

GENERALIZED SOLUTIONS OF STEADY CROSSED-FIELD MAGNETOHYDRODYNAMIC CHANNEL FLOWS

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Abstract

Solutions for the flow in an MHD interaction are presented and compared when the fluid is a perfect gas, a singly ionizing monatomic gas, and a gas for which no thermodynamic model is specified. The separate effects of a changing duct area, body force, and addition of heat are presented and interpreted.

I. INTRODUCTION

A steady quasi-one-dimensional magnetohydrodynamic (MHD) interaction is considered here. Figure 1 illustrates the interaction region. The duct area A changes slowly and the assumption is made that the electric field E and the magnetic field B are everywhere at right angles to the flow velocity v , whose direction x is along the axis of the duct. Due to the presence of E and B , a normal current j and a Hall current j_H exist. In what follows it is assumed that densities are too high for j_H to be significant. In this case Ohm's law applies,

$$j = \sigma(E - vB),$$

where σ is the ordinary scalar conductivity.

The present analysis is in three parts. In Section II the equations governing the MHD interaction are established, and solutions for the gradients of velocity and the thermodynamic parameters are derived under the assumption that the gas is calorically perfect. While these solutions are familiar, they are presented in a novel form which makes explicit the effects of duct area, body force, and addition of heat. In Section III the equations are solved on the assumption of a singly ionizing gas, and in Section IV they are solved for a general class of fluids in which no detailed assumptions are made about the thermodynamic behaviour of the fluid. From this approach it is possible to assess the effects of the common assumption of perfect gas behaviour.

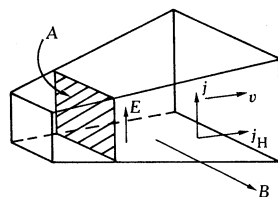


Fig. 1.—Schematic diagram of quasi-one-dimensional MHD interaction.

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II. MHD EQUATIONS AND PERFECT GAS SOLUTIONS

(a) MHD Equations

The flow is assumed to be inviscid and non-heat-conducting. It is governed by the three equations expressing conservation of mass, momentum, and energy, namely

$$d(\rho v A)/dx = 0, \quad (1)$$

where ρ is the density, v the velocity, A the duct cross sectional area, and x the distance along the duct;

$$\rho v dv/dx = -dP/dx + jB, \quad (2)$$

where P is the pressure; and

$$\rho v d(\frac{1}{2}v^2 + h)/dx = jE = j^2/\sigma + vjB, \quad (3)$$

where h is the enthalpy per unit mass.

The MHD equations (1)–(3) do not yield to closed analytic solution, except when combined with particularly simple thermodynamic relations plus restrictions such as constant v or constant P . It is well known that expressions for dv/dx , dP/dx , dT/dx , and like derivatives may be extracted when the working fluid is a perfect gas (Sutton and Sherman 1965; Turcotte 1965). Sherman (1963) has found expressions for dv/dx and dT/dx when the fluid is an ionizing gas diluted by a non-reacting diatomic gas.

(b) Perfect Gas Solutions

If the fluid is a calorifically perfect gas, P , ρ , and h can be related in familiar ways and the conservation equations can then be solved for dv/dx , dT/dx , and dP/dx . The solution for dv/dx is (see Sutton and Sherman 1965)

$$\frac{dv}{dx} = \frac{v}{M^2 - 1} \frac{1}{A} \frac{dA}{dx} + \frac{v}{(M^2 - 1)P\gamma} jB - \frac{\gamma - 1}{(M^2 - 1)P\gamma} \frac{j^2}{\sigma}, \quad (4)$$

where M is the Mach number given by v/a , a being the acoustic velocity, and γ is the ratio of the specific heats C_p at constant pressure and C_v at constant volume.

If j is eliminated from equation (4) using Ohm's law, the result may be written

$$\frac{dv}{dx} = \frac{1}{M^2 - 1} \left(\frac{v}{A} \frac{dA}{dx} - \frac{\sigma B^2}{P} (v - v_2)(v - v_1) \right), \quad (5)$$

where

$$v_1 = (\gamma - 1)E/\gamma B \quad \text{and} \quad v_2 = E/B.$$

Equation (5) has been extensively discussed in the literature (e.g. Resler and Sears 1958; Sutton and Sherman 1965; Turcotte 1965) especially for the case $dA/dx = 0$. In constant area ducts, acceleration is possible in supersonic flow only between the velocities v_1 and v_2 . In subsonic flow it is possible only outside these limits.

Equation (4) shows explicitly the effects of area change dA/dx , body force jB , and heat addition j^2/σ : (1) The effect of the first term alone is the familiar steady isentropic nozzle flow. Sonic velocity is possible only where $dA/dx = 0$, and the gas

accelerates or decelerates with increasing area depending on whether the flow is supersonic or subsonic. (2) The second term on the right-hand side of equation (4) gives the effect of a body force jB . Again the factor $(M^2-1)^{-1}$ appears and has an influence analogous to its influence in the first term. (3) The final term in (4) shows the effect of heat addition j^2/σ . Addition of heat tends to increase the pressure and, at subsonic velocities, this pressure is relieved by expansion and the velocity increases. At supersonic velocities such an expansion, which spreads roughly at sonic velocities, cannot take place. An adverse pressure gradient builds up and the velocity decreases. Equation (4) shows that, in a duct of constant area and for supersonic flow, acceleration proceeds only when

$$(\gamma-1)j^2/\gamma\sigma < vjB.$$

In subsonic flow the inequality sign is reversed. These inequalities correspond exactly to the requirements for acceleration described in connection with equation (5). They show how acceleration in a duct of constant area depends on the relative sizes of the body force and the effect of heat addition.

Equation (4) may be generalized by replacing jB with a body force F and j^2/σ with a rate of heat addition dQ/dx . It then follows that, in the absence of a body force but with heating, nozzle flow becomes sonic, not as usual where $dA/dx = 0$, but where

$$\frac{v}{A} \frac{dA}{dx} = \frac{\gamma-1}{\gamma} \frac{1}{P} \frac{dQ}{dx}.$$

Similarly, in the absence of heat addition, a body force and a changing duct area cause the flow to become sonic where

$$\frac{v}{A} \frac{dA}{dx} = -\frac{v}{P\gamma} F.$$

Thus Parker (1963) has suggested that the solar wind, which flows in a conical nozzle, passes through a "gravitational throat" where the flow is sonic.

The solutions of equations (1)–(3) for dT/dx and dP/dx are respectively (Sutton and Sherman 1965)

$$\frac{dT}{dx} = \frac{M^2(\gamma-1)T}{1-M^2} \frac{1}{A} \frac{dA}{dx} + \frac{(\gamma-1)T}{\gamma(1-M^2)P} jB - \frac{\gamma-1}{(1-M^2)C_p \rho v} \frac{j^2}{\sigma} + \frac{1}{C_p \rho v} \frac{j^2}{\sigma} \quad (6)$$

and

$$\frac{dP}{dx} = \frac{\gamma M^2 P}{1-M^2} \frac{1}{A} \frac{dA}{dx} + \frac{jB}{1-M^2} - \frac{\gamma-1}{(1-M^2)v} \frac{j^2}{\sigma} + \frac{\gamma-1}{v} \frac{j^2}{\sigma}. \quad (7)$$

(c) Interpretation of Equations

In the absence of j and B equations (6) and (7) give results for varying area of flow. Under such circumstances, their ratio is given by

$$\frac{dT/dx}{dP/dx} = \frac{(\gamma-1)T}{\gamma P} = \left(\frac{\partial T}{\partial P} \right)_s,$$

which is just the partial differential along an isentropic path. Thus, if proof were

needed, this shows that pure nozzle flow is isentropic. This result is trivial, as is the fact that, from similar considerations, the effect of the body force jB alone is also isentropic.

What is surprising, however, is that the third terms on the right-hand sides of equations (6) and (7) again stand in the isentropic ratio, as $1/C_p \rho = (\gamma-1)T/\gamma P$. Thus these terms, which represent part of the effect of heat addition, also describe an isentropic process.

The ratio of the final terms is given by

$$\frac{1}{(\gamma-1)C_p \rho} = \left(\frac{\partial T}{\partial P} \right)_\rho,$$

showing that these terms describe the effects of heating at constant density. The absence from these terms of the usual factor $(M^2-1)^{-1}$ is consistent with this interpretation, since the effects which occur without change in volume will be independent of velocity.

The last two terms in equations (6) and (7) are thus seen to divide the result of heat addition into two effects: firstly the temperature and pressure increase as though due to heating at constant volume, and then the gas expands isentropically until the pressure and velocity are matched to the mechanics of the flow. It is interesting to note that equation (4) contains terms corresponding to the first three of equations (6) and (7) but not to the last. Under the present interpretation, the velocity change is due entirely to the effects of isentropic processes.

III. SOLUTIONS OF EQUATIONS FOR IONIZING GAS

(a) Gas Model

In this section the equations of the MHD interaction (1)–(3) are solved with a model of the gas thermodynamics which allows single ionization. This gas model is taken to be a mixture of perfect gases composed of ions, electrons, and atoms respectively whose proportions vary with temperature. An account of such a gas is to be found in several places (see e.g. Cambel, Duclos, and Anderson 1963; Sutton and Sherman 1965). Equilibrium ionization is given by the Saha equation

$$\alpha^2/(1-\alpha^2) = K(t^{5/2}/P) \exp(-t^{-1}), \quad (8)$$

where α is the ratio of the number of electrons to the number of electrons and neutral atoms, referred to here as the ionization fraction; K depends on the internal partition functions of the neutral gas atoms and, in the present model, is constant; and t is a non-dimensional temperature parameter given by

$$t = (k/I)T, \quad (9)$$

k being Boltzmann's constant and I the energy of first ionization of the atoms.

If a parameter i is defined by

$$i = I/m_A,$$

where m_A is the atomic mass, which is approximately the ionic mass and much

greater than the electronic mass, then an equation of state may be written

$$P = \rho(1+\alpha)it. \quad (10)$$

Equations (8) and (10) together give an alternative form of the Saha equation

$$\alpha^2/(1-\alpha) = (Kt^{3/2}/i\rho) \exp(-t^{-1}). \quad (11)$$

Enthalpy per unit mass h is given by

$$h = \frac{5}{2}(1+\alpha)it + i\alpha. \quad (12)$$

The above properties are familiar in the literature. Some further thermodynamic properties are derived in the Appendix.

(b) Solutions of MHD Equations

The flow equations (1)–(3) are now solved with the thermodynamic equations (8)–(12). Equations (1)–(3) contain the four first-order differential coefficients dv/dx , dP/dx , $d\rho/dx$, and dh/dx . Of these, the last three may be expressed through equations (10)–(12) as quasi-linear summations of $d\alpha/dx$ and dt/dx . Equations (1)–(3) may then be written with the three differential coefficients dv/dx , $d\alpha/dx$, and dt/dx as

$$\begin{bmatrix} dv/dx \\ d\alpha/dx \\ dt/dx \end{bmatrix} [D] = \begin{bmatrix} -(\rho v/A)dA/dx \\ jB \\ j^2/\sigma + vjB \end{bmatrix}, \quad (13)$$

where

$$[D] = \begin{bmatrix} \rho & -\rho v(2-\alpha)/\alpha(1-\alpha) & \rho v(\frac{3}{2}t+1)/t^2 \\ \rho v & -2P/\alpha(1-\alpha^2) & P(\frac{5}{2}t+1)/t^2 \\ \rho v^2 & \rho vi(\frac{5}{2}t+1) & \rho vi\frac{5}{2}(1+\alpha) \end{bmatrix}. \quad (14)$$

Equations (13) may be solved by standard methods. Thus

$$dv/dx = |D'|/|D|, \quad (15)$$

where

$$|D'| = \begin{vmatrix} -(\rho v/A)dA/dx & -\rho v(2-\alpha)/\alpha(1-\alpha) & \rho v(\frac{3}{2}t+1)/t^2 \\ jB & -2P/\alpha(1-\alpha^2) & P(\frac{5}{2}t+1)/t^2 \\ j^2/\sigma + vjB & \rho vi(\frac{5}{2}t+1) & \rho vi\frac{5}{2}(1+\alpha) \end{vmatrix}. \quad (16)$$

Expansion of $|D|$ shows it to be expressible in the form

$$|D| = -\frac{\rho^2 vi P \{5t^2 + \alpha(1-\alpha)(\frac{5}{2}t+1)^2\}}{\alpha(1-\alpha)t^2} \left(1 - \frac{v^2}{\chi^2}\right), \quad (17)$$

where

$$\chi^2 = \frac{P}{\rho} \frac{5t^2 + \alpha(1-\alpha)(\frac{5}{2}t+1)^2}{3t^2 + \alpha(1-\alpha)(\frac{3}{2}t+1)(\frac{5}{2}t+1) - t}. \quad (18)$$

In the Appendix, equations (A3) and (A4) show that the right-hand side of (18) is equal to the square of the acoustic velocity. Thus equation (17) becomes

$$|D| = - \frac{\rho^2 v i P \{5t^2 + \alpha(1-\alpha)(\frac{5}{2}t+1)^2\}}{\alpha(1-\alpha)t^2} (1-M^2). \quad (19)$$

Appearance of $|D|$ in the denominator of the solutions of equations (14) thus introduces a factor $(1-M^2)^{-1}$. The variations of the flow parameters will thus be qualitatively similar to those for a perfect gas given by equations (4)–(7).

When equations (16) and (19) are substituted in (15), the result is

$$\frac{dv}{dx} = \frac{v}{M^2-1} \frac{1}{A} \frac{dA}{dx} + \frac{v}{(M^2-1)P\gamma^*} jB - \frac{\epsilon}{(M^2-1)P\gamma^* \sigma} j^2, \quad (20)$$

where γ^* and ϵ are the thermodynamic variables defined in equations (A4) and (A10) of the Appendix.

(c) Discussion of Velocity Equation

Equation (20) is similar in appearance to equation (4), its counterpart for a non-ionizing gas. The differences are that the coefficients γ^{-1} and $(\gamma-1)/\gamma$ in equation (4) are constants whereas their counterparts in equation (20), $(\gamma^*)^{-1}$ and ϵ/γ^* , are thermodynamic variables. The values of the latter lie between close limits, however, for the entire range of values of t and α . Thus γ^* ranges from $5/3$ ($\alpha = 0$ or 1 , $t \neq 0$) to 1 (the limiting case where $\alpha = 0$ or 1 , $t = 0$). The corresponding limits of ϵ are $2/3$ and 0 . For typical laboratory plasmas ($t \sim 0.1$), the values of γ^* and ϵ are close to 1 and t respectively.

It is a requirement of the gas model employed here that in the limit as ionization becomes negligible the behaviour of a monatomic perfect gas is approached. This indeed happens, with γ^* and ϵ approaching the values of γ and $\gamma-1$ respectively of the perfect gas. The same thing happens as the limit of complete ionization is approached and the gas becomes a mixture of perfect gases of constant composition.

A physical interpretation of the function ϵ emerges as follows. Consider the case of a duct of constant cross section. Equation (20) shows that acceleration is zero when

$$\epsilon j^2 / v \sigma = jB.$$

The left-hand side is the pressure gradient due to heating which is just sufficient to balance the body force. Since this heating takes place at constant density (constant velocity in a constant-area duct), it is possible to write

$$\frac{\epsilon j^2}{v \sigma} = \frac{dP}{dx} = \left(\frac{\partial P}{\partial u} \right)_\rho \frac{du}{dx} = \left(\frac{\partial P}{\partial u} \right)_\rho \frac{1}{\rho v} \frac{j^2}{\sigma},$$

whence

$$\epsilon = \rho^{-1} (\partial P / \partial u)_\rho.$$

It is seen in Section IV that if ϵ is defined in this way, an equation like (20) is generally true for a large class of fluids.

(d) Temperature and Ionization Equations

The solution for dt/dx may be found from equations (13). With the use of some of the terms defined in the Appendix and of the temperature T instead of t , this solution becomes

$$\frac{dT}{dx} = \frac{M^2 \epsilon T}{1-M^2} \frac{1}{A} \frac{dA}{dx} + \frac{\epsilon T}{(1-M^2) \gamma^* P} jB - \frac{\gamma-1}{(1-M^2) C_p \rho v} \frac{j^2}{\sigma} + \frac{1}{C_p \rho v} \frac{j^2}{\sigma}. \quad (21)$$

When it is remembered that ϵ in the ionizing gas takes the place of $(\gamma-1)$ in the non-ionizing gas, the similarity between this equation and equation (6), describing a perfect gas, is apparent.

The solution of equations (13) for $d\alpha/dx$ is found, again with the use of the Appendix, to be

$$\begin{aligned} \frac{d\alpha}{dx} = & \frac{M^2(1+\alpha)(\gamma^*-\epsilon-1)}{1-M^2} \frac{1}{A} \frac{dA}{dx} + \frac{(1+\alpha)(\gamma^*-\epsilon-1)}{(1-M^2)\gamma^*P} jB \\ & - \frac{\epsilon(1+\alpha)(\gamma^*-\epsilon-1)}{(1-M^2)\gamma^*Pv} \frac{j^2}{\sigma} + \frac{(\frac{3}{2}t+1)(1+\alpha)(\gamma^*-\epsilon-1)}{Pv} \frac{j^2}{\sigma}. \end{aligned} \quad (22)$$

The factor $(\gamma^*-\epsilon-1)$ that is common to the right-hand side disappears as α approaches 0 or 1, and thus ensures that in these limits $d\alpha/dx = 0$. There is, of course, no equation in the perfect gas theory with which to compare equation (22).

The results for dv/dx , dT/dx , and $d\alpha/dx$ presented above suffice to specify the flow. Further results may be derived from them, or from the conservation equations (1)–(3). Thus the pressure gradient is found from equations (2) and (20) to be

$$\frac{dP}{dx} = \frac{\gamma^* M^2 P}{1-M^2} \frac{1}{A} \frac{dA}{dx} + \frac{jB}{1-M^2} - \frac{\epsilon}{(1-M^2)v} \frac{j^2}{\sigma} + \frac{\epsilon}{v} \frac{j^2}{\sigma}. \quad (23)$$

The similarity between this equation and its perfect gas equivalent (7) is evident.

Examination of the terms of equations (20)–(23) results in the same conclusions as in Section II for the perfect gas. The effects of area change and body force taken by themselves are isentropic changes. The effects of heating are two-fold: heat addition at constant density and isentropic expansion.

IV. GENERAL SOLUTIONS

In this section expressions for the rates of change of the fluid properties are derived without resort to any detailed model of the thermodynamics of the fluid.

The following assumptions are made about the class of fluids being considered:

- (1) The fluid has no more than two independent intensive properties.
- (2) The following are exact differentials: the internal energy

$$du = T ds - P d(\rho^{-1}) \quad (24)$$

and the enthalpy

$$dh = T ds + \rho^{-1} dP. \quad (25)$$

Maxwell's first two equations follow from the condition for an exact differential (Zemanski 1958).

(a) *Solutions*

Equation (25) together with equations (2) and (3) yields the rate of change of entropy per unit mass as

$$ds/dx = j^2/\rho v \sigma T. \quad (26)$$

Entropy increase is always positive in the downstream direction. Then expressing dP/dx as

$$\frac{dP}{dx} = \left(\frac{\partial P}{\partial \rho} \right)_s \frac{d\rho}{dx} + \left(\frac{\partial P}{\partial s} \right)_\rho \frac{ds}{dx}$$

gives

$$\frac{dP}{dx} = a^2 \frac{d\rho}{dx} + \rho \epsilon T \frac{ds}{dx}, \quad (27)$$

in which the relation

$$\epsilon = \rho^{-1} (\partial P / \partial u)_\rho \quad (28)$$

is used as a definition.

Appropriate manipulation of equations (1), (2), (26), and (27) gives without difficulty

$$\frac{dv}{dx} = \frac{v}{M^2 - 1} \frac{1}{A} \frac{dA}{dx} + \frac{v}{(M^2 - 1)P\gamma^*} jB - \frac{\epsilon}{(M^2 - 1)P\gamma^*} \frac{j^2}{\sigma}, \quad (29)$$

where γ^* is defined by the relation $a^2 = (P/\rho)\gamma^*$. Equation (29) is familiar in the form of (4) and (20). A general solution for dv/dx not in the present form is given by Shercliff (1965).

Substitution of equation (29) into equation (2) gives

$$\frac{dP}{dx} = \frac{\gamma^* M^2 P}{1 - M^2} \frac{1}{A} \frac{dA}{dx} + \frac{jB}{1 - M^2} - \frac{\epsilon}{(1 - M^2)v} \frac{j^2}{\sigma} + \frac{\epsilon}{v} \frac{j^2}{\sigma}, \quad (30)$$

which is similar to the less general equations (7) and (23).

For a general expression for dT/dx , T is written as $T(h, P)$, so that

$$\frac{dT}{dx} = \left(\frac{\partial T}{\partial h} \right)_P \frac{dh}{dx} + \left(\frac{\partial T}{\partial P} \right)_h \frac{dP}{dx}. \quad (31)$$

The first coefficient $(\partial T / \partial h)_P$ is C_p^{-1} and the second $(\partial T / \partial P)_h$ is the Joule-Kelvin

coefficient μ . It may be proved[†] from the definitions of the terms as partial derivatives, and with Maxwell's equations, that

$$\mu = (C_p \rho)^{-1} \{(\gamma - 1)/\epsilon - 1\}. \quad (32)$$

From equations (3) and (29) an expression for dh/dx may be found, and dP/dx is given by equation (30). With these substitutions, equation (31) yields a solution for dT/dx . With the further relation[†]

$$(\gamma - 1)a^2/T\epsilon^2 C_p = 1, \quad (33)$$

this solution may be written in the form

$$\frac{dT}{dx} = \frac{M^2 \epsilon T}{1 - M^2} \frac{1}{A} \frac{dA}{dx} + \frac{\epsilon T}{(1 - M^2) \gamma^* P} jB - \frac{\gamma - 1}{(1 - M^2) C_p \rho v} \frac{j^2}{\sigma} + \frac{1}{C_p \rho v} \frac{j^2}{\sigma}, \quad (34)$$

which is similar to equations (6) and (21).

(b) Discussion

Using the general results derived in this section it is possible to describe MHD interactions involving many real gases, provided that the interrelations between the thermodynamic variables P , T , h , etc. can be stated and that the conductivity σ , normally a function of the thermodynamic state, is known.

The general results allow an assessment of the accuracy of simplifications, such as the common one of perfect-gas behaviour. The perfect gas is a degenerate case in which γ , γ^* , and $\epsilon + 1$ are all equal and constant. To take the case of the singly ionizing gas considered in Section III, γ is never much different from γ^* and both lie in the range 1 to 5/3. ϵ ranges from 2/3 to 0. Thus the greatest errors in the perfect gas assumption will be those in which ϵ plays a role. Equations (20) to (23) show that the perfect-gas assumption would lead to overestimation of the effects of heating on velocity and pressure, large overestimation of the temperature changes, and the obvious error of setting ionization equal to zero. In addition, error in the speed of sound, which depends on the value of γ^* , will result in an unreliable value for M , which will be important if M is close to unity.

Finally, the general results justify to some extent the perfect-gas assumption, which is often used because of its mathematical tractability in order to gain a qualitative understanding of a problem, since they show that it leads to results which are more or less functionally correct.

V. ACKNOWLEDGMENT

Some of the work presented here was performed at the Australian National University, and the author is indebted to Dr. R. J. Stalker for much help and critical discussion.

[†] The proofs of equations (32) and (33) are not presented here but may be obtained on application to the Editor-in-Chief, Editorial and Publications Section, CSIRO.

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APPENDIX

*Ionizing Gas Relations**Isentropic processes*

Using the exact differential

$$dh = T ds + \rho^{-1} dP,$$

it may be proved by differentiating equations (8) and (12) that, in an isentropic process,

$$t = \frac{1 + \alpha}{C_0 - \frac{5}{2}\alpha - 2 \ln\{\alpha/(1 - \alpha)\}}, \quad (A1)$$

where C_0 is a constant.

Speed of sound

By definition

$$a^2 = (\partial P / \partial \rho)_s. \quad (A2)$$

Differentiating equations (8) and (11) with respect to α along isentropic paths and dividing one result by the other gives

$$a^2 = \frac{P}{\rho} \frac{(1 - \alpha^2)(\frac{5}{2}t + 1) - 2t^2(\partial\alpha/\partial t)_s}{\alpha(1 - \alpha^2)(\frac{3}{2}t + 1) - (2 - \alpha)(1 + \alpha)t^2(\partial\alpha/\partial t)_s}.$$

An expression for $(\partial\alpha/\partial t)_s$ is found from equation (A1) and substitution gives

$$a^2 = (P/\rho)\gamma^*, \quad (A3)$$

where

$$\gamma^* = \frac{5t^2 + \alpha(1 - \alpha)(\frac{5}{2}t + 1)^2}{3t^2 + \alpha(1 - \alpha)\{(\frac{3}{2}t + 1)(\frac{5}{2}t + 1) - t\}}. \quad (A4)$$

Specific heats at constant pressure and density

By definition

$$C_p = (\partial h / \partial T)_P = (R/i)(\partial h / \partial t)_P \quad (A5)$$

By differentiating equations (8) and (12) along paths of constant P , it is found that

$$C_p = R(1+\alpha)\{5t^2 + \alpha(1-\alpha)(\frac{5}{2}t+1)^2\}/2t^2. \quad (\text{A6})$$

Similarly, differentiating equation (11) and the relation

$$u = \frac{3}{2}(1+\alpha)it + i\alpha$$

gives

$$C_p = R\{(\frac{3}{2}t+1)^2\alpha(1-\alpha) + \frac{3}{2}(1+\alpha)(2-\alpha)t^2\}/(2-\alpha)t^2. \quad (\text{A7})$$

The ratio γ of the specific heats is then given by

$$\gamma = \frac{C_p}{C_\rho} = \frac{(1+\alpha)(2-\alpha)}{2} \frac{5t^2 + \alpha(1-\alpha)(\frac{5}{2}t+1)^2}{3t^2 + \alpha(1-\alpha)\{(\frac{5}{2}t+1)(\frac{3}{2}t+1) - t\}}, \quad (\text{A8})$$

which, with equation (A4), gives

$$\gamma = \frac{1}{2}(1+\alpha)(2-\alpha)\gamma^*. \quad (\text{A9})$$

As α approaches either of its limits, γ approaches γ^* .

It is convenient here to define a new thermodynamic variable

$$\epsilon = \frac{2t^2 + \alpha(1-\alpha)(\frac{5}{2}t+1)t}{3t^2 + \alpha(1-\alpha)\{(\frac{3}{2}t+1)(\frac{5}{2}t+1) - t\}}. \quad (\text{A10})$$

This quantity is found to occur in the analysis in places where perfect gas theory has $\gamma-1$. Unless α is 0 or 1, however, ϵ is not equal to either $\gamma-1$ or γ^*-1 .

