7**, 269.

Hooke, W. M., Rothman, M. A., Avivi, P., and Adam, J. (1962).—*Phys. Fluids* **5**, 864.

Jephcott, D. F., and Malein, A. (1964).—*Proc. R. Soc.* A **278**, 243.

Kovan, I. A., Patrushev, B. I., Rusanov, V. D., Smirnov, V. P., and Frank-Kamenetsky, D. A. (1961).—I.A.E.A. Conf. Plasma Physics and Controlled Nuclear Fusion Research, Salzburg. Paper 205.

Malein, A. (1965).—*Nucl. Fusion* **5**, 352.

Millar, D. D., and Watson-Munro, C. N. (1965).—Proc. 7th Int. Conf. on Ionised Gases. Paper 47.

Sodek, B. A., Jr., and Willett, J. E. (1967).—*Plasma Phys.* **9**, 113.

Spillman, G. R. (1963).—Univ. California Lawrence Radiation Lab. Rep. No. UCRL–10990.

Woods, L. C. (1962).—*J. Fluid Mech.* **13**, 570.