MULTIPOLE ANALYSIS

I. THEORY, AND GEOMAGNETIC MULTIPOLES 1965·0

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Summary

A reduction of order procedure is outlined which allows virtually any order multipole analysis for a general spherical surface harmonic to be rapidly carried out. The geomagnetic multipoles, to order eight, are found for the epoch 1965·0 and the theory is used to obtain the spherical harmonic coefficients when the dipole axis is chosen as polar axis.

I. Introduction

The idea of using a multipole representation in potential theory appears to have been initially suggested by Gauss (1877) but the first attempt at a fuller development is due to Maxwell (1892) and the basic theory has become known as Maxwell's theory of poles. The subject has undergone further discussion and development in the works of Sylvester (1909), Courant and Hilbert (1953), and Hobson (1955). Geomagnetic multipoles have been considered by Umov (1904), Chargoy (1950, 1955), Chargoy and Alvarez (1957), Zolotov (1966), Winch and Slaucitajs (1966*a*, 1966*b*), and Winch (1967*a*, 1967*b*). Whilst multipole representations are frequently mentioned in electromagnetic theory, they are generally regarded as redundant to the normal expansions in terms of orthogonal functions. However, knowledge of the multipole axes does allow a geometrical visualization of the field and it is of interest to enquire into the behaviour and relative importance of the multipoles representing the geomagnetic field. Moreover, since the multipole strengths and axes are invariant under rotation of the coordinate frame, it is a definite advantage to have a procedure that can rapidly transfer back and forth between the multipole representation and the more usual spherical harmonic representation. For, once this can be done, we have a very practical method for carrying out the transformation of the Gaussian coefficients of the field, under a rotation of polar axis. The outline of such a method is included in this paper and it is shown how this allows evaluation of the field at any point without requiring knowledge of the explicit forms of the spherical harmonics.

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II. Basic Theory

We start from the definitions for the spherical harmonics $\mathrm{P}_n^m(\cos\theta)\cos m\phi$ and $\mathrm{P}_n^m(\cos\theta)\sin m\phi$, which can be written as

$$\mathrm{P}_n^m(\cos\theta)\cos m\phi = A_{nm}\, r^{n+1}\left\{\left(\frac{\partial}{\partial z}\right)^{n-m}\prod_i \frac{\partial}{\partial t_m^i}\right\}\frac{1}{r}, \tag{1a}$$

$$\mathrm{P}_n^m(\cos\theta)\sin m\phi = A_{nm}\, r^{n+1}\left\{\left(\frac{\partial}{\partial z}\right)^{n-m}\prod_i \frac{\partial}{\partial s_m^i}\right\}\frac{1}{r}, \tag{1b}$$

where θ is the geographic north colatitude and ϕ the Greenwich east longitude as usual. Here, the products $\prod_i$ are over all integral values of i satisfying $0 < |i| \leqslant \frac{1}{2}m$ and zero is included when m is odd. The derivatives $\partial/\partial t_m^i$ are derivatives in the directions of the vectors $\boldsymbol{t}_m^i$, which are just unit vectors in the x–y plane displaced symmetrically at intervals of π/m about the positive x axis ($\phi = 0$), so that $\boldsymbol{t}_m^{-i}$ is the reflection of $\boldsymbol{t}_m^i$ (in the x axis) and $\boldsymbol{t}_m^0$, when it exists (i.e. for odd values of m), coincides with the x direction. The derivatives $\partial/\partial s_m^i$ are in the directions of the unit vectors $\boldsymbol{s}_m^i$, which can be obtained from the $\boldsymbol{t}_m^i$ by an eastward rotation of $\pi/2m$ (this follows from $\sin m\phi = \cos m(\phi-\pi/2m)$).

Whilst these definitions are different from the usual ones, the equivalence can be easily shown (Hobson 1955). For Schmidt quasi-normalization we must take

$$A_{nm} = (-)^n\, 2^m\{(2-\delta_{m0})\,(n-m)!\,(n+m)!\}^{-\frac{1}{2}}.$$

The points where the vectors $\hat{z} = (0,0,1)$ and $\boldsymbol{t}_m^i$ (or $\boldsymbol{s}_m^i$) meet a sphere, centred at $r = 0$, are called the poles of the surface harmonic $\mathrm{P}_n^m(\cos\theta)\cos m\phi$ (or $\mathrm{P}_n^m(\cos\theta)\sin m\phi$) on the sphere. If $m = 0$, $\hat{z}$ is the only pole, and if $m = n$ then $\hat{z}$ is not a pole.

This idea may be generalized to give an analogous representation for a general surface harmonic of the nth degree given by

$$Y_n = \sum_{m=0}^{n} (g_n^m \cos m\phi + h_n^m \sin m\phi)\,\mathrm{P}_n^m(\cos\theta), \tag{2}$$

for, using equations (1), we may write

$$\begin{aligned} Y_n &= r^{n+1}\sum_{m=0}^{n}\left(\frac{\partial}{\partial z}\right)^{n-m}\left(g_n^m \prod_i \frac{\partial}{\partial t_m^i} + h_n^m \prod_i \frac{\partial}{\partial s_m^i}\right)\frac{1}{r} \\ &= r^{n+1}\sum_{m=0}^{n}\sum_{l=0}^{m} c_{ml}\left(\frac{\partial}{\partial z}\right)^{n-m}\left(\frac{\partial}{\partial x}\right)^{l}\left(\frac{\partial}{\partial y}\right)^{m-l}\frac{1}{r}, \end{aligned} \tag{3}$$

where the coefficients c_{ml} are obtained by expanding the above products. When it is taken into account that for harmonic functions

$$(\partial/\partial x)^2 + (\partial/\partial y)^2 + (\partial/\partial z)^2 = 0,$$

it can be shown (Sylvester 1909) that the ternary quantic of differential operators in (3) can be written as the product of n real linear factors (canonical form), i.e.

$$Y_n = (-)^n\, r^{n+1} M_n\left\{\prod_{i=1}^{n}\left(u_i\frac{\partial}{\partial x} + v_i\frac{\partial}{\partial y} + w_i\frac{\partial}{\partial z}\right)\right\}\frac{1}{r}, \tag{4}$$

and that this representation is unique except that the n unit vectors $\boldsymbol{u}_i = (u_i, v_i, w_i)$ may be reversed in pairs. Equation (4) is called the multipole representation of Y_n; M_n is called the multipole strength and the vectors $\boldsymbol{u}_i$ the multipole axes, which meet the sphere at the poles (θ_i, ϕ_i). A potential of the form

$$V_n = r^{-(n+1)} Y_n$$

will be called a multipole potential and can be thought of as due to a singular point of degree n at $r = 0$. It is easy to see that such a potential is formed when two singular points of degree $n-1$, having the same $n-1$ axes but opposite strengths, are brought together at $r = 0$ along an nth axis in such a manner that the product of their strength and distance apart is kept constant. This procedure is an obvious generalization of the technique employed in forming the familiar dipole potential (multipole of order one) by bringing together two equal but opposite poles (order zero).

In practice, for the geomagnetic field, the coefficients g_n^m and h_n^m in equation (2) are known from the harmonic analysis of the field at the surface of the Earth, being just the usual Gaussian coefficients. We thus have to combine (2) and (4) and solve for the axes $\boldsymbol{u}_i$ and the strength M_n.

III. Recurrence Relations

The highest order geomagnetic multipole analysis that has been carried out is the fifth (Winch 1967*b*), but the solution by any of the methods so far put forward of any higher order multipole problem is virtually, although not theoretically, impossible due to algebraic complexity and difficulties in solving the final nonlinear equations. However, using recently derived recurrence relations for P_n^m a technique has been perfected which quickly gives the solution to multipole problems of much higher orders. It has been shown (James 1967) that the equations to be solved for the $\boldsymbol{u}_i$ and M_n are of the form

$$M_n \begin{vmatrix} a_n^m \\ b_n^m \end{vmatrix} = \begin{vmatrix} g_n^m \\ h_n^m \end{vmatrix}, \qquad m = 0, 1, \ldots, n, \tag{5}$$

where the a_n^m and b_n^m are generated as functions of the $\boldsymbol{u}_i$ through the relations (starting from $a_0^0 = 1$)

$$2a_{k+1}^m = u_{k+1}\{(1+\delta_{m1})\,\alpha_{k+1}^m\,a_k^{m-1} - \beta_{k+1}^m\,a_k^{m+1}\} - v_{k+1}\{\alpha_{k+1}^m\,b_k^{m-1} + \beta_{k+1}^m\,b_k^{m+1}\} + 2w_{k+1}\,\gamma_{k+1}^m\,a_k^m, \tag{6a}$$

$$2b_{k+1}^m = u_{k+1}\{\alpha_{k+1}^m\,b_k^{m-1} - \beta_{k+1}^m\,b_k^{m+1}\} + v_{k+1}\{(1+\delta_{m1})\,\alpha_{k+1}^m\,a_k^{m-1} + \beta_{k+1}^m\,a_k^{m+1}\} + 2w_{k+1}\,\gamma_{k+1}^m\,b_k^m. \tag{6b}$$

Here

$$\alpha_{k+1}^m = \{\tfrac{1}{2}(2-\delta_{m1})(k+m)(k+m+1)\}^{\frac{1}{2}}, \qquad \beta_{k+1}^m = \{(1+\delta_{m0})(k-m)(k-m+1)\}^{\frac{1}{2}},$$

$$\gamma_{k+1}^m = \{(k-m+1)(k+m+1)\}^{\frac{1}{2}}.$$

In all this work the conventions

$$a_n^m = g_n^m = 0 \qquad \text{when either} \qquad m < 0 \quad \text{or} \quad m > n$$

and

$$b_n^m = h_n^m = 0 \qquad \text{when either} \qquad m \leqslant 0 \quad \text{or} \quad m > n$$

are to be understood and δ_{mp} is the Kronecker delta

$$\delta_{mp} = \begin{cases} 1, & m = p, \\ 0, & m \neq p. \end{cases}$$

IV. Transformed Coefficients

Using the relations (6) it is a simple matter to transform back and forth between the multipole and spherical harmonic representations. This is particularly advantageous when it is required to find the harmonic expansion of a potential in a new polar and meridional reference frame. For, suppose we have a potential

$$V = \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} (-)^n r^{-n-1} M_n \left(\prod_{i=1}^{n} (\boldsymbol{u}_{ni} \cdot \nabla) \right) \frac{1}{r},$$

where $\boldsymbol{u}_{ni}$ represents the ith axis of the nth order multipole V_n, or equivalently,

$$V = \sum_{n=1}^{\infty} r^{-n-1} \sum_{m=0}^{n} (g_n^m \cos m\phi + h_n^m \sin m\phi) \, \mathrm{P}_n^m(\cos\theta),$$

and we transfer to a new system of coordinates (θ^*, ϕ^*). We wish to determine the new coefficients $(g_n^m)^*$ and $(h_n^m)^*$ such that

$$V = \sum_{n=1}^{\infty} r^{-n-1} \sum_{m=0}^{n} \{(g_n^m)^* \cos m\phi^* + (h_n^m)^* \sin m\phi^*\} \, \mathrm{P}_n^m(\cos\theta^*).$$

If the new polar axis has direction cosines given by $\boldsymbol{U} = (U, V, W)$ and if we suppose for the moment that longitude is measured anticlockwise around the new pole, looking from above the pole, from the great semicircle containing the north geographic pole and the two extremities of the new polar axis, then, from the sine and cosine formulae of spherical trigonometry, the components of $\boldsymbol{u}_{ni}$ in the new coordinate system are easily seen to be

$$u_{ni}^* = \{w_{ni} - (\boldsymbol{u}_{ni} \cdot \boldsymbol{U}) W\}(1 - W^2)^{-\frac{1}{2}}, \qquad v_{ni}^* = \{u_{ni} V - v_{ni} U\}(1 - W^2)^{-\frac{1}{2}}, \qquad w_{ni}^* = \boldsymbol{u}_{ni} \cdot \boldsymbol{U},$$

so the new multipole representation of V, namely

$$V = \sum_{n=1}^{\infty} (-)^n r^{-n-1} M_n \left(\prod_{i=1}^{n} (\boldsymbol{u}_{ni}^* \cdot \nabla^*) \right) \frac{1}{r},$$

is immediately obtainable. Substitution of the "transformed" axes $\boldsymbol{u}_{ni}^*$ into equations (6) then quickly generates $(g_n^m)^*$ and $(h_n^m)^*$. Note that in (6) the subscript n on $\boldsymbol{u}_{ni}$ has been omitted for conciseness and i is represented by $k+1$. We see, in particular, from the canonical form (4) that if we choose our new polar axis to coincide with one of the multipole axes, say $\boldsymbol{u}_{ni}$, then the corresponding factor $(\boldsymbol{u}_{ni} \cdot \nabla)$ transforms merely to $\partial/\partial z^*$, and thus in this new system the number of coefficients required to represent the field is reduced by two. Indeed, it is easily seen from relations (6) that the coefficients $(g_n^n)^*$ and $(h_n^n)^*$ will be zero so that terms containing $\mathrm{P}_n^n(\cos\theta^*)$ will be missing from the expansion of V. If for some reason it were desired to use the point G (with direction cosines $\boldsymbol{U}_{\mathrm{G}}$) as reference for longitude, instead of the geographic north pole, it is merely necessary to replace $(g_n^m)^*$ and $(h_n^m)^*$ by

$$(g_n^m)^*\cos m\alpha+(h_n^m)^*\sin m\alpha \qquad \text{and} \qquad -(g_n^m)^*\sin m\alpha+(h_n^m)^*\cos m\alpha$$

respectively, where α is the angle between the vectors $\boldsymbol{U}\times\boldsymbol{z}$ and $\boldsymbol{U}\times\boldsymbol{U}_{\mathrm{G}}$ and can be obtained from

$$R\cos\alpha = (\boldsymbol{U}\times\hat{\boldsymbol{z}}).(\boldsymbol{U}\times\boldsymbol{U}_{\mathrm{G}}) = W_{\mathrm{G}}(U^2+V^2)-W(UU_{\mathrm{G}}+VV_{\mathrm{G}}),$$

$$R\sin\alpha = (\boldsymbol{U}\times\hat{\boldsymbol{z}})\times(\boldsymbol{U}\times\boldsymbol{U}_{\mathrm{G}}).\boldsymbol{U} = VU_{\mathrm{G}}-UV_{\mathrm{G}}.$$

For many purposes in geomagnetism it is desirable to use geomagnetic coordinates (Θ, Λ). In this case the reversed dipole axis $-\boldsymbol{u}_{11}$ is taken as polar axis and the longitude is referred to the south geographic pole. For comparison the geomagnetic coefficients are given in Table 1 together with the data of Hurwitz *et al.* (1966) which were used for the computations of the present paper.

If the transformation has to be made on a more general field that has parts due to internal, external, and non-potential sources then a separation of the field should be made first and the above procedure applied to each part separately. Whilst in the present paper external and non-potential sources are neglected, it is interesting to note a very recent method of field separation that has been put forward by Winch (1968). Winch's method involves the use of Cartesian force components of the field and, interestingly, is based on the recurrence relations (6) together with the corresponding relations for the other sources of the field.

V. Field Evaluation

Normally in order to evaluate the function V at any point P on the Earth it is necessary to find the explicit forms of the functions $\mathrm{P}_n^m(\cos\theta)$ and evaluate these at the colatitude of P. However, if we bear in mind that $\mathrm{P}_n^m(1) = \delta_{m0}$, we see that the value of V at the geographic pole is just $\sum_{n=1}^{\infty} g_n^0$. Similarly, if we make use of the transformation technique of the previous section and rotate our polar axis so that it passes through P, then the value of V at P can be obtained as $\sum_{n=1}^{\infty}(g_n^0)^*$.

VI. Relative Magnitudes

Useful magnitudes in comparing the overall importance of each of the multipoles are the r.m.s. values of the parameters X_n, Y_n, Z_n, and T_n on the surface of the Earth ($r = 1$). Here, as usual,

$$(X_n, Y_n, Z_n) = \left(-\frac{1}{r}\frac{\partial V_n}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial V_n}{\partial\phi}, -\frac{\partial V_n}{\partial r}\right), \qquad T_n = (X_n^2+Y_n^2+Z_n^2)^{\frac{1}{2}}.$$

TABLE 1

SCHMIDT SPHERICAL HARMONIC COEFFICIENTS IN γ(EARTH RADII)$^{n+2}$

Coefficients are geographic (g, h) and geomagnetic (g^*, h^*)

n		$m = 0$	1	2	3	4	5	6	7	8
1	g	−30450	−2135							
	g^*	−31066	0							
	h		5772							
	h^*		0							
2	g	−1616	2967	1585						
	g^*	−588	2969	−1883						
	h		−1999	132						
	h^*		2232	499						
3	g	1159	−1976	1300	864					
	g^*	796	−1154	−1080	−552					
	h		−411	245	−156					
	h^*		−1706	1154	−508					
4	g	923	782	494	−366	252				
	g^*	808	−476	5	294	−291				
	h		142	−285	4	−243				
	h^*		941	187	−50	−301				
5	g	−184	345	244	−37	−158	−62			
	g^*	−113	66	−305	63	−149	−4			
	h		14	124	−108	−109	65			
	h^*		332	−47	161	101	37			
6	g	34	65	5	−228	−14	13	−111		
	g^*	45	−7	46	188	−137	33	−47		
	h		−22	112	51	−24	−19	−22		
	h^*		10	−49	97	−54	−47	−93		
7	g	89	−45	6	4	−25	−22	22	10	
	g^*	66	−61	17	−18	6	4	40	−1	
	h		−40	−25	−18	3	31	−27	−25	
	h^*		−24	61	2	35	17	−1	23	
8	g	0	11	3	−13	−2	10	−14	11	6
	g^*	11	2	−4	2	−19	6	33	2	−10
	h		−10	1	5	−19	7	21	4	−19
	h^*		6	−3	3	1	5	1	12	13

Making use of the orthogonality of the harmonics we know that, when $r = 1$ Earth radius,

$$\overline{V_n^2} = \frac{1}{4\pi}\int_{4\pi} V_n^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{2n+1}\sum_{m=0}^{n}\{(g_n^m)^2+(h_n^m)^2\}$$

and

$$\overline{Z_n^2} = (n+1)^2\overline{V_n^2}.$$

Using the perhaps less familiar relation (see, for example, Copson 1962)

$$\int_0^\pi (\sin\theta)^{-1}\,\mathrm{P}_n^m(\cos\theta)\,\mathrm{P}_n^l(\cos\theta)\,\mathrm{d}\theta = \delta_{ml}/2m\,,$$

we find

$$\overline{Y_n^2} = \tfrac{1}{2}\sum_{m=0}^{n} m\{(g_n^m)^2+(h_n^m)^2\}$$

and

$$\overline{X_n^2} = \sum_{m=0}^{n}\{(g_n^m)^2+(h_n^m)^2\}\int_0^\pi (\mathrm{dP}_n^m/\mathrm{d}\theta)^2\sin\theta\,\mathrm{d}\theta$$

$$= \sum_{m=0}^{n}\{n(n+1)/(2n+1)-\tfrac{1}{2}m\}\{(g_n^m)^2+(h_n^m)^2\}\,.$$

In evaluating this last integral we have also made use of the differential equation satisfied by P_n^m. A simple addition gives us

$$\overline{T_n^2} = (n+1)\sum_{m=0}^{n}\{(g_n^m)^2+(h_n^m)^2\}\,.$$

Similar equations to the above have been listed by Lowes (1966).

TABLE 2

$\bar{X}_n$, $\bar{Y}_n$, AND $\bar{T}_n$ IN γ UNITS AND $n!M_n$ IN γ(EARTH RADII)$^{n+2}$

	$n = 1$	2	3	4	5	6	7	8
$\bar{X}_n$	24989	3550	2944	1872	729	379	219	59
$\bar{Y}_n$	4352	2988	2223	1043	498	371	118	74
$\bar{T}_n$	43934	7336	5635	3214	1310	781	364	138
$n!M_n$	31066	4888	4122	2553	976	900	482	214

Some interesting relationships between the r.m.s. values can be drawn from these equations. For instance, the r.m.s. vertical intensity $\bar{Z}_n = (\overline{Z_n^2})^{\frac{1}{2}}$, the r.m.s. horizontal intensity $\bar{H}_n = (\overline{X_n^2}+\overline{Y_n^2})^{\frac{1}{2}}$, and the r.m.s. total intensity $\bar{T}_n = (\overline{T_n^2})^{\frac{1}{2}}$ bear constant ratios to each other, independent of the field anomalies and depending only on the order of the multipole. Specifically

$$\bar{Z}_n : \bar{H}_n : \bar{T}_n = \sqrt{(n+1)} : \sqrt{n} : \sqrt{(2n+1)}\,.$$

Thus the r.m.s. vertical field is always at least slightly greater than the r.m.s. horizontal field and knowledge of any one of $\bar{V}_n = (\overline{V_n^2})^{\frac{1}{2}}$, $\bar{Z}_n$, $\bar{H}_n$, or $\bar{T}_n$ is sufficient to determine the others.

The values of $\bar{X}_n = (\overline{X_n^2})^{\frac{1}{2}}$, $\bar{Y}_n = (\overline{Y_n^2})^{\frac{1}{2}}$, $\bar{T}_n$, and $n!M_n$, are given in Table 2. The dipole is seen to be by far the most important multipole. Indeed, the total magnetostatic energy, exterior to the Earth, associated with all the other multipoles, namely $\frac{1}{2}\sum_{n=2}^{\infty}\overline{T_n^2}/(2n+1)$, amounts to less than 3% of the dipole's energy $\frac{1}{6}\overline{T_1^2}$.

VII. Solution of Equations

In previous attempts to solve the multipole problem of order n the authors generally have been forced in the end to find the solutions to a set of $2n+1$ nonlinear equations of degree $n+1$, the nonlinearity being due to products of M_n and one component taken from each of the unit vectors $\boldsymbol{u}_{ni}$ $(i = 1, \ldots, n)$, supplemented by the n equations $\boldsymbol{u}_{ni} \cdot \boldsymbol{u}_{ni} = 1$.

Zolotov (1966) has made use of Sylvester's (1909) theory and has reduced the problem to that of solving two polynomial equations of degree $2n$. Like the other techniques this is also very awkward to apply for n greater than about five. In the method of the present paper none of the difficulties associated with the nonlinearity arise since equations (6) are essentially of degree two only. Another advantage is that nowhere are the explicit forms for the spherical harmonics required to be known. The method of solving equations (5) and (6) is briefly as follows.

It is easily seen that each term on the left-hand side of equation (5), if written out in full by making use of the recurrence relations (6), contains as a factor either u_{n1}, v_{n1}, or w_{n1}. Hence, by removing the restriction that $\boldsymbol{u}_{n1}$ be a unit vector and letting it have length M_n, equations (5) can be rewritten

$$\begin{vmatrix} a_n^m \\ b_n^m \end{vmatrix} = \begin{vmatrix} g_n^m \\ h_n^m \end{vmatrix}, \qquad m = 0, 1, \ldots, n.$$

Then equations (6), with $k = n-1$, namely

$$2g_n^m = u_{nn}\{(1+\delta_{m1})\, \alpha_n^m\, a_{n-1}^{m-1} - \beta_n^m\, a_{n-1}^{m+1}\} - v_{nn}\{\alpha_n^m\, b_{n-1}^{m-1} + \beta_n^m\, b_{n-1}^{m+1}\} + 2w_{nn}\, \gamma_n^m\, a_{n-1}^m,$$

$$2h_n^m = u_{nn}\{\alpha_n^m\, b_{n-1}^{m-1} - \beta_n^m\, b_{n-1}^{m+1}\} + v_{nn}\{(1+\delta_{m1})\, \alpha_n^m\, a_{n-1}^{m-1} + \beta_n^m\, a_{n-1}^{m+1}\} + 2w_{nn}\, \gamma_n^m\, b_{n-1}^m,$$

can be regarded as a set of $2n+1$ equations of degree two in the $2n+2$ unknowns a_{n-1}^m, b_{n-1}^m, u_{nn}, v_{nn}, and w_{nn}, and the last one can be eliminated through $\boldsymbol{u}_{nn} \cdot \boldsymbol{u}_{nn} = 1$.

If any guess is made for the axis $\boldsymbol{u}_{nn}$ the above equations, with $m \neq n$, become $2n-1$ linear equations in the $2n-1$ unknowns a_{n-1}^m, b_{n-1}^m and this system is easily inverted by any of the standard methods to yield a self-consistent set of initial values. By using this technique it is only necessary to make guesses for two of the $2n+1$ unknowns and one then obtains $2n+1$ values that satisfy all but two of the $2n+1$ equations. These values can be used to start a Newton–Raphson iteration on the full set of $2n+1$ equations. This type of iteration scheme was found to converge extremely quickly with apparently any initial guess for the axis. The "reduced Gaussian coefficients" a_{n-1}^m and b_{n-1}^m are then used in the left-hand sides of (6), with $k = n-2$, and the preceding method is repeated with a guess for $\boldsymbol{u}_{n,n-1}$. This process is continued until the final set of equations "to be solved" (equations (6) with $k = 0$) are $u_{n1} = a_1^1$, $v_{n1} = b_1^1$, and $w_{n1} = a_1^0$. In practice, the geographic polar axis was used for all initial guesses and this generated the multipoles in order of increasing distance from the geographic pole.

Using

$$M_n = \{(a_1^0)^2 + (a_1^1)^2 + (b_1^1)^2\}^{\frac{1}{2}} \tag{7}$$

we can easily reconvert $\boldsymbol{u}_{n1}$ to a unit vector to complete the analysis. The resulting poles (θ_{ni}, ϕ_{ni}) are given in Table 3. Equation (7) is interesting since it describes M_n as if it were the strength of an ordinary dipole whose axis coincided with one of the multipole axes.

A fifth-order test deck was run consisting of arbitrarily chosen numbers as data and initial guesses that reproduced these data with errors as large as 100 000%. In seven iterations the first axis was obtained and all errors were less than 0·01%,

TABLE 3

POLES (θ_{ni}, ϕ_{ni}) TO NEAREST TENTH OF A DEGREE

n		$i=1$	2	3	4	5	6	7	8
1	θ	168·6							
	ϕ	110·3							
2	θ	22·9	105·0						
	ϕ	32·8	332·0						
3	θ	20·7	58·5	63·5					
	ϕ	344·7	140·8	224·3					
4	θ	24·1	36·0	56·5	67·8				
	ϕ	108·1	218·3	307·1	42·5				
5	θ	16·5	43·0	49·4	63·2	90·6			
	ϕ	270·4	348·5	189·5	51·3	354·0			
6	θ	29·4	40·1	38·8	73·0	80·7	93·2		
	ϕ	248·2	131·0	2·1	51·0	120·4	357·9		
7	θ	17·6	39·6	51·5	54·8	58·6	64·7	65·4	
	ϕ	348·4	158·8	311·7	17·3	201·1	95·8	238·6	
8	θ	25·5	31·5	60·8	61·7	69·1	70·4	79·0	96·3
	ϕ	326·7	197·9	182·8	34·5	93·7	227·3	358·1	306·5

which illustrates the speed at which multipole analysis can now be carried out. Thus we have at our disposal a very rapid and interesting means of transforming a series of spherical harmonics under change of polar axis.

In a following paper (Multipole analysis. II) the secular variation of the geomagnetic multipoles will be discussed.

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