ENERGY LOSS BY FAST TEST IONS IN A PLASMA*

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Summary

We consider the energy loss by a fast test ion in a plasma and tabulate numerical results for the corrections to the usual dominant "$\ln \Lambda$" term in the loss rate (for all values of the ratio between test ion and plasma electron thermal speeds). As a preliminary consideration, we present a derivation of the Fokker–Planck equation which treats both close collisions and cooperative plasma effects fully; the resulting equation is not new but the derivation is novel and has simplicities which facilitate calculation.

This work is motivated by some recent experiments (which are discussed), but the results are tabulated in such a form as to be generally useful.

I. Introduction

By now there exists a considerable body of work pertaining to the calculation of relaxation rates and transport coefficients in a plasma; most of this work is confined to calculation of the so-called "dominant" terms, i.e. the terms that are proportional to the ubiquitous "$\ln \Lambda$" of plasma physics.

In particular, the problem of the *rate of energy loss by a fast test ion in a plasma* has received considerable theoretical attention (see e.g. reviews by Montgomery and Tidman 1964; Shkarofsky, Johnston, and Bachynski 1966). It is only quite recently that experiments have been made which are accurate enough to measure corrections to the dominant term (Halverson 1968; Ormrod 1968), and it behoves us to provide fully accurate theoretical expressions for comparison with these experiments. The new work presented here is motivated towards this end, but the numerical results are tabulated in such a way as to be generally useful.

In simple treatments of the problem, the Coulomb collision integrals diverge logarithmically both at small distances (due to an inadequate treatment of the dynamics of close encounters) and at large distances (due to neglect of cooperative screening and plasma wave effects). These divergences are overcome by introducing phenomenological cutoffs at the plasma Debye length l_{D} for large distances and at the classical distance of closest approach $b_0 \equiv Ze^2/\kappa T$ for small distances. One is thus led in such simple theories to the omnipresent factor $\ln \Lambda \sim \ln(l_{\mathrm{D}}/b_0)$.

In more rigorous treatments the dynamics of close collisions are considered exactly, and also the cooperative effects are treated fully (via the plasma kinetic equations or otherwise), so that instead of empirical cutoffs at l_{D} and b_0 one gets a

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formalism with convergent integrals, the precise values of which depend upon the ratio between test ion speed V and the plasma electron thermal speed $(2\kappa T/m)^{\frac{1}{2}}$, that is, upon the parameter x

$$x \equiv (mV^2/2\kappa T)^{\frac{1}{2}}. \tag{1}$$

Although these accurate expressions exist in a formal sense, they have been reduced to usable form essentially only in the limit $x \gg 1$. Halverson's (1968) experiment has $x = 1\cdot 3$.

The problem is treated here in two parts:

(1) Firstly, in Section II we present a derivation of the Fokker–Planck equation for a test particle distribution function in a plasma; this derivation is accurate and free of singularities. The formal expression so obtained is not new but the outlined derivation is somewhat simpler and more physically transparent than the seminal derivations.

(2) Secondly, in Section III we reduce the formal expressions to usable form for the case of a fast test ion which is *losing energy predominantly to plasma electrons.*

The final results, representing the corrected version of the $\ln \Lambda$ of simple theories, are of necessity presented in numerical form. These results are summarized, discussed, and compared with experiment in Section IV. The discussion includes a careful account of the approximations involved; it is seen that the present results are essentially exact unless x is quite small.

The Appendix contains analytic expressions for the energy loss rate in the limiting cases when $x \gg 1$ and when $x < 1$.

It is to be appreciated that similarly accurate expressions, going beyond the dominant $\ln \Lambda$ term of simple theories, pertain to other transport and relaxation problems in a plasma. The calculation here is a prototype for such calculations, and the appropriate detailed numerical coefficients in other problems are often simple relatives of those tabulated here (Shkarofsky, Johnston, and Bachynski 1966).

II. Fokker–Planck Equation for a Test Particle in a Plasma

For the calculation of relaxation rates and transport coefficients in a plasma one can in general begin from a Fokker–Planck equation. In particular, for the case of a test particle distribution function f_t, the Fokker–Planck equation takes the form

$$\frac{\partial f_t(\boldsymbol{v}_t)}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}_t}\cdot\left(\sum_j \boldsymbol{F}_j(\boldsymbol{v}_t)\, f_t(\boldsymbol{v}_t) + \frac{1}{2}\frac{\partial^2}{\partial \boldsymbol{v}_t\,\partial \boldsymbol{v}_t} : \sum_j \boldsymbol{T}_j(\boldsymbol{v}_t)\, f_t(\boldsymbol{v}_t)\right). \tag{2}$$

The subscript t refers to the test particles, and the subscript j to the various species present in the background plasma. The vector $\boldsymbol{F}$ and the tensor $\boldsymbol{T}$ are the generalized "friction" and "diffusion" coefficients, and depend in a complicated way on the plasma properties. We assume that the test particles are few compared to the plasma particles ($n_t \ll n_j$), so that their presence does not distort the equilibrium Maxwellian distribution functions f_j of the background plasma (for a more complete discussion, see Montgomery and Tidman 1964, Ch. 2; or Shkarofsky, Johnston, and Bachynski 1966, Ch. 7).

In what follows we shall always assume that $l_D \gg b_0$, and the neglect of terms of relative order $(b_0/l_D)^2$ shall be deemed "exact".

Now the simplest version of the Fokker–Planck equation (2) can be derived directly from the Boltzmann equation (e.g. Montgomery and Tidman 1964), or alternatively from first principles (Spitzer and Härm 1953; Rosenbluth, MacDonald, and Judd 1957; Kaufman 1960) by using the ordinary Coulomb cross section for the interaction between test and plasma particles and by treating all the Coulomb collisions as distant ones (i.e. small angles of scattering). The consequent friction and dispersion coefficients can then be written

$$\boldsymbol{F}_j(\boldsymbol{v}_t) = \tfrac{1}{2}(1+m_t/m_j)\frac{\partial}{\partial \boldsymbol{v}_t}\cdot \boldsymbol{T}_j(\boldsymbol{v}_t), \tag{3a}$$

$$\boldsymbol{T}_j(\boldsymbol{v}_t) = \frac{4n_j e_j^2 e_t^2}{m_t^2}\int \frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4}\int \mathrm{d}\boldsymbol{v}\, f_j(\boldsymbol{v})\, \delta(\boldsymbol{k}.\boldsymbol{g}), \tag{3b}$$

where $\boldsymbol{g}$ is the relative velocity $\boldsymbol{v}_t - \boldsymbol{v}$. The form of equations (3) written here is *not* the usual one for this simplest approximation; we have used the above form so that the relationship between simple and complete theories (eventually equations (8) below) will be transparent. Equation (3b) is obtained from the form which is more familiar in this simple case by introducing the wave number $\boldsymbol{k}$ such that $\boldsymbol{k}.\boldsymbol{g} = kg\sin(\frac{1}{2}\theta)$, where θ is the scattering angle; the three-dimensional integral over $\boldsymbol{k}$ along with the δ-function then reduces back to the usual angle integral over the Coulomb differential cross section. (This transformation is discussed in Chapman and Cowling (1960, p. 61).) The integral over $|\boldsymbol{k}|$ diverges logarithmically both for large k (close collisions), due to the assumption that all collisions have a small scattering angle, and for small k (distant collisions) because the cooperative screening has been omitted entirely. As mentioned in the Introduction, these logarithmic divergences are overcome by the introduction of empirical cutoffs at around the classical distance of closest approach b_0 and at the Debye length l_D, leading to the ubiquitous $\ln \Lambda \sim \ln(l_D/b_0)$ of simple theories.

If the derivation of the Fokker–Planck coefficients proceeds by use of the ordinary unscreened Coulomb cross section for interactions between test and plasma particles, but treats the dynamics of binary Coulomb collisions exactly (i.e. scattering angles are not necessarily assumed all to be small), then one arrives at

$$\boldsymbol{T}_j(\boldsymbol{v}_t) = \frac{4n_j e_j^2 e_t^2}{m_t^2}\int \frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4}\int \mathrm{d}\boldsymbol{v}\, f_j(\boldsymbol{v})\, \delta\left(\boldsymbol{k}.\boldsymbol{g} - \frac{e_j e_t k^2}{\mu_{jt}\, g}\right), \tag{4}$$

with the relation (3a) still connecting $\boldsymbol{F}$ and $\boldsymbol{T}$. There is now no divergence for large k, because close collisions have been handled properly; a phenomenological cutoff is still needed for the singularity at small k (large distances). The coefficients (4) reduce to those obtained by Spitzer (1940) and Chandrasekhar (1941) (for gravitationally interacting particles) and by Frankel (1965), who has discussed some of their consequences for problems in plasma physics.

Conversely, if one treats the cooperative screening and plasma wave effects accurately, one gets expressions for $\boldsymbol{F}$ and $\boldsymbol{T}$ in which the integrals have no spurious singularities at small wave numbers. Treatments of the plasma correlations, leading to such a Fokker–Planck equation, have been given in various ways by Thompson and Hubbard (1960), Hubbard (1961*a*, 1961*b*), and Tidman, Guernsey, and Montgomery (1964), following the Balescu (1960) and Lenard (1960) approach to the equations of plasma theory, and by Kihara and Aono (1963) and Aono (1968*a*). All these treatments come to the result

$$\boldsymbol{F}_j(\boldsymbol{v}_{\mathrm{t}}) = \frac{2n_j e_j^2 e_{\mathrm{t}}^2}{m_{\mathrm{t}}}\left(\frac{1}{m_j}\int\frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4\,|D^+|^2}\cdot\frac{\partial}{\partial\boldsymbol{v}_{\mathrm{t}}}+\frac{1}{m_{\mathrm{t}}}\frac{\partial}{\partial\boldsymbol{v}_{\mathrm{t}}}\cdot\int\frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4\,|D^+|^2}\right)\int\mathrm{d}\boldsymbol{v}\,f_j(\boldsymbol{v})\,\delta(\boldsymbol{k}.\boldsymbol{g}), \quad (5a)$$

$$\boldsymbol{T}_j(\boldsymbol{v}_{\mathrm{t}}) = \frac{4n_j e_j^2 e_{\mathrm{t}}^2}{m_{\mathrm{t}}^2}\int\frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4\,|D^+|^2}\int\mathrm{d}\boldsymbol{v}\,f_j(\boldsymbol{v})\,\delta(\boldsymbol{k}.\boldsymbol{g}), \quad (5b)$$

or to some equivalent form, related by identities on D^+. The function D^+ is the plasma dispersion function, which in this instance is

$$D^+(k,\boldsymbol{k}.\boldsymbol{v}_{\mathrm{t}}) = 1-\sum_j \omega_j^2\int_{\mathrm{C}}\frac{\mathrm{d}u}{(ku+\boldsymbol{k}.\boldsymbol{v}_{\mathrm{t}})^2}\int\mathrm{d}\boldsymbol{v}\,f_j(\boldsymbol{v})\,\delta(u-\boldsymbol{k}.\boldsymbol{v}/k), \quad (6)$$

where ω_j is the plasma frequency for the jth species, $\omega_j^2 = 4\pi n_j e_j^2/m_j$. This D^+ factor in equations (5) plays the role of a plasma dielectric constant and provides the necessary dynamical shielding which need no longer be introduced via a Debye cutoff. However, the expressions (5) *do* need an empirical cutoff for large k (close collisions) because the Coulomb collisions have (as in equations (3)) been assumed to be distant ones.

Alternatively the expressions (5) can be derived directly from the Boltzmann collision integral (as in the derivation of equations (3) above) by taking the interaction between the test particle and the plasma particles to be the "dressed" interaction, as calculated by Rostoker (1960). In this event the Fourier transform of the Coulomb potential (k^{-2} in equations (3)) is replaced by

$$1/k^2 \to 1/k^2\,|D^+|. \quad (7)$$

Using this "dressed" potential, but otherwise treating the test particle trajectories as straight lines (small angle scattering), the expressions (5) are immediately obtained directly from the Boltzmann equation, as (3) were previously obtained.

It is now clear what the final and accurate step is to be. We use the "dressed" potential (7) and treat the dynamics of the test particle trajectories properly. Thus just as one went from (3) to (4) for the "undressed" simple Coulomb potential, now one proceeds from equations (5) to arrive at

$$\boldsymbol{F}_j(\boldsymbol{v}_{\mathrm{t}}) = \frac{2n_j e_j^2 e_{\mathrm{t}}^2}{m_{\mathrm{t}}}\left(\frac{1}{m_j}\int\frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4\,|D^+|^2}\cdot\frac{\partial}{\partial\boldsymbol{v}_{\mathrm{t}}}+\frac{1}{m_{\mathrm{t}}}\frac{\partial}{\partial\boldsymbol{v}_{\mathrm{t}}}\cdot\int\frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4\,|D^+|^2}\right)\int\mathrm{d}\boldsymbol{v}\,f_j(\boldsymbol{v})\,\delta\left(\boldsymbol{k}.\boldsymbol{g}-\frac{e_j e_{\mathrm{t}} k^2}{\mu_{j\mathrm{t}}\,g}\right), \quad (8a)$$

$$\boldsymbol{T}_j(\boldsymbol{v}_{\mathrm{t}}) = \frac{4n_j e_j^2 e_{\mathrm{t}}^2}{m_{\mathrm{t}}^2}\int\frac{\mathrm{d}\boldsymbol{k}\,\boldsymbol{k}\boldsymbol{k}}{k^4\,|D^+|^2}\int\mathrm{d}\boldsymbol{v}\,f_j(\boldsymbol{v})\,\delta\left(\boldsymbol{k}.\boldsymbol{g}-\frac{e_j e_{\mathrm{t}} k^2}{\mu_{j\mathrm{t}}\,g}\right). \quad (8b)$$

These equations represent a synthesis of (4) and (5); both large and small wave numbers are treated accurately, and no singularities arise.

The approximation scheme outlined in the preceding two paragraphs is presented in a more general context, and in more detail, by Frieman and Book (1963). Indeed equations (8) can alternatively be derived, after some manipulation, from the general expression given by Frieman and Book for the two-particle correlation function in a plasma.

Other authors have obtained expressions for the friction and diffusion coefficients which can be brought into forms identical with (8). Thus Guernsey (1964) has demonstrated that, if the Balescu–Lenard theory is obtained as the limit of a uniformly valid theory, then this can be shown to imply equations (8). Similarly Perkins (1965), Itikawa and Aono (1966), and Aono (1968*a*, 1968*b*) have devised formalisms which yield uniformly valid approximation schemes, leading to results equivalent to (8).

Thus the result (8) is by no means a new one. However, the presentation here, where the expressions are derived via the corrections of equations (4), (5), and (6) from the simple equations (3), is more direct than the considerably more formal presentations of most of the above authors.

Evaluation of k Integration

As a lead-in to the explicit calculation in Section III, and also to make concrete the above remarks about singularities in the k integrations, we conclude this section with an evaluation of the k integration in the diffusion coefficient $\boldsymbol{T}$ in equations (3), (4), (5), and (8).

To illustrate the procedure, consider the contraction of the tensor $\boldsymbol{T}$, disregarding the multiplicative combination of physical constants, so that we have from (8b) the integral

$$I=\int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v}) \int \frac{\mathrm{d}\boldsymbol{k}}{k^2\,|D^+|^2}\,\delta\left(\boldsymbol{k}.\boldsymbol{g}-\frac{Ze^2k^2}{\mu g}\right). \tag{9}$$

We now focus on the k integration and define intermediate wave numbers k_1 and k_2 such that

$$k_0 \gg k_1 \gg k_2 \gg k_{\mathrm{D}}\,, \tag{10}$$

where k_{D} is the inverse of the Debye length ($k_{\mathrm{D}}\equiv l_{\mathrm{D}}^{-1}$) and k_0 is the inverse of b_0 ($k_0\equiv b_0^{-1}$). Correspondingly the integration over the modulus of $\boldsymbol{k}$ is divided into three regions: (i) $\infty > k > k_1$, (ii) $k_1 > k > k_2$, and (iii) $k_2 > k > 0$.

Now in region (i), $k > k_1 \gg k_{\mathrm{D}}$ so that shielding is unimportant and we have that $D^+ \to 1+\mathcal{O}(k_{\mathrm{D}}/k_1)^2$. The k integration is then readily accomplished to yield

$$I(\mathrm{i}) = 2\pi \int \mathrm{d}\boldsymbol{v}\, \frac{f(\boldsymbol{v})}{g} \ln\left(\frac{\mu g^2}{Ze^2k_1}\right)\left\{1+\mathcal{O}\left(\frac{k_{\mathrm{D}}}{k_1}\right)^2\right\}. \tag{11}$$

Conversely, in region (iii), $k < k_2 \ll k_0$ so that only small-angle scattering is relevant and the δ-function consequently simplifies to the form of (5). The integral over the modulus of $\boldsymbol{k}$ remains complicated and we get

$$I(\mathrm{iii}) = 2\pi \int \mathrm{d}\boldsymbol{v}\, \frac{f(\boldsymbol{v})}{g} \int_0^{k_2} \frac{\mathrm{d}k}{k\,|D^+|^2}\left\{1+\mathcal{O}\left(\frac{k_2}{k_0}\right)^2\right\}. \tag{12}$$

In the intermediate region (ii), both simplifications are allowable and we arrive at

$$I(\mathrm{ii}) = 2\pi\ln\left(\frac{k_1}{k_2}\right)\int \mathrm{d}\boldsymbol{v}\,\frac{f(\boldsymbol{v})}{g}\left\{1+\mathcal{O}\left(\frac{k_\mathrm{D}}{k_2}\right)^2,\left(\frac{k_1}{k_0}\right)^2\right\}. \tag{13}$$

Reviewing these expressions we see that the integral I obtained from the final theory (8) comprises the sum $I(\mathrm{i})+I(\mathrm{ii})+I(\mathrm{iii})$. This integral is free of singularities and (provided we can assume $k_0 \gg k_\mathrm{D}$, as is usually so in practice) is independent of the choice of separation parameters k_1 and k_2. On the other hand, the simple theory, embodied in equations (3), corresponds to taking region (ii) alone; the resulting expression (13) contains empirical cutoffs k_1 and k_2 at large and small wave numbers (and simple theories use the identifications $k_1 \sim k_0$, $k_2 \sim k_\mathrm{D}$). The refinement for close collisions, leading to equation (4), corresponds to taking regions (i)+(ii), with consequently *one* empirical cutoff at k_2. Inclusion of cooperative effects (only), leading to equations (5), corresponds to taking regions (ii)+(iii), with again *one* arbitrary cutoff, this time at k_1.

III. Energy Loss of a Fast Test Ion

We apply the above methods to calculate the rate of energy loss by an energetic test ion in a plasma. By "energetic" we mean that the test particle energy is substantially in excess of that of the average plasma particle of any species. Under this assumption, the test particle distribution function may be accurately approximated by a δ-function (see May 1964 for a detailed discussion of this point)

$$f_\mathrm{t}(\boldsymbol{v}_\mathrm{t}) = \delta\big(\boldsymbol{v}_\mathrm{t}-\boldsymbol{V}(t)\big). \tag{14}$$

We notice that if the test ion is "energetic" in this sense then its speed is greater than that of the plasma ions; however, nothing has been said about the ratio between the test ion speed V and the plasma electron thermal speed $(2\kappa T/m)^{\frac{1}{2}}$; this ratio x (equation (1)) remains a parameter in our problem (m and T are of course the plasma electron mass and temperature). Previous explicit results are confined to $x \gg 1$, and some rough results for $x < 1$.

We further restrict our attention to the case when the *test ion is losing energy predominantly to plasma electrons*. As has been carefully discussed (e.g. Butler and Buckingham 1962), this will be the case provided

$$x > (m/M_\mathrm{i})^{\frac{1}{3}}(\tfrac{3}{4}\pi^{\frac{1}{2}}Z_\mathrm{i}^2\,n_\mathrm{i}/n)^{\frac{1}{3}} \tag{15a}$$

or, approximately,

$$x > (m/M_\mathrm{i})^{\frac{1}{3}}, \tag{15b}$$

where M_i, $Z_\mathrm{i}\,e$, and n_i are the mass, charge, and number density of the background plasma ions. Thus we notice that once x is in the general vicinity of unity, or greater, then indeed the test particle energy loss *is* to plasma electrons. There now ensues the excellent approximation of neglecting terms of relative order m/m_t, which simplifies things somewhat.

Using equation (14) we find from equation (2) that the rate of energy loss of a fast test ion (with energy E, speed V) which is losing energy to plasma electrons is

$$\mathrm{d}E/\mathrm{d}t = m_\mathrm{t}\big(\boldsymbol{V}.\boldsymbol{F}(\boldsymbol{V})+\tfrac{1}{2}T(\boldsymbol{V})\big), \tag{16}$$

where T is the contraction of the tensor $\boldsymbol{T}$. Substituting from the accurate equations (8) for the friction and diffusion coefficients $\boldsymbol{F}$ and $\boldsymbol{T}$, we obtain after neglecting terms of relative order m/m_t

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{2(Ze^2)^2}{m}\iiint\frac{\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega\,\delta(\omega-\boldsymbol{k}.\boldsymbol{V})}{k^4\,|D^+(k,\omega)|^2}\,\delta\left(\boldsymbol{k}.\boldsymbol{g}-\frac{Ze^2k^2}{mg}\right)\boldsymbol{k}.\boldsymbol{V}\boldsymbol{k}.\frac{\partial f(\boldsymbol{v})}{\partial\boldsymbol{v}}. \tag{17}$$

Here D^+ is defined by equation (6), $\boldsymbol{g} = \boldsymbol{V}-\boldsymbol{v}$, and $f(\boldsymbol{v})$ is the distribution function for the background plasma electrons, which we have assumed to be in equilibrium, so that

$$f(\boldsymbol{v}) = n(m/2\pi\kappa T)^{3/2}\exp(-mv^2/2\kappa T). \tag{18}$$

The dummy variable ω has been introduced in (17) for subsequent convenience.

In reducing this expression (17) to manageable form, it is convenient again to introduce the wave numbers k_1 and k_2 such that

$$k_0 \gg k_1 \gg k_2 \gg k_\mathrm{D}$$

and to consider the three regions (i), (ii), and (iii) exactly as was done at the end of Section II.

Region (ii), $k_1 > k > k_2$

Here, as discussed above, we can put $|D^+| \to 1$ and $\delta(\boldsymbol{k}.\boldsymbol{g}-Ze^2k^2/mg) \to \delta(\boldsymbol{k}.\boldsymbol{g})$, provided we neglect terms of relative order $(k_\mathrm{D}/k_2)^2$ and $(k_1/k_0)^2$.

One way of simplifying equation (17) in this region (a way that is also suitable for region (iii)) is to first differentiate $f(\boldsymbol{v})$ and perform the integration over $\boldsymbol{v}$ to get

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{ii}) = -\frac{4n(Ze^2)^2}{mV}\frac{1}{\pi^{\frac{1}{2}}}\int_{k_1>k>k_2}\frac{\mathrm{d}\boldsymbol{k}}{k^3}\int\mathrm{d}s\,s^2\exp(-s^2)\,\delta\left(\frac{s}{x}-\frac{\boldsymbol{k}.\boldsymbol{V}}{kV}\right). \tag{19}$$

Here we have used a change of variable to $s = (\omega/kV)x$; x is the parameter of the problem as defined above. This expression now comes trivially to

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{ii}) = -\frac{4\pi n(Ze^2)^2}{mV}\ln\left(\frac{k_1}{k_2}\right)\frac{4}{\pi^{\frac{1}{2}}}\int_0^x\mathrm{d}s\,s^2\exp(-s^2). \tag{20}$$

At this point it is convenient to define the function $\Psi(x)$ as

$$\Psi(x) \equiv 4\pi^{-\frac{1}{2}}\int_0^x s^2\exp(-s^2)\,\mathrm{d}s = \mathrm{erf}(x)-2\pi^{-\frac{1}{2}}x\exp(-x^2). \tag{21}$$

Alternatively, one may simplify equation (17) in the region (ii) by first performing the integrations over ω and $\boldsymbol{k}$ (this way is the one that is suitable also for region (i)). One then gets, after a partial integration with respect to $\boldsymbol{v}$,

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{ii}) = -\frac{2(Ze^2)^2}{m}\int\mathrm{d}\boldsymbol{v}\,f(\boldsymbol{v})\,\boldsymbol{V}\frac{\partial}{\partial\boldsymbol{v}}:\int_{k_1>k>k_2}\frac{\mathrm{d}\boldsymbol{k}}{k^4}\boldsymbol{k}\boldsymbol{k}\,\delta(\boldsymbol{k}.\boldsymbol{g}) \tag{22}$$

$$= +\frac{2(Ze^2)^2}{m}\int\mathrm{d}\boldsymbol{v}\,f(\boldsymbol{v})\,\boldsymbol{V}\frac{\partial}{\partial\boldsymbol{g}}:\left\{\frac{\pi}{g}\ln\left(\frac{k_1}{k_2}\right)\right\}\left(\mathbf{1}-\frac{\boldsymbol{g}\boldsymbol{g}}{g^2}\right) \tag{23}$$

$$= -\frac{4\pi(Ze^2)^2}{m}\ln\left(\frac{k_1}{k_2}\right)\int \mathrm{d}\boldsymbol{v}\ \frac{\boldsymbol{V}.\boldsymbol{g}}{g^3}f(\boldsymbol{v})\,. \tag{24}$$

The integral over $\boldsymbol{v}$ now gives exactly the expression (20) above—as it should.

This result (20) or (24) is of course the familiar one of simple theories, with phenomenological cutoffs at $k_2 \sim l_{\mathrm{D}}^{-1}$ and $k_1 \sim b_0^{-1}$. A direct and lucid derivation of this result, totally by-passing all question of the test particle distribution function, is due to Butler and Buckingham (1962); although direct, this analysis tends to obscure the implicit assumption (14) (see May 1964).

Region (iii), $k_2 > k > 0$

In this region, as discussed at the end of Section II, we may take

$$\delta(\boldsymbol{k}.\boldsymbol{g} - Ze^2k^2/mg) \rightarrow \delta(\boldsymbol{k}.\boldsymbol{g})$$

in (17), thus neglecting terms of relative order $(k_2/k_0)^2$. However, the full complication of $D^+(k,\omega)$ must be kept. Then, instead of equation (19) above, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{iii}) = -\frac{4n(Ze^2)^2}{mV}\frac{1}{\pi^{\frac{1}{2}}}\int_{k_2>k}\frac{\mathrm{d}\boldsymbol{k}}{k^3}\int\frac{\mathrm{d}s\, s^2\exp(-s^2)}{|D^+(k,s)|^2}\delta\left(\frac{s}{x} - \frac{\boldsymbol{k}.\boldsymbol{V}}{kV}\right). \tag{25}$$

At this point we pause to observe that the plasma dispersion function $D^+(k,s)$ can be written (see e.g. Kihara and Aono 1963; Abramowitz and Stegun 1964, formulae 7.1.3 and 7.1.4)

$$k^2D^+(k,s) = k^2 + k_{\mathrm{D}}^2\big(X(s) + \mathrm{i}\,Y(s)\big), \tag{26}$$

where X and Y are the functions*

$$X(s) = 1 - 2s\exp(-s^2)\int_0^s \exp t^2\,\mathrm{d}t, \tag{27a}$$

$$Y(s) = \pi^{\frac{1}{2}}s\exp(-s^2)\,. \tag{27b}$$

Then, performing the integration over the angles of $\boldsymbol{k}$, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{iii}) = -\frac{4\pi n(Ze^2)^2}{mV}\frac{4}{\pi^{\frac{1}{2}}}\int_0^x s^2\exp(-s^2)\,\mathrm{d}s\int_0^{k_2}\frac{k^3\,\mathrm{d}k}{\{k^2 + k_{\mathrm{D}}^2X(s)\}^2 + \{k_{\mathrm{D}}^2Y(s)\}^2}, \tag{28}$$

which leads to

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{iii}) = -\frac{4\pi n(Ze^2)^2}{mV}\frac{4}{\pi}\int_0^x s\,\mathrm{d}s\left[Y(s)\{\ln(k_2/k_{\mathrm{D}}) - \tfrac{1}{4}\ln(X^2+Y^2)\} - \tfrac{1}{2}X(s)\tan^{-1}(Y/X)\right]. \tag{29}$$

Here again terms of relative order $(k_{\mathrm{D}}/k_2)^2$ have been ignored.

This result, which gives an accurate account of cooperative screening and plasma wave effects, is implicit in, or related to,† the work of Hubbard (1961*a*, 1961*b*),

* Note that the identity $k_{\mathrm{D}}^2 s\exp(-s^2)\,|D^+|^{-2} \equiv -\pi^{-\frac{1}{2}}k^2\,\mathrm{Im}(1/D^+)$ serves to connect apparently differing presentations of, for example, equations (17) or (8).

† A detailed exposition of the reduction of a closely related integral is given in Shkarofsky Johnston, and Bachynski (1966, pp. 264–7), where their term X_1 corresponds to our $-X$.

Kihara and Aono (1963), Tidman, Guernsey, and Montgomery (1964), and Aono (1968*a*).

Finally we rewrite equation (29) as

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{iii}) = -\frac{4\pi n(Ze^2)^2}{mV}\Psi(x)\{\ln(k_2/k_\mathrm{D})+\Delta_1(x)\}, \tag{30}$$

where $\Psi(x)$ is the standard energy loss function defined in equation (21) and $\Delta_1(x)$ is

$$\Delta_1(x) \equiv -\pi^{-1}\{\Psi(x)\}^{-1}\int_0^x s\,\mathrm{d}s\{Y\ln(X^2+Y^2)+2X\tan^{-1}(Y/X)\}. \tag{31}$$

The inverse tangent is to be evaluated in the range 0–π.

The function $\Delta_1(x)$ is tabulated as a function of x in the second column of Table 1. Previous calculations of this correction term are confined to the limit $x \gg 1$.

TABLE 1

FUNCTION $\Delta(x)$ AND ITS COMPONENTS Δ_1 AND Δ_2

These functions represent the corrections to the $\ln\Lambda$ term of simple theories (see equation (37)). x is the (dimensionless) test ion speed

x	$\Delta_1(x)$	$\Delta_2(x)$	$\Delta(x)$	x	$\Delta_1(x)$	$\Delta_2(x)$	$\Delta(x)$
0·1	−0·49	−2·27	−2·76	2·1	0·84	−0·62	0·22
0·2	−0·48	−2·26	−2·74	2·2	0·91	−0·50	0·41
0·3	−0·46	−2·23	−2·69	2·3	0·98	−0·38	0·60
0·4	−0·43	−2·20	−2·63	2·4	1·04	−0·27	0·77
0·5	−0·39	−2·17	−2·56	2·5	1·10	−0·16	0·94
0·6	−0·34	−2·12	−2·47	2·6	1·16	−0·06	1·10
0·7	−0·29	−2·06	−2·35	2·7	1·21	+0·04	1·25
0·8	−0·23	−2·01	−2·24	2·8	1·26	0·13	1·39
0·9	−0·16	−1·94	−2·10	2·9	1·30	0·22	1·52
1·0	−0·09	−1·86	−1·95	3·0	1·35	0·30	1·65
1·1	−0·01	−1·78	−1·78	4·0	1·68	0·98	2·66
1·2	+0·08	−1·68	−1·61	5·0	1·92	1·46	3·39
1·3	0·16	−1·58	−1·42	6·0	2·12	1·85	3·96
1·4	0·25	−1·47	−1·22	7·0	2·28	2·17	4·44
1·5	0·34	−1·36	−1·02	8·0	2·41	2·44	4·86
1·6	0·43	−1·24	−0·81	9·0	2·53	2·68	5·22
1·7	0·52	−1·12	−0·60	10·0	2·64	2·90	5·54
1·8	0·61	−1·00	−0·39				
1·9	0·69	−0·87	−0·18	$x \gg 1$	$\ln x+(0\cdot34)$	$2\ln x-(1\cdot69)$	$3\ln x-(1\cdot35($
2·0	0·77	−0·74	+0·03				

Region (i), $\infty > k > k_1$

In this region, which includes close collisions, we can put $D^+ \to 1$ if we are prepared to neglect terms of relative order $(k_1/k_0)^2$. Then, following the procedure which led to equation (24), we get

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{i}) = \frac{2(Ze^2)^2}{m}\int \mathrm{d}\boldsymbol{v}\,f(\boldsymbol{v})\,\boldsymbol{V}\frac{\partial}{\partial \boldsymbol{g}} : \int\limits_{k>k_1}\frac{\mathrm{d}\boldsymbol{k}}{k^4}\boldsymbol{k}\,\boldsymbol{k}\,\delta\!\left(\boldsymbol{k}.\boldsymbol{g}-\frac{Ze^2k^2}{mg}\right) \tag{32}$$

$$= -\frac{4\pi(Ze^2)^2}{m}\int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v})\,\frac{\boldsymbol{V}.\boldsymbol{g}}{g^3}\left\{\ln\left(\frac{mg^2}{Ze^2k_1}\right)-1\right\}. \tag{33}$$

Alternatively (see Montgomery and Tidman 1964, pp. 17–20; or Kihara and Aono 1963) exactly this result can be derived directly from the Coulomb collision integrals, provided the angle integrations are done properly. (In his investigation of the corrections to the dominant $\ln\Lambda$ term arising from the dynamics of close collisions, Frankel (1965) used a somewhat overly simple approach, to arrive at equation (33) with $\{\ln(\;)\}$ instead of $\{\ln(\;)-1\}$, thus missing a significant "non-dominant" term.)

Performing the angle integrals in (33), we arrive at a result which can be written

$$\frac{\mathrm{d}E}{\mathrm{d}t}(\mathrm{i}) = -\frac{4\pi n(Ze^2)^2}{mV}\,\Psi(x)\,\{\ln(4k_0/k_1)+\Delta_2(x)\}. \tag{34}$$

As before, $k_0 = \kappa T/Ze^2$ and $4k_0$ is the inverse of the average impact parameter for 90° Coulomb scattering (Shkarofsky, Johnston, and Bachynski 1966, p. 246), while $\Delta_2(x)$ is defined as*

$$\Delta_2(x) = \ln(\tfrac{1}{2}x^2)-1+4\{\pi^{\frac{1}{2}}\,\Psi(x)\}^{-1}\,G(x), \tag{35a}$$

$$G(x) = \int_0^x s^2\exp(-s^2)\ln\left(\frac{x^2-s^2}{x^2}\right)\mathrm{d}s + \int_x^\infty s^2\exp(-s^2)\left\{\ln\left(\frac{s+x}{s-x}\right)-\frac{2x}{s}\right\}\mathrm{d}s. \tag{35b}$$

The correction function $\Delta_2(x)$ is tabulated in the third column of Table 1.

Adding together the contributions from regions (i), (ii), and (iii), as expressed in equations (34), (20), and (30) respectively, we arrive at the total energy loss rate

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{4\pi n(Ze^2)^2}{mV}\,\Psi(x)\,\{\ln(4k_0/k_\mathrm{D})+\Delta_1(x)+\Delta_2(x)\}. \tag{36}$$

The most significant terms which have been neglected in obtaining this result are those of order $(k_1/k_0)^2$ and $(k_\mathrm{D}/k_2)^2$ relative to the non-logarithmic (non-dominant) terms $\Delta(x)$. Suppose we now make the choice $k_1 \sim k_2 \sim (k_0\,k_\mathrm{D})^{\frac{1}{2}}$; then the corrections to our calculation are uniformly of relative order (k_D/k_0). Since, by definition, a "plasma" has $k_\mathrm{D}/k_0 \ll 1$, neglect of such terms represents an excellent approximation.

IV. Discussion

Summarizing the results of Section III, we see that a fully accurate expression for the rate of energy loss of a test ion, losing energy predominantly to the background plasma electrons, can be written as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{4\pi n(Ze^2)^2}{mV}\,\Psi(x)\,\{\ln\Lambda+\Delta(x)\}, \tag{37}$$

where $\Psi(x)$ is the familiar energy loss function (equation (21)), x is the ratio between

* $G(x)$ in equations (35) is in fact $\frac{1}{2}x$ times Frankel's (1965) G, which in turn is an appropriate specialization of Chandrasekhar's (1941) stellar dynamical function G.

test ion speed and plasma electron thermal speed (equation (1)), and $\{\ln\Lambda+\Delta(x)\}$ replaces the $\ln\Lambda$ term of simple theories; we have used the definition*

$$\ln\Lambda \equiv \ln(4\kappa T l_{\mathrm{D}}/Ze^2) \equiv \ln(4l_{\mathrm{D}}/b_0)\,. \tag{38}$$

The quantity $\Delta(x)$, defined as simply

$$\Delta(x) = \Delta_1(x)+\Delta_2(x)\,, \tag{39}$$

where $\Delta_1(x)$ and $\Delta_2(x)$ are in turn defined by equations (31) and (35) respectively, represents the difference between the simple theory, with its empirical cutoffs, and the complete theory which gives an accurate account both of the dynamics of binary collisions and of collective screening and plasma wave effects. This correction function $\Delta(x)$ is tabulated in the last column of Table 1.

In this work, summarized in equation (37) and Table 1, our approximations have been (1) neglect of terms of relative order m/m_{t}, (2) neglect of terms of relative order (b_0/l_{D})—both of which are excellent approximations—and (3) neglect of energy loss to plasma ions, which is a valid assumption (equations (15)) provided $x > (m/M_1)^{\frac{1}{2}}$. Thus in general the above results should pertain accurately once x is roughly in the neighbourhood of, or greater than, unity.

Other assumptions implicit in the work are that classical mechanics (rather than quantum mechanics) provides an adequate description of close collisions, and that no magnetic fields are present. Quantum mechanics does not dominate close encounters until the electron temperature is of the order of 100 eV; at such elevated plasma temperatures it is necessary to use a quantum mechanical formulation, such as that of Wyld and Pines (1962) or Harris (1969), to describe close collisions (i.e. large k, region (i)). This point has received attention from Akhiezer (1961), Frankel (1965), and Hines and Budwine (in preparation). Such considerations will of course lead to a different correction term, replacing our $\Delta_2(x)$, for close collisions (but our $\Delta_1(x)$, representing distant effects, will still apply). Conversely, if magnetic fields are present they will in general affect the screening and plasma wave effects, modifying the dielectric constant D^+, and thus leading to an altered correction term, replacing our $\Delta_1(x)$, for distant collisions (Akhiezer 1961; Honda, Aono, and Kihara 1963; May and Cramer 1969). However, usually the presence of a magnetic field will not affect $\Delta_2(x)$. For a formal discussion including both complications, i.e. quantum mechanics and strong magnetic fields, see Walters and Harris (1968). We have displayed $\Delta_1(x)$ and $\Delta_2(x)$ separately for this reason, namely that one may remain useful even if the other becomes irrelevant.

These results have been presented in such a way as to be generally useful; however, the calculations were motivated largely by some recent experiments. In particular, Ormrod (1968) has found the relaxation rate for protons in an argon plasma to be a half that predicted by the simple theory, and the careful experiments of Halverson (1968) yield an energy loss rate for protons in a lithium plasma of about 60% that calculated on the basis of the simple theory. Halverson's experiment is for the relaxation of a 5 keV proton in a lithium plasma of electron density

* We follow the definition of $\ln\Lambda$ used by Halverson (1968) (see also Shkarofsky, Johnston, and Bachynski 1966, p. 247). Other authors define $\ln\Lambda = \ln(l_{\mathrm{D}}/b_0)$ or, most commonly, $\ln\Lambda = \ln(3l_{\mathrm{D}}/b_0)$.

$n \sim 4\times 10^{12}\,\mathrm{cm}^{-3}$ and temperature $T \sim 1\cdot 5\,\mathrm{eV}$; this corresponds to $x = 1\cdot 3$ which is well in excess of $(m/M_{\mathrm{i}})^{\frac{1}{3}} \sim 0\cdot 04$, and also $b_0/l_{\mathrm{D}} \sim 10^{-4}$, so that all our assumptions are well fulfilled (except the neglect of magnetic fields).

Halverson expresses his experimental result in the form (cf. equation (37))

$$\{\ln \Lambda + \Delta(x)\}_{\mathrm{exp}} = 5\cdot 8\,, \tag{40}$$

whereas the simple theory, quoted for comparison by Halverson, gives

$$\ln \Lambda = 9\cdot 7\,. \tag{41}$$

From Table 1 we see that for $x = 1\cdot 3$ the correction term is negative, leading to

$$\{\ln \Lambda + \Delta(x)\}_{\mathrm{theory}} = 8\cdot 3\,. \tag{42}$$

Thus the accurate theory diminishes, but preserves, the discrepancy.

It can be shown that the magnetic fields in Halverson's experiment do not alter the energy loss rate to any substantial degree; the discrepancy remains even when this effect is accounted for (May and Cramer 1969).

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Appendix

We give analytic expressions here for the energy loss rate in the limiting cases when the test ion speed is much larger than, or smaller than, the plasma electron thermal speed; i.e. for $x \gg 1$ and $x < 1$. These results are expressed in terms of the correction function $\Delta(x)$ defined in equation (37).

Case (*a*) $x \gg 1$

Here we have $\Psi(x) \to 1+\mathcal{O}\{\exp(-x^2)\}$. Turning to $\Delta_1(x)$, and noticing that (see definitions (27)) $X(s) \to -(1/2s^2)$ and $Y(s) \sim \exp(-s^2)$ for large s, we may show that for $x \gg 1$

$$\Delta_1(x) = \ln x + \tfrac{1}{2}\ln 2 - \tfrac{3}{4}x^{-2} + \mathcal{O}(x^{-4}). \tag{A1}$$

For the close-collisions correction, the expression (35b) for $G(x)$ is readily simplified, leading to

$$\Delta_2(x) = 2\ln x - \ln 2 - 1 - \tfrac{3}{2}x^{-2} + \mathcal{O}(x^{-4}). \tag{A2}$$

Thus the overall correction in the limit $x \gg 1$ is

$$\Delta(x) = 3\ln x - (1\cdot 347) - \tfrac{9}{4}x^{-2} + \mathcal{O}(x^{-4}). \tag{A3}$$

Case (*b*) $x < 1$

In this event the expressions for $\Psi(x)$ and $\Delta_1(x)$ simplify easily, to give

$$\Delta_1(x) = -\tfrac{1}{2} + (\tfrac{3}{5} - \tfrac{1}{20}\pi)x^2 + \mathcal{O}(x^4). \tag{A4}$$

On the other hand, for small x the expression for $G(x)$ gives

$$4\pi^{-\frac{1}{2}}\Psi^{-1}G = -2\ln x - C + \tfrac{2}{5}x^2 + \mathcal{O}(x^4) \tag{A5}$$

and consequently

$$\Delta_2(x) = -(1 + C + \ln 2) + \tfrac{2}{5}x^2 + \mathcal{O}(x^4). \tag{A6}$$

Collecting these results, the total correction in the limit $x < 1$ becomes

$$\Delta(x) = -(2\cdot 7704) + (1 - \tfrac{1}{20}\pi)x^2 + \mathcal{O}(x^4). \tag{A7}$$

It is comforting to note that the numerical results in Table 1 conform to these limiting expressions when x is small or large.