CHANDRASEKHAR'S X- AND Y-FUNCTIONS FOR ISOTROPIC SCATTERING IN THICK SLABS

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Summary

Approximate forms are developed for the X- and Y-functions of isotropic scattering of Chandrasekhar in terms of the well-tabulated H-function. The approximations that are asymptotically correct for slabs of large thickness are compared with available tabulated values.

I. Introduction

The functions $X(\mu)$ and $Y(\mu)$ were introduced by Chandrasekhar (1947, 1948) for study of the transfer of radiation through finite atmospheres. They were applied to problems of neutron transport by Auerbach (1961). Some numerical tables were given by Chandrasekhar (1952) and these tables were recalculated with improved accuracy and over an extended range by Mayers (1962), who discussed the numerical difficulties encountered when calculating the functions.

In the present paper approximations are derived (valid for thick media) in terms of the simpler and well-tabulated H-function.

II. Application of the Functions to Neutron Transport Theory

The X- and Y-functions were originally defined as the solutions of the pair of simultaneous nonlinear integral equations

$$X(\mu) = 1+\tfrac{1}{2}c\mu \int_0^1 \frac{X(\mu)\,X(\mu')-Y(\mu)\,Y(\mu')}{\mu+\mu'}\,\mathrm{d}\mu', \tag{1}$$

$$Y(\mu) = \exp(-\tau/\mu)+\tfrac{1}{2}c\mu \int_0^1 \frac{Y(\mu)\,X(\mu')-X(\mu)\,Y(\mu')}{\mu-\mu'}\,\mathrm{d}\mu'. \tag{2}$$

They make their appearance in neutron transport theory when solutions are sought for the angular flux reflected from or transmitted through an infinite slab that has thickness τ and unit total cross section, and that isotropically scatters neutrons with scattering cross section c.

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In the interior of the slab the one-velocity transport equation is then

$$\phi(x,\mu)+\mu\frac{\partial\phi}{\partial x} = \tfrac{1}{2}c\int_{-1}^{1}\phi(x,\mu')\,\mathrm{d}\mu', \qquad 0\leqslant x\leqslant\tau, \tag{3}$$

in standard notation. If a monodirectional beam of neutrons impinges on the slab at $x = 0$ and if the plane $x = \tau$ is a free boundary we have the boundary conditions

$$\phi(0,\mu) = \delta(\mu-\mu_0), \quad \phi(\tau,-\mu) = 0, \qquad 0<\mu,\mu_0\leqslant 1, \tag{4}$$

$\delta(\mu-\mu_0)$ being a Dirac delta function.

The emerging angular flux distributions can then be written

$$\begin{aligned}\phi(0,-\mu) &= \tfrac{1}{2}\mu^{-1}\int_0^1 S(\mu,\mu')\,\phi(0,\mu')\,\mathrm{d}\mu' \\ &= \tfrac{1}{2}\mu^{-1}S(\mu,\mu_0),\end{aligned} \tag{5}$$

$$\begin{aligned}\phi(\tau,\mu) &= \exp(-\tau/\mu)\,\delta(\mu-\mu_0)+\tfrac{1}{2}\mu^{-1}\int_0^1 T(\mu,\mu')\,\phi(0,\mu')\,\mathrm{d}\mu' \\ &= \exp(-\tau/\mu)\,\delta(\mu-\mu_0)+\tfrac{1}{2}\mu^{-1}T(\mu,\mu_0),\end{aligned} \tag{6}$$

so that determination of these fluxes involves evaluation of the reflection and transmission functions $S(\mu,\mu')$ and $T(\mu,\mu')$. These also depend implicitly on the parameters c and τ.

The X- and Y-functions now appear, following Auerbach (1961), as

$$X(\mu) = 1+\int_0^1 (2\mu')^{-1}S(\mu,\mu')\,\mathrm{d}\mu' \tag{7}$$

and

$$Y(\mu) = \exp(-\tau/\mu)+\int_0^1 (2\mu')^{-1}T(\mu,\mu')\,\mathrm{d}\mu', \tag{8}$$

whilst, in turn,

$$S(\mu,\mu') = \frac{c\mu\mu'}{\mu+\mu'}\Big(X(\mu)\,X(\mu')-Y(\mu)\,Y(\mu')\Big) \tag{9}$$

and

$$T(\mu,\mu') = \frac{c\mu\mu'}{\mu-\mu'}\Big(Y(\mu)\,X(\mu')-X(\mu)\,Y(\mu')\Big). \tag{10}$$

Elimination of S and T from equations (7)–(10) leads to the defining equations (1) and (2).

III. Variational Formula

Integrals of the angular fluxes emerging from the slab have the general form

$$I = \int_0^1 S^*(\mu)\,\phi(0,-\mu)\,\mathrm{d}\mu + \int_0^1 T^*(\mu)\,\phi(\tau,\mu)\,\mathrm{d}\mu. \tag{11}$$

If in particular $T^*(\mu)$ and $S^*(\mu)$ are chosen as zero and $\mu\delta(\mu-\mu_1)$ then

$$I \equiv \mu_1 \phi(0, -\mu_1),$$

while interchanging these forms for S^* and T^* gives

$$I = \mu_1 \phi(\tau, \mu_1).$$

To calculate I we use instead the variational expression

$$\begin{aligned}\mathscr{L} = &\int_0^1 S^*(\mu)\,\psi(0,-\mu)\,\mathrm{d}\mu + \int_0^1 T^*(\mu)\,\psi(\tau,\mu)\,\mathrm{d}\mu \\ &- \int_0^1 \phi^*(0,\mu)\{\psi(0,\mu) - \delta(\mu-\mu_0)\}\,\mathrm{d}\mu - \int_0^1 \phi^*(\tau,-\mu)\,\psi(\tau,-\mu)\,\mathrm{d}\mu + \mathscr{L}_0,\end{aligned} \tag{12}$$

with

$$\mathscr{L}_0 = \int_0^\tau \mathrm{d}x \int_{-1}^1 \mathrm{d}\mu\,\phi^*(x,\mu)\left(\tfrac{1}{2}c\int_{-1}^1 \psi(x,\mu')\,\mathrm{d}\mu' - \psi(x,\mu) - \frac{\partial\psi}{\partial x}(x,\mu)\right).$$

$\mathscr{L}$ is clearly identical with I if the trial function $\psi(x,\mu)$ satisfies equations (3) and (4) for $\phi(x,\mu)$, and it is easy to show that $\mathscr{L}$ has zero variation for first-order errors in $\psi(x,\mu)$ about the correct value when the function $\phi^*(x,\mu)$ satisfies the adjoint transport equation

$$\phi^*(x,\mu) - \mu\frac{\partial\phi^*}{\partial x} = \tfrac{1}{2}c\int_{-1}^1 \phi^*(x,\mu')\,\mathrm{d}\mu', \tag{13}$$

with boundary conditions

$$\phi^*(0,-\mu) = \mu^{-1}S^*(\mu) \qquad \text{and} \qquad \phi^*(\tau,\mu) = \mu^{-1}T^*(\mu). \tag{14}$$

In addition $\mathscr{L}$ has zero variation for first-order errors in the adjoint function $\phi^*(x,\mu)$.

The standard procedure for use of the variational expression (12) is to choose for $\psi(x,\mu)$ and $\phi^*(x,\mu)$ parameter-dependent trial functions and then to select the parameters in such a way that the expression (12) is stationary with respect to variation of the parameters. This technique can be applied if numerical values are sought either for the emergent distributions (5) and (6) or for the reflection and transmission functions S and T. In the present work, however, we are interested in obtaining semi-analytic forms for the X- and Y-functions and use of the standard method leads to forms for S and T that are not amenable to the integrations shown in equations (7) and (8).

We therefore follow an alternative course by simply selecting for the trial functions expressions that are reasonable approximations to the true neutron flux and adjoint function, substituting these into equation (12), and accepting the results as approximations to the integral (11).

We are seeking approximations that are valid for slabs of large thickness τ and for such systems numerical studies show that the angular flux $\phi(x,\mu)$ for the finite slab is well approximated (except near $x=\tau$) by the angular flux appropriate to the semi-infinite system $0 \leqslant x < \infty$. Physically the flux near the source plane $x=0$ is not strongly affected by the presence or absence of material beyond $x=\tau \gg 0$. Since the equivalent result is true for solutions of the adjoint equation an appropriate set of trial functions will be the flux and adjoint functions for correctly chosen semi-infinite medium problems and these we now describe.

IV. Solutions for Semi-infinite Media

If $\psi(x,\mu)$ satisfies the transport equation (3) in $0 \leqslant x \leqslant \infty$ with boundary condition

$$\psi(0,\mu) = \delta(\mu-\mu_0), \qquad 0<\mu<1, \tag{15}$$

then

$$\psi(0,-\mu) = \tfrac{1}{2}\mu^{-1}S_\infty(\mu,\mu_0), \tag{16}$$

where the semi-infinite medium reflection function S_∞ is given in terms of the H-function by

$$S_\infty(\mu,\mu_0) = \{c\mu\mu_0/(\mu+\mu_0)\}H(\mu)\,H(\mu_0).$$

For large x the angular flux $\psi(x,\mu)$ settles down into an asymptotic distribution which (provided $c<1$) decays exponentially as $\exp(-Kx)$ where the inverse diffusion length K is the solution of

$$K = \tfrac{1}{2}c\log\{(1+K)/(1-K)\}, \qquad 0<K<1.$$

In terms of K the angular flux for large x is given by Auerbach (1961) as

$$\psi(x,\mu) = \{c\alpha(\mu_0)/2(1-K\mu)\}\exp(-Kx), \tag{17}$$

where

$$\alpha(\mu) = \frac{\mu H(\mu)}{1-K\mu}\,\frac{1-K^2}{(K^2+c-1)H(K^{-1})}.$$

Near the boundary $x=0$ other terms have to be added to (17) but these transients die away rapidly as x increases.

Since this trial function $\psi(x,\mu)$ is chosen to satisfy the transport equation (3) in $0 \leqslant x < \infty$ and thus throughout the slab $0 \leqslant x \leqslant \tau$ (albeit with an incorrect boundary value at $x=\tau$), the term $\mathscr{L}_0$ in the variational expression (12) vanishes identically. In calculating $\mathscr{L}$ we need only evaluate the trial function and the adjoint trial function $\phi^*(x,\mu)$ at the boundaries $x=0$ and $x=\tau$.

It is convenient at this stage to investigate the function $\phi^*(x,\mu)$ which satisfies the adjoint transport equation (13) in the semi-infinite region $0 \leqslant x < \infty$ with a boundary condition

$$\phi_1^*(0,-\mu) = \delta(\mu-\mu_1), \qquad 0<\mu<1. \tag{18}$$

On comparison of the regular equation (3) with the adjoint equation (13) and of the boundary condition (15) with the adjoint boundary condition (18) it is clear that $\phi_1^*(x,\mu)$ depends on μ in the same way that $\psi(x,\mu)$ depends on $-\mu$. In particular we have at $x = 0$

$$\phi_1^*(0,\mu) = \tfrac{1}{2}\mu^{-1}S_\infty(\mu,\mu_1), \qquad 0 < \mu < 1, \tag{19}$$

while

$$\phi_1^*(x,\mu) \simeq \tfrac{1}{2}c\,\alpha(\mu_1)\exp(-Kx)/(1+K\mu) \qquad \text{for large } x. \tag{20}$$

V. Approximation for $X(\mu)$

If in equation (11) we set

$$T^*(\mu) = 0 \qquad \text{and} \qquad S^*(\mu) = \mu\delta(\mu-\mu_1)$$

then

$$I = \mu_1\phi(0,-\mu_1).$$

To estimate I we evaluate $\mathscr{L}$ from equation (12) using the trial function $\psi(x,\mu)$ studied in Section IV. If we substitute the present forms for $S^*(\mu)$ and $T^*(\mu)$ into (12), use equations (15) and (16) for $\psi(0,\pm\mu)$, and note that $\mathscr{L}_0$ is zero for our trial function $\psi(x,\mu)$ then equation (12) becomes

$$\mathscr{L} = \tfrac{1}{2}c\frac{\mu_0\mu_1}{\mu_0+\mu_1}H(\mu_0)H(\mu_1) - \int_0^1 \mu\phi^*(\tau,-\mu)\psi(\tau,-\mu)\,d\mu. \tag{21}$$

The expression $\mathscr{L}$ will then approximate I if $\phi^*(x,\mu)$ is chosen as an approximate solution of the adjoint problem defined by equation (13) with boundary conditions (14), which here become

$$\phi^*(0,-\mu) = \delta(\mu-\mu_1), \qquad \phi^*(\tau,\mu) = 0.$$

We saw in Section IV that the function $\phi_1^*(x,\mu)$ with properties given in equations (18)–(20) will serve as a suitable approximation.

If the further assumption is now made that the asymptotic formulae (17) and (20) provide at $x = \tau$ sufficiently accurate expressions for the trial functions ψ and ϕ^*, then equation (21) becomes

$$\mu_1\phi(0,-\mu_1) \simeq \tfrac{1}{2}c\{\mu_0\mu_1/(\mu_0+\mu_1)\}H(\mu_0)H(\mu_1)$$
$$+(c^2/8K^2)\alpha(\mu_0)\alpha(\mu_1)\exp(-2K\tau)\log(1-K^2). \tag{22}$$

Equations (5) and (7) imply

$$X(\mu) = 1+\int_0^1 (\mu/\mu_0)\phi(0,-\mu)\,d\mu_0$$

and use of equation (22) gives

$$X(\mu) \simeq 1+\tfrac{1}{2}c\mu H(\mu)\int_0^1 \frac{H(\mu_0)}{\mu+\mu_0}\,d\mu_0$$
$$+(c^2/8K^2)\alpha(\mu)\exp(-2K\tau)\log(1-K^2)\int_0^1 \frac{\alpha(\mu_0)}{\mu_0}\,d\mu_0 .$$

From the integral equation for the H-function

$$H(\mu) = 1+\tfrac{1}{2}c\mu H(\mu)\int_0^1 \frac{H(\mu_0)}{\mu+\mu_0}\,d\mu_0, \tag{23}$$

and the normalization integral

$$\int_0^1 \frac{H(\mu)}{1-K\mu}\,d\mu = \frac{2}{c}, \tag{24}$$

it follows that

$$X(\mu) \simeq H(\mu)+\tfrac{1}{4}c\log(1-K^2)\frac{H(\mu)}{1-K\mu}\left(\frac{\exp(-K\tau)}{H(K^{-1})\{c/(1-K^2)-1\}}\right)^2 . \tag{25}$$

As τ tends to infinity this expression tends to $H(\mu)$, as it should.

VI. Approximation for $Y(\mu)$

If in equation (11) we set

$$S^*(\mu) = 0 \qquad \text{and} \qquad T^*(\mu) = \mu\delta(\mu-\mu_1)$$

then

$$I = \mu_1\phi(\tau,\mu_1),$$

which also will be estimated by the variational expression (12).

As a trial function ψ the solution for a semi-infinite medium is again used, when it follows that

$$\mu_1\phi(\tau,\mu_1) \simeq \mu_1\psi(\tau,\mu_1)-\int_0^1 \mu\phi^*(\tau,-\mu)\psi(\tau,-\mu)\,d\mu, \tag{26}$$

where $\phi^*(\tau,\mu)$ has now to satisfy the adjoint transport equation with boundary conditions

$$\left.\begin{aligned} &\phi^*(\tau,\mu) = \delta(\mu-\mu_1), && 0<\mu<1, \\ \text{and}\quad &\phi^*(0,-\mu) = 0, && 0<\mu<1. \end{aligned}\right\} \tag{27}$$

This particular adjoint problem is similar to the adjoint problem posed in the preceding section, but reflected completely about the plane $x = \tfrac{1}{2}\tau$. For the present problem we therefore use the mirror image of the approximate solution $\phi_1^*(x,\mu)$, i.e. we take

$$\phi^*(x,\mu) = \phi_1^*(\tau-x,-\mu).$$

In particular on the boundary $x = \tau$ this gives

$$\phi^*(\tau, -\mu) = \tfrac{1}{2}\mu^{-1} S_\infty(\mu, \mu_1).$$

If at the same time we approximate $\psi(\tau, \mu)$ by the asymptotic solution (17) then equation (16) reduces to

$$\mu_1 \phi(\tau, \mu_1) \simeq \tfrac{1}{2} c\mu_1 \alpha(\mu_0) \exp(-K\tau)/(1-K\mu_1)$$
$$-\tfrac{1}{4}c^2 \mu_1 \alpha(\mu_0) \exp(-K\tau)\, H(\mu_1) \int_0^1 \frac{\mu H(\mu)}{(\mu+\mu_1)(1+K\mu)} \mathrm{d}\mu .$$

The integral term is dealt with by a partial fraction expansion and use of the relations (23) and (24). Some elementary manipulations lead finally to

$$\mu_1 \phi(\tau, \mu_1) \simeq \tfrac{1}{2} c \frac{\mu_1 H(\mu_1)}{1-K\mu_1} \frac{\mu_0 H(\mu_0)}{1-K\mu_0} \frac{K \exp(-K\tau)}{\{H(K^{-1})\}^2 \{c/(1-K^2)-1\}} .$$

From equations (8) and (10) it then follows that

$$Y(\mu) \simeq \frac{K\mu H(\mu)}{1-K\mu} \frac{\exp(-K\tau)}{\{H(K^{-1})\}^2 \{c/(1-K^2)-1\}}, \tag{28}$$

which tends to zero as τ tends to infinity.

It is interesting to note that equation (28) can be obtained by an independent argument if the thickness τ is large enough to permit the asymptotic distribution (17) being set up at the centre $x = \frac{1}{2}\tau$. The procedure is to solve the Milne problem for the region $-\infty < x \leqslant \tau$ and to match the corresponding asymptotic distribution at $x = \frac{1}{2}\tau$ to the distribution (17). It is found that the emergent distribution at $x = \tau$ is identical with that found by the variational treatment.

VII. Conservative Case

If there is no absorption and c becomes unity, the solutions (25) and (28) become indeterminate, since K becomes zero. Two procedures can then be followed. We can use as trial functions the semi-infinite medium solutions appropriate to the conservative case or alternatively the solutions (25) and (28) can be evaluated by a limiting procedure as K tends to zero. Either procedure leads to the expressions

$$X(\mu) = (1-\tfrac{3}{4}\mu) H(\mu) \tag{29}$$

and

$$Y(\mu) = 0, \tag{30}$$

which are certainly unsatisfactory as they are independent of thickness. This poor result is not, however, surprising since the alternative procedure described at the

TABLE 1

COMPARISON OF $X(\mu)$ AND $Y(\mu)$ VALUES FROM EQUATIONS (25) AND (28) WITH ACCURATE VALUES FROM MAYERS (1962)

μ	$X(\mu)$ Exact	$X(\mu)$ Eqn (25)	Error (%)	$Y(\mu)$ Exact	$Y(\mu)$ Eqn (28)	Error (%)
			$c = 0\cdot80$, $\tau = 2\cdot5$			
0·0	1·0	1·0	—	0·0	0·0	—
0·1	1·1384	1·1384	0	0·0088	0·0083	−6·0
0·2	1·2276	1·2276	0	0·0208	0·0195	−6·5
0·3	1·2988	1·2988	0	0·0365	0·0337	−8·3
0·4	1·3583	1·3583	0	0·0578	0·0517	−10·6
0·5	1·4092	1·4092	0	0·0861	0·0745	−13·4
0·6	1·4532	1·4533	0	0·1213	0·1038	−14·4
0·7	1·4917	1·4918	0	0·1622	0·1420	−12·5
0·8	1·5256	1·5253	0	0·2071	0·1936	−6·5
0·9	1·5556	1·5540	−0·1	0·2544	0·2663	+4·7
1·0	1·5824	1·5778	−0·3	0·3030	0·3754	+23·9
			$c = 0\cdot90$, $\tau = 2\cdot0$			
0·0	1·0	1·0	—	0·0	0·0	—
0·1	1·1689	1·1682	−0·1	0·0211	0·0198	−6·2
0·2	1·2839	1·2822	−0·1	0·0496	0·0463	−6·7
0·3	1·3783	1·3756	−0·2	0·0868	0·0794	−8·5
0·4	1·4585	1·4546	−0·3	0·1352	0·1200	−11·2
0·5	1·5274	1·5224	−0·3	0·1941	0·1691	−12·9
0·6	1·5872	1·5805	−0·4	0·2606	0·2283	−12·4
0·7	1·6392	1·6297	−0·6	0·3312	0·2998	−9·5
0·8	1·6848	1·6705	−0·8	0·4031	0·3865	−4·1
0·9	1·7249	1·7028	−1·3	0·4743	0·4927	+3·9
1·0	1·7605	1·7459	−0·8	0·5437	0·6248	+14·9
			$c = 0\cdot90$, $\tau = 5\cdot0$			
0·0	1·0	1·0	0·0	0·0	0·0	0·0
0·1	1·1720	1·1720	0·0	0·0041	0·0041	−0·0
0·2	1·2911	1·2912	+0·0	0·0096	0·0096	−0·0
0·3	1·3908	1·3907	−0·0	0·0165	0·0164	−0·6
0·4	1·4776	1·4775	−0·0	0·0250	0·0248	−0·8
0·5	1·5548	1·5546	−0·0	0·0352	0·0350	−0·6
0·6	1·6242	1·6239	−0·0	0·0477	0·0472	−1·0
0·7	1·6871	1·6868	−0·0	0·0628	0·0620	−1·3
0·8	1·7446	1·7441	−0·0	0·0807	0·0799	−1·0
0·9	1·7973	1·7966	−0·0	0·1016	0·1019	+0·3
1·0	1·8458	1·8448	−0·1	0·1254	0·1292	+3·0
			$c = 0\cdot95$, $\tau = 2\cdot0$			
0·0	1·0	1·0	—	0·0	0·0	—
0·1	1·1879	1·849	−0·3	0·0276	0·0250	−9·4
0·2	1·3203	1·3132	−0·5	0·0647	0·0583	−9·7
0·3	1·4312	1·4191	−0·8	0·1121	0·0996	−11·2
0·4	1·5268	1·5091	−1·2	0·1718	0·1492	−13·2
0·5	1·6098	1·5856	−1·5	0·2423	0·2078	−14·2
0·6	1·6822	1·6502	−1·9	0·3201	0·2761	−13·7
0·7	1·7457	1·7035	−2·4	0·4016	0·3554	−11·5
0·8	1·8015	1·7459	−3·1	0·4838	0·4469	−7·6
0·9	1·8508	1·7774	−4·0	0·5646	0·5524	−2·2
1·0	1·8946	1·7974	−5·1	0·6428	0·6742	+4·9

TABLE 1 (*Continued*)

μ	$X(\mu)$ Exact	$X(\mu)$ Eqn (25)	Error (%)	$Y(\mu)$ Exact	$Y(\mu)$ Eqn (28)	Error (%)
			$c = 0\cdot95$, $\tau = 4\cdot0$			
0·0	1·0	1·0	—	0·0	0·0	—
0·1	1·1937	1·1930	−0·1	0·0118	0·0117	−0·8
0·2	1·3338	1·3320	−0·1	0·0276	0·0273	−1·1
0·3	1·4545	1·4514	−0·2	0·0472	0·0466	−1·3
0·4	1·5621	1·5574	−0·3	0·0708	0·0699	−1·3
0·5	1·6594	1·6529	−0·4	0·0989	0·0973	−1·6
0·6	1·7481	1·7396	−0·5	0·1318	0·1293	−1·9
0·7	1·8296	1·8186	−0·6	0·1697	0·1664	−1·9
0·8	1·9046	1·8907	−0·7	0·2123	0·2092	−1·5
0·9	1·9737	1·9563	−0·9	0·2589	0·2586	−0·1
1·0	2·0376	2·0158	−1·1	0·3090	0·3156	+2·1
			$c = 0\cdot95$, $\tau = 10\cdot0$			
0·0	1·0	1·0	—	0·0	0·0	—
0·1	1·1952	1·1952	—	0·0013	0·0012	—
0·2	1·3373	1·3373	—	0·0028	0·0028	—
0·3	1·4603	1·4604	—	0·0047	0·0048	+2·0*
0·4	1·5708	1·5708	—	0·0071	0·0072	+1·4*
0·5	1·6716	1·6716	—	0·0099	0·0100	+1·1*
0·6	1·7644	1·7644	—	0·0132	0·0133	+0·8*
0·7	1·8506	1·8506	—	0·0169	0·0171	+1·2*
0·8	1·9309	1·9309	—	0·0213	0·0215	+0·9*
0·9	2·0061	2·0060	—	0·0263	0·0265	+0·8*
1·0	2·0766	2·0765	—	0·0321	0·0323	+0·6*

* Although percentage errors are quoted in this example, the difference between the approximation (28) and the "exact" values of Mayers (1962) are all within the quoted errors in Mayers's compilation.

conclusion of the preceding section cannot be followed for the conservative case. The asymptotic distribution set up far from the source plane will be

$$\psi(x,\mu) = \tfrac{1}{2}\sqrt{3}\,\mu_0 H(\mu_0)\,, \qquad x \gg 0\,,$$

which is independent of position. On the other hand, the asymptotic distribution for the Milne problem ($-\infty < x < \tau$) has the form

$$\psi(x,\mu) = A(0\cdot7104\ldots+\mu+\tau-x)\,, \qquad x \ll \tau\,,$$

and the two distributions cannot be matched.

VIII. Comparison with Tabulated Values

To test the accuracy of the approximations embodied in equations (25) and (28) their values have been compared with those tabulated by Mayers (1962) for:

$c = 0\cdot80$	$0\cdot90$	$0\cdot95$
$\tau = 2\cdot5$	$2\cdot0,\ 5\cdot0$	$2\cdot0,\ 4\cdot0,\ 10\cdot0$

The results are displayed in Table 1, which also gives the percentage error in the approximations. Provided $\tau \geqslant 5 \cdot 0$ the agreement is satisfactory and the approximations may be used with some confidence. For $\tau < 5 \cdot 0$ the approximations would be of value as initial guesses in an iterative scheme for calculating $X(\mu)$ and $Y(\mu)$.

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