THE PROPERTIES OF A DRIFTING AND DIFFUSING PULSE OF ELECTRONS AND TOWNSEND'S COEFFICIENT OF IONIZATION α_T†

By L. G. H. Huxley‡

When diffusion is active in an ionizing stream of electrons the experimental Townsend coefficient of ionization α_T is not simply the ratio ν_i/W of ν_i, the average rate of volume ionization per electron, to the drift speed W of the electrons, as was implied by Townsend's interpretation of α_T, but is given by the formula (Huxley 1959)

$$\alpha_T = \lambda - \mu = \lambda\{1-(1-2\alpha_i/\lambda)^{\frac{1}{2}}\}, \tag{1}$$

in which $2\lambda = W/D$, where D is the coefficient of diffusion in the gas, $\alpha_i = \nu_i/W$, and $\mu = (\lambda^2 - 2\lambda\alpha_i)^{\frac{1}{2}}$. Also

$$\text{as} \quad D \rightarrow 0, \qquad \lambda \rightarrow \infty \quad \text{and} \quad \alpha_T \rightarrow \alpha_i = \nu_i/W, \tag{2}$$

in conformity with Townsend's interpretation of α_T.

These matters have been discussed recently in the literature (see Crompton 1967), and the theory of steady currents in gases has been given (Burch and Huxley 1967) in which the effects both of diffusion and of the electrodes are incorporated.

In what follows the properties of an ionizing and travelling pulse of electrons, spreading by diffusion as it advances through the gas, are examined, and a measure of insight into the physical basis of formula (1) is thus achieved.

It is assumed throughout that D and W are constant and thus independent of the spatial coordinates (x, y, z) and of the time t.

Equation of Continuity

The analysis is based on the equation of continuity, which here is

$$\mathrm{d}n/\mathrm{d}t = D.\nabla^2 n - \overline{W}.\mathrm{grad}\, n + \nu_i\, n, \tag{3}$$

in which $n \equiv n(x, y, z, t)$ is the number density of the electrons. When $|W| = W_z = W$, then $\overline{W}.\mathrm{grad}\, n = W\,\partial n/\partial z$. It is therefore assumed without loss of generality that $\overline{W}$ is parallel to the axis $+0z$.

The one-dimensional number density is

$$q(z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n\,\mathrm{d}x\,\mathrm{d}y\,,$$

and $q(z,t)\,\mathrm{d}z$ is the number of electrons contained between the planes $z =$ constant and $z+\mathrm{d}z =$ constant. It is assumed that the stream is of finite lateral extent such that n and $\mathrm{grad}\, n$ are zero when $|x|$ and $|y|$ are large.

† Manuscript received June 24, 1968.

‡ 19 Glasgow Place, Hughes, Canberra, A.C.T. 2605.

Apply the operator $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y$ to all terms of equation (3) to obtain

$$\frac{1}{D}\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial^2 q}{\partial z^2} - 2\lambda\frac{\partial q}{\partial z} + 2\lambda\alpha_i q\,, \tag{4}$$

with

$$2\lambda = W/D\,, \qquad \alpha_i = \nu_i/W\,, \qquad q \equiv q(z,t)\,.$$

The flux F of particles across a plane $z =$ constant is (in unit time)

$$F = (W - D\,\partial/\partial z)\,q(z,t)\,.$$

Growth and Progress of a Pulse

Consider the following solution of equation (4)

$$q(z,t) = \tfrac{1}{2}N(\pi Dt)^{-\frac{1}{2}}\exp(\alpha_i Wt)\exp\{-(z-Wt)^2/4Dt\}\,, \tag{5}$$

which represents a moving and diffusing pulse that begins from the plane $z = 0$ at $t = 0$ as a δ-function containing N particles.

Note that:

(1) The total population of the pulse at time t is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(z,t)\,\mathrm{d}z = N\exp(\alpha_i Wt)\,;$$

its growth factor is $\exp(\alpha_i Wt) = \exp(\nu_i t)$.

(2) The relative distribution of particles in the pulse is the same as that when $\nu_i = 0$; that is to say, the "shape" of the pulse is unaffected by the process of ionization by collision when D, W, and ν_i are independent of position, as has been assumed.

(3) The distance z_{max} at which $q = q_{max}$, its greatest value at a given time t, is $z = z_{max} = Wt$; thus the "crest" of the pulse, which is also its centroid, progresses at speed W. Conversely, the time $t = t_{max}$ at which the crest crosses the plane $z =$ constant is $t_{max} = z/W$. Consequently, the pulse travels in the same manner as a pulse in which $\nu_i = 0$, apart from the overall growth factor $\exp(\nu_i t)$.

The solution of equation (3) that is adopted in equation (5) was used by Lucas (1964) to interpret the growth of continuous currents between plane electrodes in terms of a succession of pulses and their images.

Total Number of Particles that cross the Geometrical Plane $z =$ constant

This number is

$$T(z) = \int_0^{\infty} (W - D\,\partial/\partial z)\,q(z,t)\mathrm{d}t\,. \tag{6}$$

The number is not $N\exp(\alpha_i z_{\max})$ except in the absence of diffusion, when the pulse travels as a δ-function whose growth factor is the same for all its particles, namely, $\exp(\alpha_i z_{\max}) = \exp(\alpha_T z)$. In a pulse that spreads by diffusion as it travels, particles cross the fixed geometrical plane at different times t, and those that cross ahead of the crest do so with smaller values of the growth factor than the values of those that cross after the crest. There is no *a priori* reason why the mean value of these growth factors should have the value of the growth factor of the centroid. In fact, as shown below, the growth factor of $T(z)$ in equation (6) is indeed $\exp\{(\lambda-\mu)z\}$ in agreement with equation (1).

Consider

$$I(z) = \int_0^\infty q(z,t)\,\mathrm{d}t = \frac{N}{2\pi^{\frac{1}{2}}}\int_0^\infty \exp(\nu_i t)\,\frac{\exp\{-(z-Wt)^2/4Dt\}}{(Dt)^{\frac{1}{2}}}\,\mathrm{d}t\,.$$

Put $s = Dt$, $\mu = (\lambda^2-2\lambda\alpha_i)^{\frac{1}{2}}$, and expand and regroup the terms of the exponent to obtain

$$I(z) = \frac{N\exp(\lambda z)}{2\pi^{\frac{1}{2}}D}\int_0^\infty \exp(-\mu s)\,\frac{\exp(-z^2/4s)}{s^{\frac{1}{2}}}\,\mathrm{d}s\,.$$

But (Watson 1944)

$$\int_0^\infty \exp(-ps)\,\frac{\exp(-z^2/4s)}{s^{\nu+1}}\,\mathrm{d}s = 2^{\nu+1}\frac{p^{\frac{1}{2}\nu}}{z^\nu}K_\nu(p^{\frac{1}{2}}z)\,.$$

Put $\nu = -\frac{1}{2}$; $p = \mu$ to find

$$I(z) = (N/2D)\,\mu^{-1}\exp\{(\lambda-\mu)z\}\,,$$

and from equation (6)

$$T(z) = (W-D\,\mathrm{d}/\mathrm{d}z)\,I(z) = N\{(\lambda+\mu)/2\mu\}\exp\{(\lambda-\mu)z\}\,. \tag{7}$$

Steady Current

The emission of particles at a uniform rate $\bar{\imath}$ at the plane $z = 0$ can be considered as a sequence of pulses of content $\bar{\imath}\,\mathrm{d}t$ emitted at short intervals $\mathrm{d}t$. Thus if in equation (7) N is replaced by $\bar{\imath}\,\mathrm{d}t$ and the summation is made over an interval of time t, the total number of particles crossing the plane z = constant in this time is

$$it = Nt\{(\lambda+\mu)/2\mu\}\exp\{(\lambda-\mu)z\}\,,$$

and consequently the current across the plane is

$$i = N\{(\lambda+\mu)/2\mu\}\exp\{(\lambda-\mu)z\} = i_0\exp(\alpha_T z)\,, \tag{8}$$

where

$$i_0 = N(\lambda+\mu)/2\mu \qquad \text{and} \qquad \alpha_T = \lambda-\mu = \lambda\{1-(1-2\alpha_i/\lambda)^{\frac{1}{2}}\}\,.$$

The same formula (8) is obtained by summing the instantaneous fluxes of particles across the plane z = constant, contributed by pulses $\bar{\imath}\,\mathrm{d}t$ that have travelled for times ranging from $t = 0$ to $t = \infty$.

Direct Method

Equation (8) is, of course, more readily derived from the steady state form of equation (4) ($\mathrm{d}q/\mathrm{d}t = 0$), namely,

$$\mathrm{d}^2q/\mathrm{d}z^2 - 2\lambda\,\mathrm{d}q/\mathrm{d}z + 2\lambda\alpha_{\mathrm{i}}\,q = 0\,,$$

the general solution of which is

$$q = A\exp\{(\lambda-\mu)z\} + B\exp\{(\lambda+\mu)z\}\,.$$

The solution appropriate to a stream proceeding to $z = \infty$ is that with $B = 0$. Consequently

$$i = (W - D\,\mathrm{d}/\mathrm{d}z)\,q(z) = A(\lambda+\mu)\exp\{(\lambda-\mu)z\}\,.$$

Let i_0 be the value of i given by this formula when $z = 0$. It follows that

$$i = i_0\exp\{(\lambda-\mu)z\} = i_0\exp(\alpha_{\mathrm{T}}\,z)\,.$$

Evidently the relationship $\alpha_{\mathrm{T}} = \lambda-\mu = \lambda\{1-(1-2\alpha_{\mathrm{i}}/\lambda)^{\frac{1}{2}}\}$ is a property of the stream and is not to be attributed to the influence of the electrodes.

Acknowledgment

I am greatly indebted to Dr. R. W. Crompton for discussions on the subject.

References

Burch, D. S., and Huxley, L. G. H. (1967).—*Aust. J. Phys.* **20**, 625.
Crompton, R. W. (1967).—*J. appl. Phys.* **38**, 4093.
Huxley, L. G. H. (1959).—*Aust. J. Phys.* **12**, 174.
Lucas, J. (1964).—*J. Electron. Control* **17**, 43.
Watson, G. N. (1944).—"Theory of Bessel Functions." 2nd Ed. Sec. 6.23, Eqn. (15). (Cambridge Univ. Press.)