INELASTIC COLLISIONS IN THE TOWNSEND–HUXLEY DIFFUSION EXPERIMENT

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Abstract

The Boltzmann equation, including density gradients, is solved for the electron distribution function in the Townsend–Huxley experiment. Elastic and inelastic collisions with constant cross sections are assumed to occur, the inelastic energy loss per collision being small compared with the mean energy. The inelastic energy loss and the electron mean energy are calculated and tabulated over a range of values of E/P.

I. Introduction

The transport equation for electrons in steady state interaction with a neutral gas, at low pressure, in a uniform field has been described and analysed previously, assuming elastic collisions, by Francey (1969a, 1969b). A similar analysis is used here to find the electron distribution function when some inelastic collision processes are included.

The equation of change for the spherically symmetric part f_0 of the velocity distribution function may be written (Holstein 1946)

$$\tfrac{1}{3} u \lambda_e \nabla_r^2 f_0 + \tfrac{2}{3} a \lambda_e \frac{\partial}{\partial Z}\left(u \frac{\partial f_0}{\partial u} + \frac{\partial (u f_0)}{\partial u}\right) + \tfrac{4}{3} a^2 \lambda_e \frac{\partial}{\partial u}\left(u \frac{\partial f_0}{\partial u}\right)$$

$$+ \frac{2m}{M \lambda_e} \frac{\partial}{\partial u}\left\{ u^2\left(f_0 + \frac{2kT}{m}\frac{\partial f_0}{\partial u}\right)\right\} + \sum_h \left((u+u_h)\frac{f_0(u+u_h)}{\lambda_h(u+u_h)} - \frac{u f_0(u)}{\lambda_h(u)}\right)$$

$$+ u^{\frac{1}{2}}\left(\frac{\partial f_0}{\partial t}\right)_{\text{ion}} = -u^{\frac{1}{2}} S. \qquad (1)$$

In this equation u is the square of the electron post-collision velocity, m is the electronic mass, M is the molecular mass, a is the magnitude of the acceleration due to the electric field which is in the Z direction, λ_e and λ_h are mean free paths for elastic and inelastic collisions respectively, h denotes a particular level of molecular excitation, k is Boltzmann's constant, and T is the gas temperature. The quantity S is a source of electrons which maintains the distribution in a steady state. The last term on the left-hand side of (1) refers to ionization of molecules by electron impact; such ionization is assumed to occur infrequently here so that this term can henceforth be neglected.

Equation (1) can be further simplified by considering the case where the energy lost in inelastic collisions is small compared with the electron energy, that is,

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$u_h \ll u$. In this case the term which is summed over h reduces to

$$(u_{in}/\lambda_e)u\, \partial f_0/\partial u\,,$$

where

$$u_{in} = \sum_h \alpha_h u_h \qquad \text{and} \qquad \alpha_h = \lambda_e/\lambda_h\,.$$

Now, assuming cylindrical symmetry about the electric field direction and using the dimensionless variables

$$y = \left(\frac{6m}{M\lambda_e^2}\right)^{\frac{1}{2}} Z\,, \qquad R = \left(\frac{6m}{M\lambda_e^2}\right)^{\frac{1}{2}} \rho\,, \qquad \omega = \frac{mu}{2kT} = \frac{\epsilon}{kT}\,,$$

ρ, Z being cylindrical polar coordinates and ϵ the electron energy, equation (1) becomes

$$\frac{\partial^2 f_0}{\partial y^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f_0}{\partial R}\right) + 2B^{\frac{1}{2}}\frac{\partial^2 f_0}{\partial \omega\, \partial y} + \frac{B^{\frac{1}{2}}}{\omega}\frac{\partial f_0}{\partial y}$$

$$+ (B+\omega)\frac{\partial^2 f_0}{\partial \omega^2} + \left(\frac{B}{\omega} + \omega + 2 + b\right)\frac{\partial f_0}{\partial \omega} + 2f_0 = -\frac{MS\lambda_e}{2(m\epsilon)^{\frac{1}{2}}}\,, \qquad (2)$$

where

$$B = \frac{ma^2 M\lambda_e^2}{6(kT)^2}\,, \qquad b = \frac{M\epsilon_{in}}{2mkT}\,.$$

II. RESULTS

(a) Distribution Function

When λ_e and λ_h are taken to be constants, i.e. independent of electron energy, equation (2) can be solved in two stages. The first stage is to eliminate two of the terms involving the energy variable ω by considering $B \gg u$ and $B/u \gg u$. The resulting equation can be separated and solved by the method described by Francey (1969b). This solution is valid over a restricted part of the electron energy range but contains the space variation of f_0 in full. The second stage is to omit the space variation of f_0 from (2) and solve for the complete energy distribution.

The solution resulting from the first stage is given by

$$f_0 = \frac{\Omega}{\epsilon_0 - \epsilon + maZ}\exp\left(-\frac{(ma\rho)^2(2+b)}{4BkT(\epsilon_0-\epsilon+maZ)} + \frac{2(\epsilon_0-\epsilon+maZ)/kT}{2+b}\right)\,, \qquad (3)$$

where

$$\Omega = \left(\frac{2+b}{2}\right)^2 \frac{6^{3/2} m^{5/2}\exp\{(2+b)\epsilon_0/kTB\}}{M^{\frac{1}{2}}16\pi^2 kTB^{3/2}\lambda_e^2} \qquad (4)$$

and ϵ_0 is the energy of the source electrons. This distribution is Maxwellian and gives, for the ratio of the diffusion coefficient D to the drift velocity W, when maZ is taken to be large compared with $\epsilon - \epsilon_0$,

$$D/W = kT/ma\,.$$

Equation (3) also reduces to the corresponding solution for elastic collisions only when b is set equal to zero.

The solution for the second stage is

$$f_0 = \exp\left(-\int \frac{\omega \, d\omega}{B+\omega} - b\int \frac{d\omega}{B+\omega}\right). \tag{5}$$

For large values of B, corresponding to large values of E/P (electric field strength to gas pressure), equation (5) reduces to

$$f_0 = \exp\{-B + (B-b)\ln B - b\omega B^{-1} + \tfrac{1}{2}b\omega^2 B^{-2} - \tfrac{1}{2}\omega^2 B^{-1}\}. \tag{6}$$

By taking equations (3) and (6) together and considering $maZ \gg \epsilon - \epsilon_0$, the full solution for large E/P becomes

$$f_0 = \frac{\Omega}{maZ}\exp\left(-\frac{(2+b)ma\rho^2}{4kTBZ} + \frac{2maZ}{(2+b)kT} - B + (B-b)\ln B\right.$$

$$\left. -\frac{b(\epsilon-\epsilon_0)}{BkT} + \frac{b(\epsilon-\epsilon_0)^2}{2B^2(kT)^2} - \frac{(\epsilon-\epsilon_0)^2}{2B(kT)^2}\right). \tag{7}$$

(b) Transport Coefficients

The transport coefficients D and W can be evaluated using (7) in the following manner. The electron number density is found from

$$n = \int f_0 \, dv,$$

the integration being carried out over all velocity space, with f_0 going to zero at the limits. The electron flux in the Z direction is found from

$$\Gamma_Z = \int f_1(v_Z)\cos\theta \, v_Z \, dv$$

where

$$f_1 = -\lambda_e \, \partial f_0/\partial Z - (a\lambda_e/v)\partial f_0/\partial v$$

and θ is the angle between v and the Z direction. By equating coefficients in the equation

$$\Gamma_Z = Wn - D \, \partial n/\partial Z,$$

D and W are found to be given by

$$1\cdot 73\, D = \frac{\tfrac{1}{2}\lambda_e \phi^{-1} - \tfrac{1}{4}\lambda_e \psi \phi^{-1}(\pi^{\frac{1}{4}}\phi^{-\frac{1}{4}})}{0\cdot 62\,\phi^{-3/4} - 0\cdot 45\,\psi\phi^{-5/4}}, \quad 1\cdot 73\, W = \frac{\tfrac{1}{3}a\lambda_e(\pi^{\frac{1}{4}}\phi^{-\frac{1}{4}} - \psi\phi^{-1})}{0\cdot 62\,\phi^{-3/4} - 0\cdot 45\,\psi\phi^{-5/4}}, \tag{8}$$

where

$$\phi = \frac{9m^2}{4}\left(\frac{1}{2B(kT)^2} - \frac{b}{2(kTB)^2}\right), \qquad \psi = \frac{3m}{2}\left(\frac{b}{kTB} + \frac{b\epsilon_0}{(kTB)^2} - \frac{\epsilon_0}{B(kT)^2}\right).$$

These expressions reduce to those found for elastic collisions when b is set equal to zero. The first Townsend energy coefficient K_1 is related to the ratio D/W through

$$K_1 = (eE/kT)D/W \qquad (9)$$

By using the expressions (8) for D and W and the numerical values $m = 9 \cdot 11 \times 10^{-31}$ kg, $M = 3 \cdot 34 \times 10^{-27}$ kg, $T = 293$ K, and $\lambda_e = 4 \cdot 04 \times 10^{-2}/P$ m (cross section for elastic collisions $= 10^{-19}$ m^2), K_1 can be determined as a function

TABLE 1

COMPARISON OF CALCULATED ELECTRON ENERGIES IN HYDROGEN WITH EXPERIMENTAL VALUES

(1) E/P (V cm^{-1} torr^{-1})	(2) E/N (Td)	(3) Measured K_1 (293 K)*	(4) ϵ_{in} (eV) (293 K)	(5) $\bar{\epsilon}_e$ (eV) from (12)	(6) $\bar{\epsilon}$ (eV) from (14)	(7) $\bar{\epsilon}$ (eV) from (15)	(8) TB (288 K)	(9) Measured $\bar{\epsilon}$ (eV)† G (77 K)	(10) L (273 K)
0·07	0·21	1·57	0·000036	0·05	0·06	0·05			
0·10	0·3	1·82	0·000047	0·08	0·09	0·06		0·05	
0·20	0·6	2·64	0·00008	0·16	0·12	0·09		0·07	
0·5	1·5	5·3	0·00018	0·39	0·29	0·17			
0·8	2·4	7·8	0·00030	0·62	0·44	0·25			
1·0	3·0	9·2	0·00036	0·78	0·56	0·30		0·27	
2·0	6·0	14·8	0·00074	1·56	1·11	0·48		0·44	
3·0	9·0	19·12	0·00116	2·34	1·59	0·62	0·70		
4·0	12	22·8	0·00158	3·12	1·82	0·74		0·68	
5·0	15	26·2	0·00199	3·90	2·51	0·85	0·96		
6·0	18	29·3	0·00240	4·68	3·00	0·95		0·90	
7·0	21	32·4	0·00296	5·46	3·50	1·05			
8·0	24	35·5	0·00324	6·24	3·92	1·15			
9·0	27	38·9	0·00368	7·02	4·41	1·26			
10	30	42·0	0·00410	7·80	5·00	1·36	1·62	1·36	
15	45	62·2	0·00614	11·70	7·59	2·02			
20	60	82	0·00822	15·60	9·31	2·66	2·90		3·24
25	75	95	0·01038	19·50	10·91	3·08			
30	90	105	0·01259	23·40	14·64	3·40	3·90		
35	105	113	0·01472	27·30	16·58	3·66			
40	120	122	0·0170	31·20	18·85	3·95	4·63		6·04
45	135	129	0·0192	35·10	21·15	4·18			
50	150	137	0·0215	39·0	22·58	4·44			
55	165	143	0·0238	42·9	25·45	4·63			
60	180	150	0·0260	46·8	29·00	4·86			7·74
65	195	156	0·0281	50·7	29·75	5·05			
70	210	162	0·0306	54·6	30·91	5·25			
75	235	170	0·0328	58·5	34·20	5·51			
80	240	175	0·0349	62·4	37·60	5·67			9·23

* Experimental values of K_1 from R. W. Crompton (personal communication).
† Experimental values from: TB, Townsend and Bailey (1921); G, Gibson (1970); L, Lucas (1970).

of E/P and the inelastic parameter ϵ_{in}. Thus

$$K_1 = 23 \cdot 6(E/P) + (19 \cdot 7 \times 10^{21} \epsilon_{in} - 20 \cdot 5 \times 10^{44} \epsilon_{in}^2 - 22 \cdot 4 \times 10^{-3})/(E/P)$$

$$+ 0 \cdot 54 - 16 \cdot 4 \times 10^{22} \epsilon_{in}. \qquad (10)$$

Values of K_1 for a wide range of E/P have been determined experimentally and these values can be used to find ϵ_{in} from equation (10). The data used here and the resulting calculated values of ϵ_{in} for hydrogen at 293 K are shown in Table 1. It may be seen from this table that ϵ_{in} varies linearly with E/P.

(c) Electron Mean Energy

The mean energy of the electron swarm is

$$\bar{\epsilon} = \int f\epsilon \, d\boldsymbol{v} \Big/ \int f \, d\boldsymbol{v} \, .$$

Approximate methods may be used to evaluate the integrals, and a first approximation to $\bar{\epsilon}$ is given by

$$\bar{\epsilon} = \tfrac{3}{2} m \{ \Gamma(\tfrac{5}{4}) - \tfrac{3}{4} \psi \phi^{-\frac{1}{2}} \Gamma(\tfrac{3}{4}) \} / \{ \phi^{\frac{1}{4}} \Gamma(\tfrac{3}{4}) - \tfrac{1}{4} \psi \Gamma(\tfrac{1}{4}) \} \, . \tag{11}$$

By retaining only the leading terms in both the numerator and denominator of (11) and setting $b = 0$, there results for elastic collisions

$$\bar{\epsilon}_e = \Gamma(\tfrac{5}{4}) \, a \lambda_e (Mm)^{\frac{1}{2}} / \sqrt{3} \, \Gamma(\tfrac{3}{4}) \, . \tag{12}$$

This is precisely the result for elastic collisions and constant cross section given by Druyvesteyn and Penning (1940) and it leads to $\bar{\epsilon}_e = 0 \cdot 78 \, E/P$, where $\bar{\epsilon}$ is in electron-volts and E/P in V cm^{-1} torr^{-1}. By comparing the result (11) with the expressions (8) for D and W above, there results as a first approximation including inelastic collisions

$$D/\mu = 0 \cdot 78 \, \bar{\epsilon}/e \, . \tag{13}$$

This result may be compared with the constant collisional frequency result for elastic collisions (Allis 1956)

$$D/\mu = 0 \cdot 67 \, \bar{\epsilon}/e \, .$$

A better approximation to the mean energy is given by

$$\bar{\epsilon} = \frac{10^{-12}(7 \cdot 77 - 7 \cdot 88 \, \psi \phi^{-\frac{1}{2}} + 2 \cdot 9 \, \psi^2 \phi^{-1} - 0 \cdot 33 \, \psi^3 \phi^{-3/2} - 0 \cdot 06 \, \psi^4 \phi^{-2})}{1 \cdot 23 \, \phi^{\frac{1}{4}} - 0 \cdot 91 \psi + 0 \cdot 15 \, \psi^2 \phi^{-\frac{1}{2}} + 0 \cdot 04 \, \psi^3 \phi^{-1} - 0 \cdot 01 \, \psi^4 \phi^{-3/2}} \, , \tag{14}$$

with

$$\phi = \frac{6 \cdot 5 \times 10^{-23}}{(E/P)^2} \left(1 - \frac{5 \cdot 15 \times 10^{20} \, \epsilon_{in}}{(E/P)^2} \right)$$

and

$$\psi = \frac{1 \cdot 74 \times 10^{11} \, \epsilon_{in}}{(E/P)^2} - \frac{5 \cdot 76 \times 10^{-13}}{(E/P)^2} + \frac{2 \cdot 97 \times 10^8 \, \epsilon_{in}}{(E/P)^4} \, ,$$

where $\bar{\epsilon}$ is again given in electron-volts and E/P in V cm^{-1} torr^{-1} and ϵ_{in} is in joules. Equation (14) enables calculation of mean energies in the presence of inelastic collisions within the present approximations. Measured values of K_1 are used to evaluate ϵ_{in} and hence $\bar{\epsilon}$ is calculated at each value of E/P. The results are shown together in Table 1. Using measured values of K_1, column 7 gives the mean energy as calculated from (9) and (13), that is,

$$\bar{\epsilon} = K_1 kT / 0 \cdot 78 \, , \tag{15}$$

while columns 8, 9, and 10 give experimental values of the mean energy in hydrogen as measured by Townsend and Bailey (1921), Gibson (1970), and Lucas (1970) respectively.

III. Discussion

Column 4 of Table 1 shows values of ϵ_{in} calculated from measured values of K_1 in hydrogen at 293 K. This is not a satisfactory method of calculating ϵ_{in} but direct calculations from cross section data are hampered by lack of such data. Very rough calculations appear to show that the tabulated results are of the correct order.

In column 5 are shown mean energies which would arise if all collisions were elastic. These do not arise from measured values of K_1. In fact for elastic collisions the theory shows that $K_1 = 23 \cdot 6 \, E/P$ and Townsend factors very much greater than those shown in column 3 would have been measured.

The mean energies in column 6 show the effect of inelastic collisions in reducing the mean energy of the electrons. The mean energies given here are still much higher than those shown in column 7 so that it may be that the effect of inelastic collisions is not being sufficiently weighted in this approximation. It is of interest to note that the values in column 7 agree quite well with those published by Townsend and Bailey (1921) and Gibson (1970).

Further calculations need to be done and other approximations tried to establish the value of these results. Such work is proceeding.

IV. Acknowledgments

V. References

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