SHORT COMMUNICATIONS

STUDY OF COMBINED EFFECT OF KELVIN–HELMHOLTZ INSTABILITY AND GRAVITATIONAL INSTABILITY ON A SELF-GRAVITATING FLUID LAYER*

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The instability of a self-gravitating fluid layer of finite thickness surrounded by another fluid of different density has been studied recently by Uberoi (1963) and Tassoul (1967) § under varying conditions. Now the condition can arise when the fluid inside the layer and the surrounding material are in relative horizontal motion. It is interesting to study the combined effects of the Kelvin–Helmholtz (KH) instability associated with short wavelengths and the gravitational instability associated with long wavelengths on this layer.

We consider a homogeneous distribution of gravitating ideally conducting fluid mass with constant density ρ in the form of a plane layer of thickness $2h$. The XOY plane is taken to coincide with the unperturbed middle level of the layer and the positive z axis in the upward direction normal to the unperturbed fluid surfaces. This layer is surrounded by a nonconducting fluid of uniform density ρ_0. In the equilibrium state we assume that the system is pervaded by a uniform magnetic field H_1 in the conducting layer and H_2 in the nonconducting fluid, both directed in the x direction. We further assume that initially the conducting and nonconducting fluids are moving with velocities V_1 and V_2 respectively in the x direction. As the disturbances along the direction of the streaming velocities of the fluids and the magnetic field are most sensitive to the KH instability (Chandrasekhar 1961), we shall consider the wave propagation in the x direction only. Hence, we assume that the perturbed quantities depend on time t and spatial coordinate x as $\exp\{\mathrm{i}(\omega t+kx)\}$. As the mathematical procedure is well known (e.g. Uberoi 1963; Tassoul 1967), we shall not give the details here. Following the procedure given in Uberoi (1963), we obtain the dispersion relation, which factorizes into the following two factors corresponding to the asymmetric and symmetric perturbations:

$$\{(\omega+kV_1)^2-k^2V_{\mathrm{A}}^2\}\tanh\theta+\tau\{(\omega+kV_2)^2-k^2V_{\mathrm{B}}^2\}+4\pi G\rho(1-\tau)\left(-1+\frac{1-\tau}{1+\coth\theta}\right)=0 \tag{1}$$

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§ In Chakraborty (1964, quoted from Tassoul 1967) the presence of surrounding material of density ρ_0 is taken into account in deriving the dispersion relation but is discussed only for the case $\rho_0 = 0$.

and

$$\{(\omega+kV_1)^2-k^2V_A^2\}\coth\theta+\tau\{(\omega+kV_2)^2-k^2V_B^2\}+4\pi G\rho(1-\tau)\left(-1+\frac{1-\tau}{1+\tanh\theta}\right)=0\,, \tag{2}$$

where

$$\tau=\rho_0/\rho\,,\qquad \theta=kh\,,\qquad V_A^2=\frac{\mu_0H_1^2}{4\pi\rho}\frac{1}{4\pi G\rho h^2}\,,\qquad V_B^2=\frac{\mu_0H_2^2}{4\pi\rho_0}\frac{1}{4\pi G\rho h^2}\,.$$

In the following analysis we shall assume $\tau<1$ and finally point out some important results that appear due to the presence of relative motion in the case $\tau>1$.

Symmetric Perturbations

(i) $H_1=H_2=0$

The dispersion relation (2) in this case, gives

$$\omega=-k\frac{V_1+V_2\tau\tanh\theta}{1+\tau\tanh\theta}\pm\frac{(4\pi G\rho\tanh\theta)^{\frac{1}{2}}\varDelta^{\frac{1}{2}}}{1+\tau\tanh\theta}\,, \tag{3}$$

where

$$\varDelta=\{(1-\tau)(1+\tau\tanh\theta)G(\theta)-\theta^2\tau U^2\}\,,$$

$$G(\theta)=\left(\theta-\frac{1-\tau}{1+\tanh\theta}\right),\qquad U^2=\frac{(V_2-V_1)^2}{4\pi G\rho h^2}\,.$$

When $U=0$, and if θ' denotes the root of the equation $G(\theta)=0$, then the system is stable for $\theta>\theta'$ and unstable for $\theta<\theta'$ (Uberoi 1963), the instability arising due to self-gravitational force. Considering the case when $U\neq 0$, we find that the presence of the relative streaming velocities of the fluids, however small, alter the stability criterion as seen below.

From (3) we note that $\varDelta$ is negative for all values of $\theta\leqslant\theta'$. For $\theta>\theta'$ the function $G(\theta)$ is a positive monotonically increasing function, increasing as θ. As $\tau U^2\theta^2$ increases as θ^2, $\varDelta$ is negative for large θ. Hence, depending on the value of τU^2 either $\varDelta$ will be negative for all θ or it will have two roots θ_1,θ_2 such that $\varDelta$ is positive for $\theta_1<\theta<\theta_2$ and negative for all other values of θ. Thus, for fixed τ, depending on the value of U^2 either the system is overstable for all wavelengths or there exists a range of values of wave numbers $\theta_1\leqslant\theta\leqslant\theta_2$ for which the system is stable. We note that as U^2 increases, i.e. as the kinetic energy due to the relative motion of the two fluids increases in comparison to the gravitational energy, this range of stability decreases. The range of instability is $0<\theta<\theta_1$ and $\theta>\theta_2$, which shows that the waves are either very long or very short to be stabilized. Here we point out that the instability of long waves is due to self-gravitational force and of small waves is due to the relative motion of the fluid layers.

Table 1 gives critical values θ_1, θ_{1m}, $\omega_m(\theta=\theta_{1m})$, and θ_2 for various values of τ and U^2. We note the following points: (1) As U^2 and τ increase, the range of stability decreases until this range becomes zero and the system becomes overstable for all θ. (2) As U^2 increases, the critical wavelength $\lambda^*=(2\pi h/\theta_1)$ associated with the long wavelength instability decreases. Hence, the presence of the relative speed $(V_2 - V_1)$ enhances the instability mainly arising from self-gravitational forces. (3) The maximum wavelength $\lambda^* = (2\pi h/\theta_2)$, associated with the short wavelength KH instability, increases as the value of U^2 and τ is increased. (4) For the case of long wavelength instability due to self-gravitational force, the wave number θ_{1m} associated with the maximum mode of instability and the characteristic frequency ω_m at θ_{1m} increase with the increase in U^2. Hence, the dimensions of the fragments into which the layer tends to break up gravitationally and the characteristic time $t = (2\pi/\omega_m)$ of this breakup are decreased by the presence of the relative motion.

TABLE 1

θ_1, θ_{1m}, ω_m, AND θ_2 AS FUNCTIONS OF τ AND U^2 FOR SYMMETRIC CASE IN ABSENCE OF MAGNETIC FIELD

θ_1 and θ_{1m} are the wave numbers at which the instability occurs and at which it is a maximum, ω_m is the frequency at θ_{1m}, and θ_2 is the minimum wave number at which the KH instability sets in

U^2	$\tau = 0\cdot1$				$\tau = 0\cdot3$			
	θ_1	θ_{1m}	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$	θ_2	θ_1	θ_{1m}	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$	θ_2
0	0·58801	0·27001	0·31974	—	0·48401	0·22201	0·22491	—
0·2	0·59400	0·27201	0·32031	49·04595	0·49800	0·22601	0·22619	14·80799
0·5	0·60401	0·27401	0·32119	19·33999	0·52400	0·23401	0·22825	5·69400
1	0·62001	0·27801	0·32270	9·42800	0·58601	0·24801	0·23209	2·62000
2	0·66201	0·29001	0·32592	4·45001	—	—	—	—
5	—	—	—	—	—	—	—	—
	$\tau = 0\cdot5$				$\tau = 0\cdot7$			
0	0·37001	0·17201	0·14153	—	0·24201	0·11601	0·07002	—
0·2	0·39200	0·17801	0·14309	7·23999	0·26800	0·12201	0·07141	3·48599
0·5	0·43400	0·19001	0·14573	2·71599	0·33200	0·14001	0·07405	1·15600
1	0·63601	0·22001	0·15141	0·96801	—	—	—	—
2	—	—	—	—	—	—	—	—
5	—	—	—	—	—	—	—	—

(ii) $H_1 \neq 0$, $H_2 \neq 0$

From the dispersion relation (2) we obtain

$$\omega = -k\frac{(V_1 \coth\theta + V_2\tau)}{\tau + \coth\theta} \pm \frac{(4\pi G\rho)^{\frac{1}{2}}(\tanh\theta)^{\frac{1}{2}}\Delta^{\frac{1}{2}}}{\tau\tanh\theta + 1}, \tag{4}$$

where

$$\Delta = \theta^2\Phi + G(\theta), \tag{5}$$

$$\Phi = \coth\theta\,[V_B^2\tau^2\tanh^2\theta - \{U^2 - (V_A^2 + V_B^2)\}\tau\tanh\theta + V_A^2],$$

$$G(\theta) = (1-\tau)(\tau\tanh\theta + 1)\left(\theta - \frac{1-\tau}{1+\tanh\theta}\right).$$

As Φ is quadratic in $\tanh\theta$, it is easy to show that when $U^2 < (V_A+V_B)^2$, Φ is positive for all positive values of θ; when

$$(V_A+V_B)^2 < U^2 < U^{*2} \equiv (\tau+1)/\tau(V_A^2+\tau V_B^2), \qquad \text{and} \qquad \tau > V_A/V_B,$$

Φ is positive for $\theta < \theta_1$ and $\theta > \theta_2$, while it is negative when $\theta_1 < \theta < \theta_2$; when

$$(V_A+V_B)^2 < U^2 < U^{*2}, \qquad \text{but} \qquad \tau < V_A/V_B,$$

Φ is positive for all θ; when $U^2 > U^{*2}$, Φ is positive if $\theta < \theta_1$ and negative if $\theta > \theta_1$, where θ_1 and θ_2 are given by

$$\tanh\theta_{1,2} = (2V_B^2\tau)^{-1}[\{U^2-(V_A^2+V_B^2)\}\mp\{U^2-(V_A-V_B)^2\}^{\frac{1}{2}}\{U^2-(V_A+V_B)^2\}^{\frac{1}{2}}].$$

TABLE 2

θ', θ'_m, ω_m, AND θ'' AS FUNCTIONS OF τ, U^2, V_A^2, AND V_B^2 FOR SYMMETRIC CASE IN PRESENCE OF MAGNETIC FIELD

θ' and θ'_m are the wave numbers at which the instability occurs and at which it is a maximum, ω_m is the frequency at θ'_m, and θ'' is the minimum wave number at which the KH instability sets in

U^2	$\tau = 0\cdot5$				$\tau = 0\cdot7$			
	θ'	θ'_m	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$	θ''	θ'	θ'_m	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$	θ''
				$V_A^2 = 0\cdot5$, $V_B^2 = 2$				
0	0·18001	0·09001	0·10616	—	0·09001	0·04601	0·04539	—
2	0·20401	0·10001	0·10967	—	0·10001	0·05001	0·04691	—
4	0·24601	0·11401	0·11441	—	0·12201	0·05801	0·04892	—
4·7	0·27401	0·12201	0·11657	7·24200	0·13601	0·06001	0·04985	8·34800
4·9	0·28601	0·12401	0·11726	3·47800	0·14001	0·06201	0·05014	2·23600
5·0	0·29201	0·12601	0·11762	2·71600	0·14401	0·06201	0·05030	1·28601
5·2	0·30601	0·12801	0·11838	1·81600	0·15201	0·06401	0·05062	0·65201
5·4	0·32601	0·13201	0·11918	1·29001	0·16001	0·06601	0·05096	0·47401
				$V_A^2 = 2$, $V_B^2 = 1$				
0	0·09001	0·04401	0·07434	—	0·03801	0·01801	0·02884	—
2	0·09201	0·04601	0·07493	—	0·03801	0·01801	0·02897	—
4	0·09601	0·04601	0·07555	—	0·03801	0·02001	0·02914	—
6	0·10001	0·04801	0·07624	—	0·04001	0·02001	0·02933	—
8	0·10401	0·05001	0·07696	2·86199	0·04201	0·02001	0·02951	0·92601
10	0·11001	0·05201	0·07775	0·89001	0·04201	0·02001	0·02969	0·51801
12	0·11801	0·05401	0·07862	0·56401	0·04401	0·02001	0·02988	0·36401
14	0·12801	0·05601	0·07460	0·40001	0·04601	0·02201	0·03009	0·27801

The analytical behaviour of the function Φ and the numerical estimates of $\Phi(\theta)$ and $G(\theta)$, for a few chosen values of V_A^2, V_B^2, and τ, show that for $0 < U^2 < U^{*2}$ there exists a critical wave number θ_c such that $\Delta(\theta)$ is negative for $\theta > \theta_c$. Hence, from (5) we note that the system is stable for wavelengths $\lambda < \lambda_c$ $(= 2\pi h/\theta_c)$ and is overstable for wavelengths greater than λ_c.

When $U^2 > U^{*2}$, we find that there exists a range of wave numbers, say $\theta' < \theta < \theta''$ for which $\Delta(\theta)$ is positive and is negative for all other values. But as U^2 is increased, the range of wave numbers for which $\Delta(\theta)$ is positive decreases and for higher values of U it becomes negative for all θ. Hence, depending on the values of U^2, the system is either stable for $\theta' < \theta < \theta''$ and overstable for $\theta < \theta'$ and $\theta > \theta''$ or is overstable for all values of θ. Once again we find that the waves are very long or very short to be stabilized.

TABLE 3

MINIMUM WAVE NUMBER θ_c AT WHICH INSTABILITY SETS IN AS FUNCTION OF τ AND U^2 FOR ASYMMETRIC CASE IN ABSENCE OF MAGNETIC FIELD

U^2	θ_c				
	$\tau = 0\cdot1$	$0\cdot2$	$0\cdot4$	$0\cdot6$	$0\cdot8$
0·2	49·04595	23·59397	10·18999	5·12599	2·17199
0·5	19·33999	9·18200	3·87601	1·94800	0·94801
1	9·42800	4·36001	1·78600	0·98801	0·56401
2	4·45001	1·92600	0·84201	0·55401	0·35801
5	1·39801	0·58401	0·36801	0·29001	0·20601

Hence, we find that the presence of a magnetic field suppresses the instability associated with short wavelengths when $U^2 < U^{*2}$, but due to self-gravitational forces the instability at long wavelengths persists. When the relative speed exceeds the critical speed U^*, the short wavelength instability sets in and we again have two ranges of instability for very short and very long wavelengths. As U is increased,

TABLE 4

MINIMUM WAVE NUMBER θ_c AT WHICH INSTABILITY SETS IN AS A FUNCTION OF τ, U^2, V_A^2, AND V_B^2 FOR ASYMMETRIC CASE IN PRESENCE OF MAGNETIC FIELD

U^2	θ_c			
	$\tau = 0\cdot5$	$0\cdot7$	$0\cdot5$	$0\cdot7$
	$V_A^2 = 0\cdot5$, $V_B^2 = 2$		$V_A^2 = 2$, $V_B^2 = 1$	
0	—	—	—	—
4	—	—	—	—
6	0·93001	0·80601	—	—
8	0·49801	0·46201	2·46000	0·43601
10	0·36601	0·34401	0·37201	0·30001
12	0·29601	0·28001	0·28401	0·24201
14	0·25401	0·24001	0·23801	0·20601
16	0·22401	0·21201	0·20801	0·18201

the range of wave numbers for which the system is stable decreases till the system is overstable for all wavelengths. We note from equations (3) and (4) that the real frequency of oscillation is not affected by the magnetic field. Table 2 gives the critical values θ', θ'_m, $\omega_m(\theta = \theta'_m)$, and θ'' for various values of τ, U^2, V_A^2, and V_B^2. We note that the increase in relative motion decreases the dimensions of the fragments, into which the layer tends to break due to gravitational instability, and also the characteristic time associated with this breakup, whereas the increase in total magnetic pressure has the tendency to increase the value of λ_m and $t = (2\pi/\omega_m)$. We also note that for values of U^2 very close to U^{*2} the maximum wavelength $(= 2\pi h/\theta'')$ for which the system is unstable due to KH instability has a very small value.

TABLE 5

θ_c, θ_m, AND ω_m AS FUNCTIONS OF τ, U^2, V_A^2, AND V_B^2

(a) *Symmetric Case*

U^2	$\tau = 2$ θ_c	θ_m	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$	$\tau = 4$ θ_c	θ_m	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$
			$V_A^2 = 0\cdot5$, $V_B^2 = 2$			
0	0·44201	0·25801	0·33231	0·99601	0·30601	0·97123
2	0·61401	0·35801	0·37940	1·15601	0·40601	1·00795
4	1·07201	0·55801	0·47755	1·35601	0·50601	1·04908
6	2·53801	1·05601	0·71090	1·70601	0·60601	1·10692
8	—	—	—	2·41001	0·80601	1·19082
10	—	—	—	7·30601	2·20601	1·37074
12	—	—	—	—	—	—
			$V_A^2 = 2$, $V_B^2 = 1$			
0	0·38801	0·15401	0·27868	1·10601	0·40601	0·99548
2	0·55401	0·25401	0·31017	1·40601	0·40601	1·00925
4	1·06401	0·55401	0·39105	1·90601	0·60601	1·06945
6	—	—	—	3·59000	1·10601	1·18304
8	—	—	—	—	—	—
10	—	—	—	—	—	—
12	—	—	—	—	—	—

(b) *Asymmetric Case*

U^2	$\tau = 2$ θ_c	θ_m	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$	$\tau = 4$ θ_c	θ_m	$\omega_m(4\pi G\rho)^{-\frac{1}{2}}$
			$V_A^2 = 0\cdot5$, $V_B^2 = 2$			
0	0·40201	0·15801	0·29811	0·90601	0·40601	0·73512
2	0·47001	0·25801	0·30876	1·00601	0·40601	0·75287
4	0·61801	0·25801	0·32951	1·10601	0·50601	0·77781
6	2·21801	1·25601	0·36544	1·50601	0·60601	0·81170
8	—	—	—	2·30601	0·90601	0·87634
10	—	—	—	7·20601	3·30601	1·32504
12	—	—	—	—	—	—
			$V_A^2 = 2$, $V_B^2 = 1$			
0	0·54001	0·25401	0·36037	1·13601	0·50601	0·83563
2	0·69401	0·35401	0·38380	1·39000	0·60601	0·87318
4	1·07401	0·45401	0·43263	1·89000	0·80601	0·93047
6	—	—	—	3·50601	1·50601	1·07484
8	—	—	—	—	—	—
10	—	—	—	—	—	—
12	—	—	—	—	—	—

Asymmetric Perturbations

We do not wish to present the detailed mathematical analysis for asymmetric perturbations as the results follow from physical interpretation. As seen in the symmetric case, the KH instability and the gravitational instability both act independently. Hence, as the layer in the case $U = 0$ is gravitationally stable for asymmetric perturbations (Uberoi 1963), the only instability that appears when $U \neq 0$ is the short wavelength KH instability. The results, which are similar to those of Chandrasekhar (1961) are as follows.

In the absence of a magnetic field we have a critical wave number θ_c such that for $\theta < \theta_c$ the system is stable, while for $\theta > \theta_c$ the system is overstable. Table 3 gives the values of θ_c for various values of τ and U^2. We note that as U^2 increases the range of stable wave numbers decreases, i.e. the maximum wavelength for which the system is unstable increases with U. Increase in τ also increases the maximum wavelength of instability. In the presence of a magnetic field we conclude that when $0 \leqslant U^2 < U^{*2}$, the system is stable for all wave numbers but, when $U^2 > U^{*2}$, it is stable for all wave numbers $\theta < \theta_c$ and overstable for $\theta > \theta_c$. Hence, the magnetic field is able to suppress the KH instability when

$$(V_2 - V_1)^2 < (\rho_0 + \rho)(H_1^2 + H_2^2)/4\pi\rho_0\,\rho\,.$$

Table 4 gives the values of θ_c for various values of V_A, V_B, τ, and U^2.

Considering the case $\tau > 1$ in the presence of a magnetic field, we find both for symmetric and asymmetric perturbations that when $U^2 < U^{*2}$, the system is overstable for θ less than a certain θ_c but, when $U^2 > U^{*2}$, the system becomes overstable for all wavelengths. Thus, when the surrounding material is denser we do not have two ranges of instability for any value of U. The magnetic field suppresses the short wavelength instability for $U^2 < U^{*2}$ and makes the potentially unstable equilibrium arrangement stable for wavelengths $\lambda < \lambda_c$; but as the relative speed between the two fluid layers increases, the system acquiring the extra source of kinetic energy becomes overstable. We note that for $\tau > 1$ the system behaves alike both in the presence of symmetric and asymmetric perturbations. Table 5 gives the values of θ_c and θ_m and ω_m, the wave number and characteristic frequency at θ_m associated with the maximum mode of instability, for various values of V_A, V_B, $\tau > 1$, and U^2. We note that the wavelength $\lambda_m = 2\pi h/\theta_m$ and the characteristic time $t = 2\pi/\omega_m$ decrease as the relative motion is increased, and as the value of U^2 approaches the critical value U^{*2} the values of λ_m and t become strikingly small.

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