

ON THE PROBLEM OF UNRESOLVED RESONANCES IN NEUTRON SPECTROSCOPY

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Summary

A direct method of calculating the probability for missing levels in slow neutron spectroscopy is described. The method gives a criterion for missing levels based on the Breit-Wigner resonance profile between adjacent levels and does not depend on correcting an observed distribution of resonance parameters to the theoretically expected one.

I. INTRODUCTION

In neutron cross section measurements, statistical errors in the accumulated counts and the finite instrumental resolution impose a practical limit on the ability to detect weak resonances. In addition, when overlapping between levels occurs, quite large resonances may remain unresolved. The probability of missing levels below the detectability limit depends only on the incident neutron energy while the overlapping probability depends primarily on the ratio of average level width $\langle \Gamma \rangle$ to average level spacing $\langle D \rangle$. When calculating average parameters such as the s-wave strength function and the average level spacing, an important correction must be applied to the raw data for these unresolved levels.

Fuketa and Harvey (1965) have described a method of calculating the proportion of levels lying below the detectability limit, based on the Porter-Thomas distribution of the reduced neutron widths. The experimental resolution limit is proportional to a power of the neutron energy, and integration of the Porter-Thomas distribution up to this limit over the energy range considered yields the relative probability of missing the weak levels. This method does not take into account the levels that are missed by overlapping since these may have reduced neutron widths in excess of the detectability limit.

The Fuketa and Harvey method is not rigorously correct (Wilkins 1967) even for the calculation of the probability of missing weak levels, since it is assumed that resonances are randomly distributed on the energy axis instead of obeying a distribution such as that proposed by Wigner (1957), where small spacings are rare for levels of the same spin sequence. The effect of the assumption about the random placement of levels is to overestimate the number of levels missed in the energy range.

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In fissile nuclei, where the ratio $\langle \Gamma \rangle / \langle D \rangle$ is large, overlapping of levels cannot be ignored and Musgrove (1967) developed a method of calculating the number of levels missed in these nuclei. The method in effect corrected the observed distribution of resonance peak heights and, since it was insensitive to the precise experimental resolution limit, it was expected to contain a contribution from overlapping levels. A calculation performed on ^{233}U , which has the largest known value of $\langle \Gamma \rangle / \langle D \rangle$, indicated that approximately 30% of the levels in that nucleus were undetected.

The correction method described here is an improvement over those outlined above since it no longer depends on correcting an experimental distribution of parameters, but gives the total probability for missing levels directly. Both classes of missed resonances are treated identically and weak levels are assumed to be missed owing to "overlapping", no matter what the inter-resonance spacing.

II. METHOD OF CORRECTION FOR NONFISSILE NUCLIDES

The Doppler-broadened capture cross section at an energy E between two adjacent resonance levels at energies E_1 and E_2 is given approximately by the sum of two Breit-Wigner terms and a contribution from distant levels:

$$\sigma_c(E) = 4\pi\lambda^2 \left(\frac{g_1 \Gamma_{n1} \Gamma_{\gamma 1}}{\Gamma_1^2} \psi(x_1, t_1) + \frac{g_2 \Gamma_{n2} \Gamma_{\gamma 2}}{\Gamma_2^2} \psi(x_2, t_2) \right) + \sigma_{\text{dist}}, \quad (1)$$

where Γ_n , Γ_γ , and Γ are respectively the neutron, radiation, and total widths of a level, λ is the neutron wavelength, and g is the appropriate spin weight factor. Also

$$\psi(x, t) = \frac{1}{2}(\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (1+y^2)^{-1} \exp\{-(x-y)^2/4t\} dy$$

is the usual Voigt profile with arguments defined by

$$x = 2(E - E_r)/\Gamma, \quad t = 4\mu E_r T / \Gamma^2,$$

where μ is the mass ratio and T the temperature of the absorber atom in energy units.

Radiation widths vary little from level to level since radiative decay proceeds through a large number of independent channels, and it is usual to assume that the neutron decay widths are negligible in comparison. Under this assumption the total widths are approximately constant and equal to the radiation width, and equation (1) can be written as

$$\sigma_c \simeq \text{const} \{ \psi(x, t) + R\psi(x-a, t) \} + \sigma_{\text{dist}}, \quad (2)$$

where $a = 2(E_2 - E_1)/\Gamma$, $R = \Gamma_{n2}^0/\Gamma_{n1}^0$ is the ratio of the peak heights, and we have further assumed for simplicity that $g_1 = g_2$ and $t_1 = t_2$. When $R = 0$, the second resonance is missed for all values of a , but, for a particular nonzero value of R , a limiting separation exists, say $a_L(R, t)$, where the two levels are just resolved. We adopt the criterion that the resonances are just resolved at the first formation of a "shoulder" between them, i.e. when two points of inflexion first occur in the resonance profile of equation (2) in the interval $0 < x < a$. Using the expression for $\partial^2\psi/\partial x^2$

given by Clancy and Keane (1964), the second derivative of the cross section with respect to x can be written as

$$\begin{aligned} d^2\sigma/dx^2 = \text{const}[1 - (1+x^2+2t)\psi(x) - 2x(x\psi(x) - \phi(x)) \\ + R\{1 - (1+(x-a)^2+2t)\psi(x-a) - 2(x-a)((x-a)\psi(x-a) - \phi(x-a))\}] \\ + d^2\sigma_{\text{dist}}/dx^2, \end{aligned} \quad (3)$$

where

$$\phi(x, t) = \frac{1}{2}(\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} y(1+y^2)^{-1} \exp\{-(x-y)^2/4t\} dy,$$

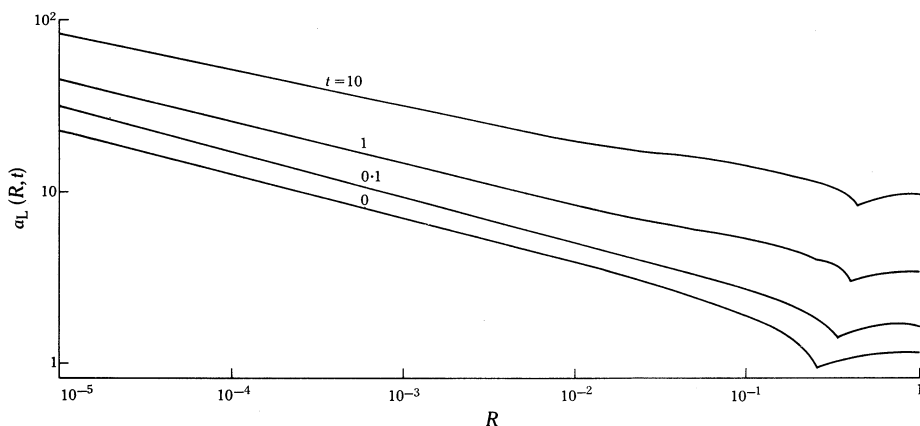


Fig. 1.—Limiting separation of two resonances $a_L(R, t)$ (in units of half the level width) for which the levels are just resolved plotted against the peak height ratio R .

and the argument t has been suppressed in $\psi(x, t)$ and $\phi(x, t)$. If it is now assumed that σ_{dist} is a slowly varying quantity between the resonances and consequently that $d^2\sigma_{\text{dist}}/dx^2 \simeq 0$, the positions of the zeros of the $d^2\sigma/dx^2$ can be found numerically. The limiting separation of the two resonances $a_L(R, t)$ for which less than two points of inflexion occur in $0 < x < a$ is shown in Figure 1 plotted against R for four values of t . For small values of R ($\leq 10^{-2}$) it is seen that $a_L(R, t)$ is proportional to a power of R , the exact dependence being given by

$$\left. \begin{aligned} a_L(R, 0) &= 1.213 R^{-0.257}, & a_L(R, 0.1) &= 1.546 R^{-0.257}, \\ a_L(R, 1) &= 2.266 R^{-0.257}, & a_L(R, 10) &= 7.482 R^{-0.208}. \end{aligned} \right\} \quad (4)$$

The total probability of failing to resolve the two levels is the product of the probability for finding a relative peak height R and a level separation less than $\frac{1}{2}\Gamma a_L(R, t)$, integrated over all R . The two distributions are readily obtained for a nonfissile nucleus that forms two level sequences when bombarded with slow neutrons.

The frequency function for the peak height ratio R is given simply by the F distribution with (1, 1) degrees of freedom, since reduced neutron widths are distributed about their mean as χ^2 variables with one degree of freedom:

$$R(r) dr = \{r^{-1/2}/\pi(1+r)\} dr. \quad (5)$$

This function is plotted against the ratio $r = \Gamma_{n2}/\Gamma_{n1}$ in Figure 2.

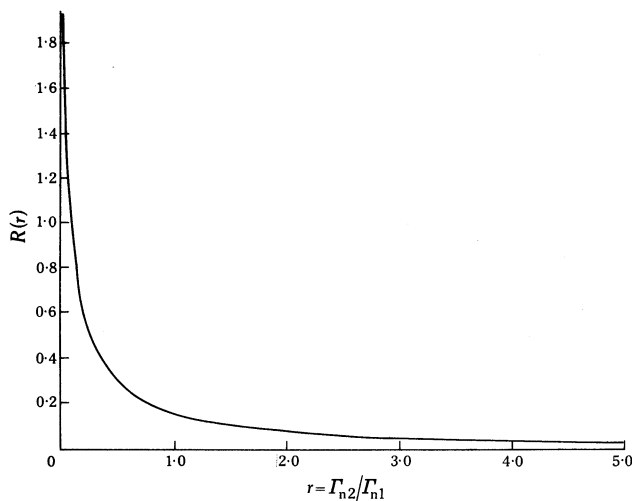


Fig. 2.—Probability density function $R(r)$ in equation (5) versus the ratio $r = \Gamma_{n2}/\Gamma_{n1}$.

The spacing distribution law for a level sequence of the same spin and parity and average level spacing $\langle D \rangle$ is given to a good approximation by the surmise of Wigner (1957)

$$p_1(x) dx = \frac{1}{2} \pi x \exp(-\frac{1}{4} \pi x^2) dx, \quad (6)$$

where $x = D/\langle D \rangle$. Owing to the level repulsion effect, overlapping between levels of the same spin sequence is negligible and the correction technique of Fuketa and Harvey (1965) is adequate. However, since the spacing distributions for different spin states are independent, the probability of overlapping becomes appreciable when two spin states are superimposed. The probability density of spacings resulting from the random superposition of two level sequences both having equal average level spacings $\langle D \rangle$ and spacing distributions given by equation (6) is given by (e.g. Mehta 1967)

$$q_1(x) = \frac{1}{2} \pi x \exp(-\frac{1}{4} \pi x^2) \operatorname{erfc}(\frac{1}{2} \pi^{\frac{1}{2}} x) + \exp(\frac{1}{2} \pi x^2), \quad (7)$$

where $x = D/\langle D \rangle$; and finally the distribution function for such a sequence is

$$P(y \leq x) = 1 - \exp(-\frac{1}{4} \pi x^2) \operatorname{erfc}(\frac{1}{2} \pi^{\frac{1}{2}} x), \quad (8)$$

where

$$\operatorname{erfc} z = 2\pi^{-\frac{1}{2}} \int_z^\infty \exp(-t^2) dt.$$

The frequency and distribution functions for the spacings of two superimposed level sequences given by equations (7) and (8) are shown in Figure 3.

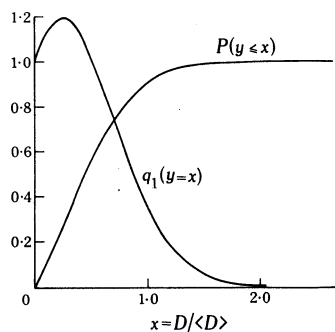


Fig. 3.—Probability density function $q_1(y = x)$ and distribution function $P(y \leq x)$ for spacings of two superimposed level sequences versus $x = D/\langle D \rangle$. $\langle D \rangle$ is the average level spacing per spin state, assumed the same for each sequence.

The total probability for failing to resolve the two levels can now be written as

$$P_1(\text{overlap}) = \int_0^\infty R(r) P(x \leq x_L(r)) dr, \quad (9)$$

where $x_L(r) = \Gamma a_L(r)/2\langle D \rangle$. The integration is performed numerically and Figure 4

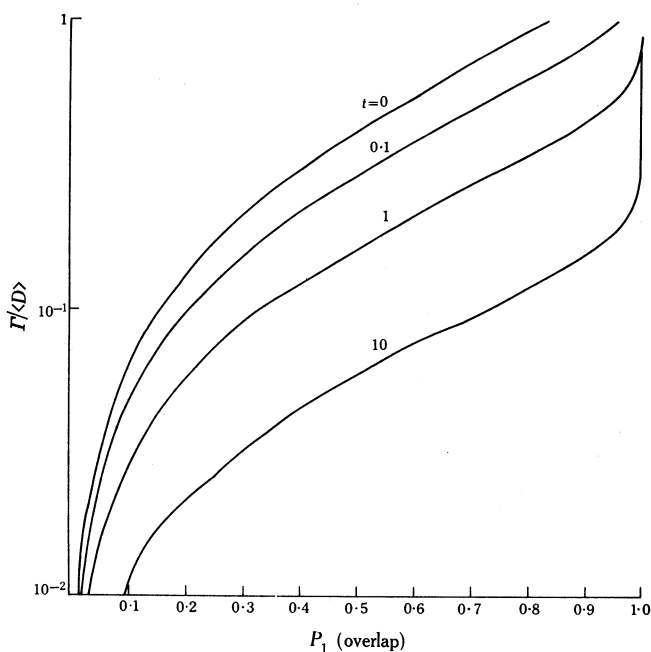


Fig. 4.—Probability of two levels overlapping $P_1(\text{overlap})$ in nonfissile nuclei versus the ratio $\Gamma/\langle D \rangle$.

shows the overlapping probability versus the ratio $\Gamma/\langle D \rangle$ for four values of t . There are further small corrections to be applied to the missing probability of equation (9) from multiple overlapping of levels. The contribution from levels overlapping three

at a time can be calculated as for the two-level case, but since this will be found to be small no higher order corrections will be attempted.

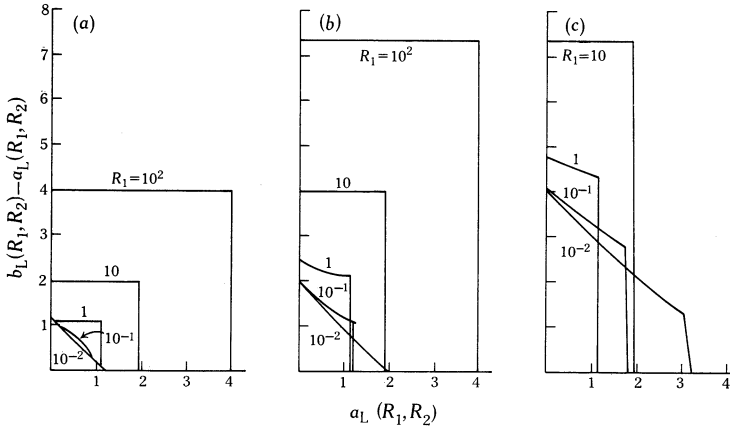


Fig. 5.—Limiting separations of three adjacent levels $a_L(R_1, R_2, t)$ and $b_L(R_1, R_2, t) - a_L(R_1, R_2, t)$ shown for $t = 0$, various values of R_1 , and (a) $R_2 = 1$, (b) $R_2 = 0.1$, and (c) $R_2 = 0.01$.

III. CORRECTION FOR TRIPLE OVERLAPPING OF LEVELS

The capture cross section corresponding to equation (2) for three adjacent levels can be written as

$$\sigma_c \simeq \text{const} \{ \psi(x, t) + R_1 \psi(x - a, t) + R_2 \psi(x - b, t) \} + \sigma_{\text{dist}}, \quad (10)$$

where

$$R_1 = \Gamma_{n2}^0 / \Gamma_{n1}^0, \quad R_2 = \Gamma_{n3}^0 / \Gamma_{n1}^0, \\ a = 2(E_2 - E_1) / \Gamma, \quad b = 2(E_3 - E_1) / \Gamma, \quad 0 \leq a \leq b.$$

The second derivative of the cross section corresponding to equation (3) is readily found and again the limiting values of a and b for which less than two points of inflexion occur in $0 < x < b$ are found numerically. Figure 5 shows the limiting values $a_L(R_1, R_2)$ and $b_L(R_1, R_2)$ for several values of R_1 and R_2 and $t = 0$. Because of the symmetry between the configurations $(1, R_1, R_2)$ and $(1, R_1/R_2, 1/R_2)$ only values of $R_2 \leq 1$ need be considered. The overlapping probability in this case is a double integral over R_1 and R_2 and some simplification of the limiting level separations is desirable for computation. Comparison with the limiting values of a in Figure 1 for the two-level problem allows the following approximations to be made to the limiting three-level separations:

$$R_1 \geq 1, R_2 \leq 1 \quad a_L(R_1, R_2) \simeq a_L(R_1), \quad b_L(R_1, R_2) \simeq a_L(R_1) + a_L(R_1/R_2), \quad (11)$$

$$R_1 \leq 1, R_2 \leq 1, R_1 > R_2 \quad a_L(R_1, R_2) \simeq a_L(R_1), \quad b_L(R_1, R_2) \simeq a_L(R_2), \quad (12)$$

$$R_1 \leq 1, R_2 \leq 1, R_2 > R_1 \quad a_L(R_1, R_2) \simeq a_L(R_2), \quad b_L(R_1, R_2) \simeq a_L(R_2), \quad (13)$$

where the argument t has been suppressed. Owing to symmetry, the three regions of R_1, R_2 above are sufficient to include all possible configurations of three levels.

To proceed further, we require the probability that if a resonance occurs at E , which is taken to be the origin, then the second level occurs within an interval $x_1 \leq a$ and the next level occurs within an interval $x_2 \leq b$. The two level sequences are assumed to have equal average level spacings, and without loss of generality the resonance at E is assumed to belong to the first sequence. The probability that the third resonance occurs in $x, x+dx$, conditional upon the second having occurred at $x_1 = k$, can be written as (Wilkins and Musgrove, to be published)

$$\begin{aligned}
 P(x | x_1 = k) dx = & h_1(k) \left\{ p_1(x-k) \int_x^\infty \exp(-\tfrac{1}{4}\pi y^2) dy \left(\int_k^\infty \exp(-\tfrac{1}{4}\pi y^2) dy \right)^{-1} \right. \\
 & \left. + \exp(-\tfrac{1}{4}\pi x^2) \left(\int_k^\infty \exp(-\tfrac{1}{4}\pi y^2) dy \right)^{-1} \int_x^\infty p_1(y-k) dy \right\} dx \\
 & + h_2(k) \left\{ p_1(x-k) \int_x^\infty p_1(y) dy \left(\int_k^\infty p_1(y) dy \right)^{-1} \right. \\
 & \left. + p_1(x) \left(\int_k^\infty p_1(y) dy \right)^{-1} \int_x^\infty p_1(y-k) dy \right\} dx, \quad (14)
 \end{aligned}$$

where $p_1(x)$ is the Wigner distribution in equation (6) and $h_i(k)$ is the probability that the resonance at k is of the i th sequence and

$$\left. \begin{aligned} h_1(k) &= \tfrac{1}{2}\pi k \exp(-\tfrac{1}{4}\pi k^2) \operatorname{erfc}(\tfrac{1}{2}\pi^{\frac{1}{2}} k) / q_1(k), \\ h_2(k) &= \exp(-\tfrac{1}{2}\pi k^2) / q_1(k), \end{aligned} \right\} \quad (15)$$

with $q_1(k)$ given by equation (7). The probability that the third resonance occurs anywhere in the interval $k \leq x_2 \leq b$ can then be written as

$$\begin{aligned}
 P(x_2 \leq b | x_1 = k) &= \int_k^b P(x | x_1 = k) dx \\
 &= \tfrac{1}{2}\pi k \frac{\exp(-\frac{1}{4}\pi k^2)}{q_1(k)} \left(\operatorname{erfc}(\tfrac{1}{2}\pi^{\frac{1}{2}} k) - \exp(-\tfrac{1}{4}\pi(b-k)^2) \operatorname{erfc}(\tfrac{1}{2}\pi^{\frac{1}{2}} b) \right) \\
 &\quad + \frac{\exp(-\frac{3}{8}\pi k^2)}{q_1(k)} \left(\exp(-\tfrac{1}{8}\pi k^2) - \exp(-\tfrac{1}{2}\pi(b-\tfrac{1}{2}k)^2) \right), \quad (16)
 \end{aligned}$$

and the required distribution is

$$P(x_2 \leq b, x_1 \leq a) = \int_0^k q_1(k) P(x_2 \leq b | x_1 = k) dk, \quad (17)$$

where $a \leq b$. Figure 6 shows the slightly different distribution function $P(x_2 \leq (b-a), x_1 \leq a)$ versus x_1 and x_2 , where in this case x_2 is the spacing between the second and third levels. The integration of equation (17) was performed numerically.

Since R_1 and R_2 are dependent variables, the probability element in their joint distribution is required. The distribution of reduced neutron widths about their mean is given by

$$P(x) = (2\pi x)^{-\frac{1}{2}} \exp(-\frac{1}{2}x),$$

and putting $x_i = \Gamma_{ni}^0 / \langle \Gamma_n^0 \rangle$, $i = 1, 2, 3$, the joint element for x_1, x_2 , and x_3 can be written as

$$P(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \prod_{i=1}^3 (2\pi x_i)^{-\frac{1}{2}} \exp(-\frac{1}{2}x_i) dx_i. \quad (18)$$

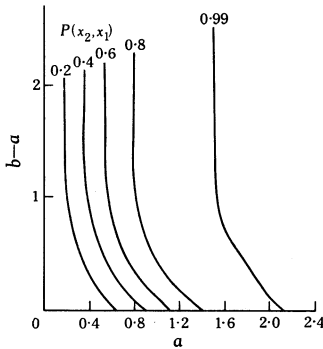


Fig. 6.—Distribution function $P(x_2 \leq (b-a), x_1 \leq a)$ shown for first resonance spacing x_1 and second resonance spacing x_2 for a superposition of two simple sequences having equal average level spacings.

The required element is obtained by substituting $r_1 = x_2/x_1$ and $r_2 = x_3/x_1$ in equation (18) and integrating over x_1 to obtain

$$R(r_1, r_2) dr_1 dr_2 = \{2\pi(r_1 r_2)^{\frac{1}{2}} (1+r_1+r_2)^{3/2}\}^{-1}. \quad (19)$$

The probability of failing to resolve the three levels is therefore

$$P_2(\text{overlap}) = \int_0^\infty dr_1 \int_0^\infty R(r_1, r_2) P(x_2 \leq x_{2L}(r_1, r_2), x_1 \leq x_{1L}(r_1, r_2)) dr_2, \quad (20)$$

where

$$x_{1L} = \Gamma a_L(r_1, r_2) / 2\langle D \rangle \quad \text{and} \quad x_{2L} = \Gamma b_L(r_1, r_2) / 2\langle D \rangle.$$

When the appropriate values for a_L and b_L are inserted from equations (11), (12), and (13) the integration can be carried out, and Figure 7 shows the probability for triple overlapping versus $\Gamma/\langle D \rangle$ for four values of t . For most nuclei the correction can be ignored for small energies; however, in ^{233}U for which $\langle \Gamma \rangle / \langle D \rangle \simeq 0.2$ the correction amounts to approximately 2%.

IV. METHOD OF CORRECTION FOR FISSILE NUCLIDES

In fissile nuclides the calculation of the correction for missed levels is complicated by the fact that the fission widths, and hence the total widths, fluctuate from level to level and the peak height ratio of two resonances is no longer given by the ratio

of two reduced neutron widths. The fission cross section between two adjacent levels is given in the single-level Breit-Wigner approximation by

$$\sigma_f(E) = 4\pi\lambda^2 \left(\frac{g_1 \Gamma_{n1} \Gamma_{f1}}{\Gamma_1^2} \psi(x, t) + \frac{g_2 \Gamma_{n2} \Gamma_{f2}}{\Gamma_2^2} \psi(x-a, t) \right) + \sigma_{\text{dist}}, \quad (21)$$

where Γ_f is the fission width of the level and all other symbols have been defined previously. Multilevel effects have been ignored in equation (21) since, owing to the level repulsion effect, overlapping is much more probable for levels of opposite

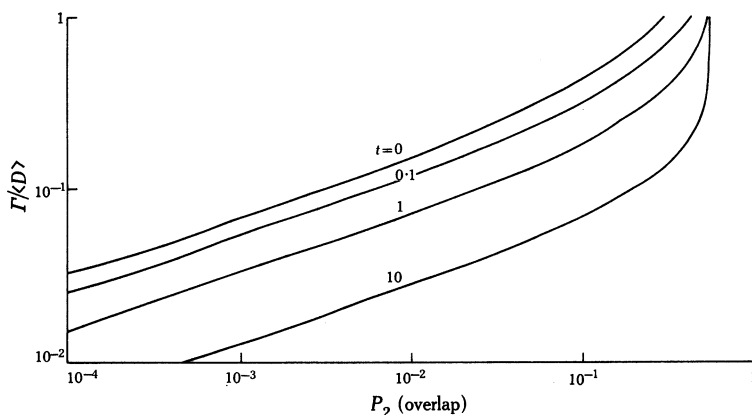


Fig. 7.—Probability of three levels overlapping $P_2(\text{overlap})$ versus the ratio $\Gamma/\langle D \rangle$ for four values of t .

spin than for levels of the same spin and parity. In any case, it is expected that the limiting separation at which two levels with peak height ratio R are just resolved will, at least on *average*, be given by the $a_L(R)$ found in the nonfissile case. In analogy with equation (2), the fission cross section can be written as

$$\sigma_f \simeq \text{const} \{ \psi(x, t) + RS\psi(x-a, t) \} + \sigma_{\text{dist}}, \quad (22)$$

where

$$R = \Gamma_{n2}^0 / \Gamma_{n1}^0 \quad \text{and} \quad S = (\Gamma_{f2} / \Gamma_2^2) \div (\Gamma_{f1} / \Gamma_1^2).$$

The fission process proceeds through a small number of open channels and the distribution of fission widths about their mean is usually represented by a chi-squared distribution with n degrees of freedom, where n usually lies between 2 and 4:

$$P(z) = (n^{1/2} / \{2^{1/2} \Gamma(\frac{1}{2}n)\}) \exp(-\frac{1}{2}nz) z^{1/2} n^{-1}, \quad (23)$$

with $z = \Gamma_f / \langle \Gamma_f \rangle$. It should be noted that it is purely a matter of convenience to represent this distribution by a member of the χ^2 family since the fission process typically proceeds through a different number of channels from each spin state and the average fission widths for each spin state are usually different.

To obtain the probability element for the variate Γ_1/Γ^2 , the substitution $y = z/(z+c)^2$ is made in equation (23). Thus

$$c = \Gamma_\gamma / \langle \Gamma_1 \rangle \quad \text{is constant,} \quad z = \{1 - 2cy \pm (1 - 4cy)^{\frac{1}{2}}\}/2y,$$

and

$$P(y) dy = \frac{n^{\frac{1}{2}n}}{2^{\frac{1}{2}n} \Gamma(\frac{1}{2}n)} \frac{f(y)}{y(1-4cy)^{\frac{1}{2}}} dy, \quad 0 \leq y \leq 1/4c, \quad (24)$$

where

$$f(y) = \exp(-\frac{1}{2}n\zeta_1)\zeta_1^{\frac{1}{2}n} + \exp(-\frac{1}{2}n\zeta_2)\zeta_2^{\frac{1}{2}n},$$

with

$$\zeta_1 = \{1 - 2cy + (1 - 4cy)^{\frac{1}{2}}\}/2y \quad \text{and} \quad \zeta_2 = \{1 - 2cy - (1 - 4cy)^{\frac{1}{2}}\}/2y.$$

The element in the joint distribution of the two independent variables Γ_{11}/Γ_1^2 and Γ_{12}/Γ_2^2 is clearly

$$P(x, y) dx dy = \frac{n^n}{2^n (\Gamma(\frac{1}{2}n))^2} \frac{f(x)f(y)}{xy\{(1-4cx)(1-4cy)\}^{\frac{1}{2}}} dx dy, \quad (25)$$

where $x = \Gamma_{11}/\Gamma_1^2$ and $y = \Gamma_{12}/\Gamma_2^2$. The probability density function for the ratio x/y is obtained by substituting $u = x/y$ and $v = y$ in equation (25) and integrating over v . Thus

$$P(u) du = \frac{n^n}{2^n (\Gamma(\frac{1}{2}n))^2} \int_0^{1/4c} \frac{f(uv)f(v)}{uv\{(1-4cv)(1-4cuv)\}^{\frac{1}{2}}} dv du, \quad u < 1, \quad (26a)$$

$$= \frac{n^n}{2^n (\Gamma(\frac{1}{2}n))^2} \int_0^{1/4cu} \frac{f(uv)f(v)}{uv\{(1-4cv)(1-4cuv)\}^{\frac{1}{2}}} dv du, \quad u > 1. \quad (26b)$$

Putting $v = \sin^2\theta/4c$ in equation (26a) and $v = \sin^2\theta/4cu$ in (26b) we obtain

$$P(u) du = \frac{K}{u} \int_0^{\frac{1}{2}\pi} \frac{f(\sin^2\theta/4c)f(u\sin^2\theta/4c)}{\sin\theta(1-u\sin^2\theta)^{\frac{1}{2}}} d\theta du, \quad u < 1, \quad (27a)$$

$$= \frac{K}{u^{\frac{1}{2}}} \int_0^{\frac{1}{2}\pi} \frac{f(\sin^2\theta/4c)f(u\sin^2\theta/4c)}{\sin\theta(u-\sin^2\theta)^{\frac{1}{2}}} d\theta du, \quad u > 1, \quad (27b)$$

where

$$K = n^n/2^{n-1}(\Gamma(\frac{1}{2}n))^2.$$

The joint probability element in the distribution of $S = (\Gamma_{11}/\Gamma_1^2) \div (\Gamma_{12}/\Gamma_2^2)$ and $R = \Gamma_{n1}^0/\Gamma_{n2}^0$ is now

$$P(r, u) du dr = \{r^{-\frac{1}{2}}P(u)/\pi(1+r)\} du dr,$$

and the element for the product ur is obtained by substituting $x = ur$ and $y = u$ and integrating over y . Thus

$$Q(x) = K \int_0^1 \frac{(x/y)^{-\frac{1}{2}}P(y)}{\pi(x+y)} dy + K \int_1^\infty \frac{(x/y)^{-\frac{1}{2}}P(y)}{\pi(x+y)} dy \quad (28)$$

and ultimately the required probability element in the distribution of $x = RS$ becomes

$$Q(x) dx = \frac{K}{\pi} \int_0^1 \frac{1}{(xy)^{\frac{1}{2}}} \left(\frac{1}{x+y} + \frac{1}{xy+1} \right) dy \int_0^{\frac{1}{2}\pi} \frac{f(\sin^2 \theta/4c) f(y \sin^2 \theta/4c)}{\sin \theta (1-y \sin^2 \theta)^{\frac{1}{2}}} d\theta dx, \quad (29)$$

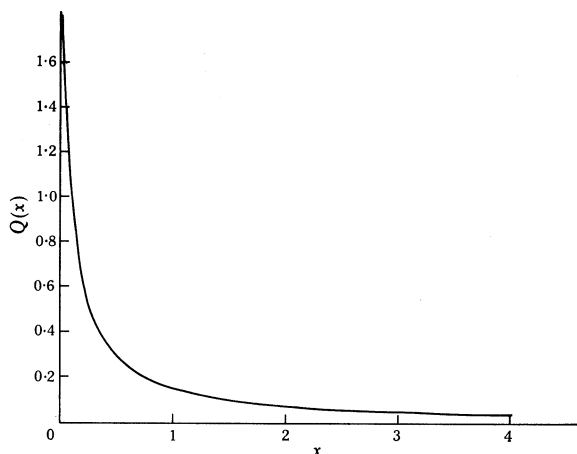


Fig. 8.—Probability density function $Q(x)$ in equation (29) versus $x = (\Gamma_{n2}^0 \Gamma_{t2}/\Gamma_2^2) \div (\Gamma_{n1}^0 \Gamma_{t1}/\Gamma_1^2)$ for $c = 0.1$ and $n = 3$.

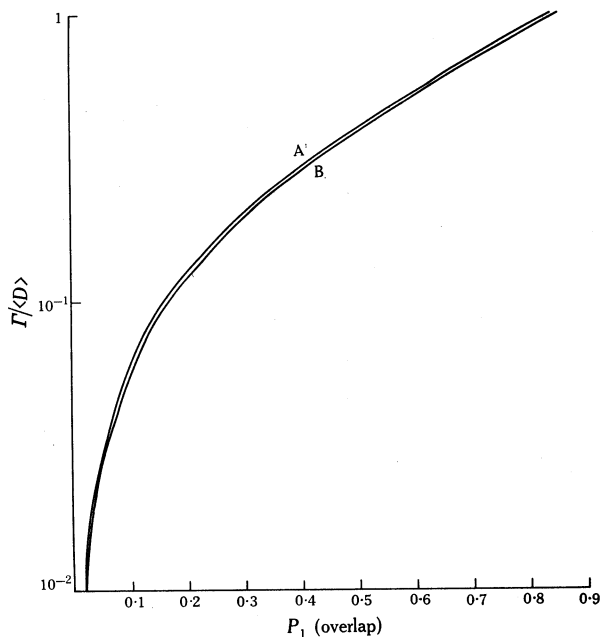


Fig. 9.—Probability of two-level overlap versus the ratio $\Gamma/\langle D \rangle$ for $t = 0$ in (A) nonfissile nuclides, and (B) fissile nuclides.

upon manipulating the second integral in equation (28). Figure 8 gives $Q(x)$ versus x for $c = 0.1$ and an assumed value of $n = 3$ for the number of degrees of freedom in the fission width distribution. The total probability for missing either of two adjacent levels in fissile nuclei can now be obtained by replacing the R distribution

by Q in equation (9):

$$P_1(\text{overlap}) = \int_0^\infty Q(r) P(x \leq x_L(r)) dr. \quad (30)$$

The overlapping probability is shown in Figure 9 against $I/\langle D \rangle$ for $t = 0$ and is compared with that found for the nonfissile nuclides. The overlapping probability for fissile nuclei is at most 5% greater than that for nonfissile nuclei and for most purposes can be ignored. Because of this we do not attempt to calculate the three-level overlapping probability for fissile nuclides but will be content to approximate it by the correction already calculated in the nonfissile case. We now give a calculation of the probability for missing levels in ^{233}U and compare it with an earlier result of Musgrove (1967).

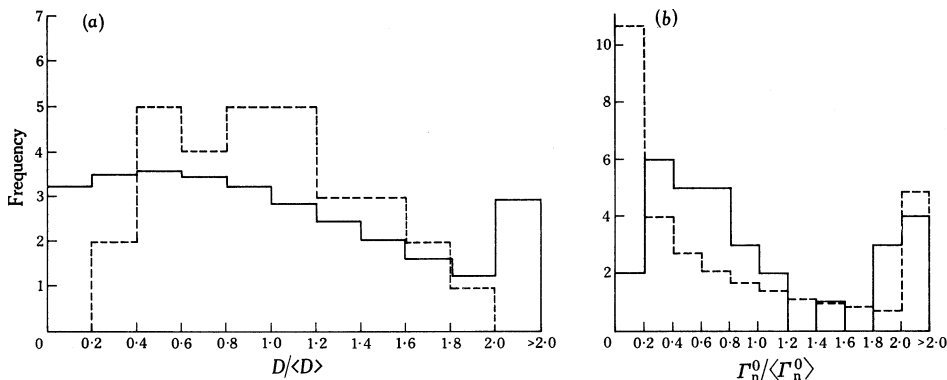


Fig. 10.—Histograms for ^{233}U of (a) first 30 level spacings compared with the expectation values (dashed lines) from equation (7) and (b) 31 values of $\Gamma_n^0/\langle \Gamma_n^0 \rangle$ compared with the expectation values.

V. CALCULATION OF PROBABLE NUMBER OF MISSED LEVELS IN ^{233}U

There is a considerable body of indirect experimental evidence for the existence of undetected resonances in ^{233}U . Small inter-resonance spacings are observed much less frequently than is expected when two independent level sequences are superimposed. This is seen in Figure 10(a) where a histogram of the first 30 level spacings (normalized to unit mean) is shown and compared with the expectation values of equation (7).

In the experimental distribution of reduced neutron widths, once again, small values are lacking as is seen in the histogram plotted in Figure 10(b) with expectation values. In fact, Nifenecker (1964) found that this distribution is fitted quite well by a χ^2 distribution with four degrees of freedom instead of the expected one degree of freedom. Since there is no reason to believe that ^{233}U has other than a single neutron decay channel, or that the distributions of resonances of different spin states are not independent in that nucleus, it appears that unresolved levels may cause the discrepancies.

In order to calculate the probable number of levels missed for ^{233}U , care must be taken to select an "unbiased" sample of observed levels. Our criterion for failing to resolve two levels with peak height ratio R is that the levels are separated by less than $a_L(R)\langle\Gamma\rangle/2\langle D\rangle$. In fissile nuclei where Γ fluctuates considerably from level to level it is possible for two levels to be resolved experimentally with separations less than the limit (e.g. when $\frac{1}{2}(\Gamma_1 + \Gamma_2) \leq \langle\Gamma\rangle$). In ^{233}U , the levels reported by Nifenecker (1964) at 7.65, 16.82, 18.75, and 26.50 eV fall into this category and are therefore omitted from the "observed" resonances for this calculation.

Suppose that N_0 levels are detected experimentally in an energy range $E_1 \leq E \leq E_n$, where E_1 and E_n are the energies of the first and last levels in the sequence, and further suppose that ν levels with average total width $\langle\Gamma\rangle'$ are missed in this range. The actual average level spacing per spin state $\langle D\rangle$ and the actual average total width $\langle\Gamma\rangle$ may be written as

$$\langle D\rangle = 2(E_n - E_1)(N_0 + \nu - 1)^{-1}, \quad \langle\Gamma\rangle = \left(\sum_{i=1}^{n_0} \Gamma_i + \nu\langle\Gamma\rangle' \right) (N_0 + \nu)^{-1}, \quad (31)$$

where Γ_i is the total width of the i th level and we have assumed equal average level spacings in each spin state. For this calculation it will be assumed that the missed levels have the same average total width as the observed levels. The expected number of levels missed in ^{233}U can now be obtained by iteration. Thus

$$\nu^{(m)} = (N_0 + \nu^{(m-1)}) \{P_1(\gamma^{(m-1)}) + P_2(\gamma^{(m-1)})\}, \quad (32)$$

where m is the order of the iteration, γ is the ratio $\langle\Gamma\rangle/\langle D\rangle$, and P_1 and P_2 are respectively the probabilities for two- and three-level overlap given in equations (9) and (20). The parameters used in the present calculation were taken from Nifenecker (1964), where 31 levels are reported with average total width 0.38 eV between energies 1.799 and 29.51 eV. Since four of these levels are "unresolved" by our criterion, we solve equation (32) for ν using $N_0 = 27$. The probable number of levels missed in the range was found to be 16.8 after 12 iterations. The positions of 4 of these are known already, but the remaining 13 levels are undetected. This value agrees very well with the earlier result of Musgrove (1967) where 11 ± 2 was the expected number of levels missed for the Nifenecker data in a slightly smaller energy range. The corrected average level spacing of ^{233}U for the present calculation is 0.64 eV compared with the earlier corrected value of 0.65 eV.

VI. CONCLUSIONS

In the direct method of calculating the probability for missing levels in slow neutron spectroscopy as described above, the probability depends only on the observed ratio $\langle\Gamma\rangle/\langle D\rangle$, unlike earlier methods which require the (unknown) average reduced neutron width of the missed levels as a parameter. The present method treats overlapping levels and weak levels identically and it has been shown that for practical purposes only two-level overlapping contributes to the total probability for missing levels. For nonfissile nuclides the correction is small ($\leq 10\%$) at low energies but a calculation of the number of missed levels in ^{233}U indicates that

approximately 30% of the levels in that nucleus are undetected. This value agrees quite well with that reported by Musgrove (1967).

VII. REFERENCES

- CLANCY, B. E., and KEANE, A. (1964).—Australian Atomic Energy Commission Rep. No. AAEC/E117.
- FUKETA, T., and HARVEY, J. A. (1965).—*Nucl. Instrum. Meth.* **33**, 107.
- MEHTA, M. L. (1967).—"Random Matrices." p. 216. (Academic Press: New York.)
- MUSGROVE, A. R. (1967).—*Aust. J. Phys.* **20**, 617.
- NIFENECKER, H. (1964).—*J. Physique* **25**, 877.
- WIGNER, E. P. (1957).—Oak Ridge National Lab. Rep. No. ORNL 2309, p. 59.
- WILKINS, C. A. (1967).—*J. nucl. Sci. Technol.* **4**, 37.