

A GENERAL FORMULATION OF THE TRANSFER EQUATION

II.* LINE FORMATION WITH GENERAL REDISTRIBUTION

By C. J. CANNON†

[Manuscript received 11 October 1971]

Abstract

The multidimensional equation of transfer for spectral line radiation under a general redistribution law is studied. It is shown that the equation may be rewritten as a system of equations of the "Feautrier" form, which are known to be exceedingly stable and efficient under numerical reduction. It is also shown that the inclusion of a multidimensional differential macroscopic velocity field does not alter the functional form of the equations obtained and therefore may also be treated by the general Feautrier technique.

I. INTRODUCTION

In Part I of this series (Cannon 1971*a*) the family of transfer equations describing single anisotropic scattering, with and without polarization, was shown to belong to a class of second-order differential equations with two-point boundary conditions. The equations so obtained are of the well-known "Feautrier" type (Feautrier 1964) where the new unknown to be determined is the average of the radiation field in the upward and downward directions. The advantage gained by such a representation is twofold. First, the directional radiation field, which is necessarily an increasing function of position (depth) away from the boundary of the atmosphere, results in increasingly severe instabilities with depth under numerical reduction if the original first-order differential form of the transfer equation is used. The Feautrier technique, by the very nature of the equations to be solved, obviously overcomes this difficulty. The second-order differential Feautrier equations have explicit constraints not only at the surface of the atmosphere but, in the case of a semi-infinite medium, at some specified depth away from the boundary, and this latter condition ensures numerical stability of the solution throughout. Secondly, these equations are readily reduced, using well-known difference techniques, to a form suitable for numerical solution. In fact, an exponentially increasing depth grid may be chosen (see e.g. Auer 1967; Cuny 1967) which therefore significantly increases the efficiency of the method relative to those using the original form of the transfer equation.

The Feautrier technique has been used extensively in the study of simple problems occurring in spectral line transfer theory. One-dimensional media have been studied, for example, by Cuny (1967) and Auer and Mihalas (1969) under the restrictive assumption of complete redistribution in the rest frame of the atom. Multidimensional situations have been considered by Cannon (1970*a*, 1970*b*, 1971*b*) and macroscopic differential velocity field calculations have been given by Rees (1970), Cannon (1971*c*), and Cannon and Rees (1971) all under the same assumption.

* Part I, *Aust. J. Phys.*, 1971, **24**, 341-9.

† Department of Applied Mathematics, University of Sydney, Sydney, N.S.W. 2006.

The derived Feautrier system of equations for these cases relies to a very large extent, however, on the simple form of the redistribution function for complete redistribution, and it is not obvious that the technique may be generalized further to study more complicated (and more physically realistic) line formation problems. Other methods have been used in the study of the above-mentioned problems but, because of numerical difficulties, such methods have only been further generalized to determine solutions for very simple departures from complete redistribution (Hearn 1963, 1964; Hummer 1969). It is for this reason that the applicability of the Feautrier technique, with all its inherent numerical advantages, is examined in this paper for a *general* redistribution law.

In order to apply the technique it is essential that the equation of transfer be defined as an explicit functional of the redistribution function, and this then necessitates the derivation of the "exact" line transfer equation from a consideration of the fundamental microscopic processes involved. This is given in Section II. The resulting equation, although highly nonlinear in the radiation field, may be reduced to the Feautrier system (Section III). However, the nonlinearity suggests that the computation of its solutions, even using the Feautrier technique, could be prohibitive. Thus an alternative specification of the transfer equation which is more amenable to numerical solution, and which is quite frequently discussed in the literature, is given in Section IV. Problems involving nonzero multidimensional macroscopic differential velocity fields are considered in Section V.

II. EQUATION OF TRANSFER

The equation of transfer for spectral line radiation in an atmosphere exhibiting zero mass motion may be written in the form

$$(\mathbf{\Omega} \cdot \nabla) I_\nu(\mathbf{r}, \mathbf{\Omega}) = -\kappa_\nu(\mathbf{r}, \mathbf{\Omega}) \{I_\nu(\mathbf{r}, \mathbf{\Omega}) - S_\nu(\mathbf{r}, \mathbf{\Omega})\}, \quad (1)$$

where

$$\kappa_\nu(\mathbf{r}, \mathbf{\Omega}) = (h\nu_0/4\pi) \{N_L(\mathbf{r}) B_{LU} \phi_\nu(\mathbf{r}, \mathbf{\Omega}) - N_U(\mathbf{r}) B_{UL} \psi_\nu(\mathbf{r}, \mathbf{\Omega})\} \quad (2)$$

and

$$S_\nu(\mathbf{r}, \mathbf{\Omega}) = \frac{A_{UL} j_\nu(\mathbf{r}, \mathbf{\Omega})}{\{N_L(\mathbf{r})/N_U(\mathbf{r})\} B_{LU} \phi_\nu(\mathbf{r}, \mathbf{\Omega}) - B_{UL} \psi_\nu(\mathbf{r}, \mathbf{\Omega})}. \quad (3)$$

We have assumed, for clarity in exposition only, a model two-level atom with no continuum. These are not restrictive assumptions as a relaxation of them may be incorporated into the analysis in the usual manner. In equations (1)–(3), $I_\nu(\mathbf{r}, \mathbf{\Omega})$ is the specific intensity of the radiation field at position \mathbf{r} in a direction $\mathbf{\Omega}$ and at a frequency ν measured from the line centre frequency ν_0 . The populations of the upper and lower levels which give rise to the radiation field in question are $N_U(\mathbf{r})$ and $N_L(\mathbf{r})$ respectively, while B_{LU} , B_{UL} , and A_{UL} are the Einstein rate coefficients for absorption, stimulated emission, and spontaneous emission respectively and $\phi_\nu(\mathbf{r}, \mathbf{\Omega})$, $\psi_\nu(\mathbf{r}, \mathbf{\Omega})$, and $j_\nu(\mathbf{r}, \mathbf{\Omega})$ are the corresponding probabilities (profiles) of that event occurring. The latter probabilities are, of course, normalized to unity.

To proceed further, we require the ratio N_L/N_U and the form of the respective profiles. Oxenius (1965) has discussed these quantities in some detail and, since these results are pertinent to the present analysis, a summary of his work is given

here. Hummer (1969) has also treated the problem in some detail, but studies an approximate form for the opacity $\kappa_\nu(\mathbf{r}, \mathbf{\Omega})$. This alternative formulation is considered in Section IV.

First, let $q(\gamma)$ be the probability per unit solid angle and frequency interval of an atom absorbing a photon of frequency γ in the rest frame of the atom. The normalized absorption profile is therefore given by

$$\phi_\nu(\mathbf{r}, \mathbf{\Omega}) = \int F_L(\mathbf{r}, \mathbf{\Omega}) q(\gamma) d^3v, \quad (4)$$

where, to first order in $\Delta \equiv v_0/c$,

$$\gamma = \nu - \Delta \mathbf{\Omega} \cdot \mathbf{v}. \quad (5)$$

Here, \mathbf{v} is the velocity of the atom in the observer's rest frame and $F_L(\mathbf{r}, \mathbf{\Omega})$ is the velocity distribution of the de-excited atoms. The function $F_L(\mathbf{r}, \mathbf{\Omega})$ is usually taken to be Maxwellian and, along with $q(\gamma)$, must be normalized to unity.

When considering an emission of a photon we assume that elastic collisions during the lifetime of the excited state, and the effect of the absorption of photons, will not change the velocity of the atom. Thus, neglecting polarization effects, the probability of an atom spontaneously emitting a photon of frequency $\nu' \pm d\nu'$ in direction $\mathbf{\Omega}' \pm d\mathbf{\Omega}'$ after absorbing a photon of frequency ν and direction $\mathbf{\Omega}$ is

$$g(\mathbf{\Omega}, \mathbf{\Omega}') p(\gamma, \gamma') d\nu' d\mathbf{\Omega}', \quad (6)$$

where

$$\gamma' = \nu' - \Delta \mathbf{\Omega}' \cdot \mathbf{v}, \quad (7)$$

$p(\gamma, \gamma')$ is the probability of a photon of frequency γ being re-emitted at frequency γ' , and $g(\mathbf{\Omega}, \mathbf{\Omega}')$ is the corresponding angular probability. Both $p(\gamma, \gamma')$ and $g(\mathbf{\Omega}, \mathbf{\Omega}')$ are normalized to unity. It is important to note that $g(\mathbf{\Omega}, \mathbf{\Omega}')$ may be written as $g(\mathbf{\Omega} \cdot \mathbf{\Omega}')$, that is, $g(\mathbf{\Omega}, \mathbf{\Omega}')$ is a function only of the angle between $\mathbf{\Omega}$ and $\mathbf{\Omega}'$.

Thus, using a slight variation of Hummer's (1962) notation, we have

$$R_\nu(\nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}') = g(\mathbf{\Omega}, \mathbf{\Omega}') q(\gamma) p(\gamma, \gamma') / 4\pi, \quad (8)$$

$$R(\mathbf{r}; \nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}') = \int F_L(\mathbf{r}, \mathbf{\Omega}) R_\nu(\nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}') d^3v, \quad (9)$$

where $R(\mathbf{r}; \nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}') d\nu d\mathbf{\Omega} d\nu' d\mathbf{\Omega}'$ is the total probability of an atom absorbing a photon of frequency $\nu \pm d\nu$ and direction $\mathbf{\Omega} \pm d\mathbf{\Omega}$ and spontaneously re-emitting a photon of frequency $\nu' \pm d\nu'$ and direction $\mathbf{\Omega}' \pm d\mathbf{\Omega}'$; $R(\mathbf{r}; \nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}')$ and $R_\nu(\nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}')$ are both normalized to unity. Thus an alternative form of the absorption coefficient given by equation (4) is

$$\phi_\nu(\mathbf{r}, \mathbf{\Omega}) = 4\pi \int_{\nu_c} \int_{\mathbf{\Omega}_c} R(\mathbf{r}; \nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}') d\nu' d\mathbf{\Omega}', \quad (10)$$

where the above integrals are taken over the complete range of ν' and $\mathbf{\Omega}'$ specified by ν_c and $\mathbf{\Omega}_c$ respectively.

Following Oxenius (1965), we may further define a function $\chi(\mathbf{r}, \mathbf{v}; \mathbf{\Omega}, \nu)$, normalized to unity and analogous to $q(\gamma)$, such that

$$\psi_\nu(\mathbf{r}, \mathbf{\Omega}) = \int F_U(\mathbf{r}, \mathbf{v}) \chi(\mathbf{r}, \mathbf{v}; \mathbf{\Omega}, \nu) d^3v, \quad (11)$$

where $F_U(\mathbf{r}, \mathbf{v})$ is the velocity distribution of the excited state. This is also usually taken to be Maxwellian. Oxenius shows that $\chi(\mathbf{r}, \mathbf{v}; \mathbf{\Omega}, \nu)$ is given by

$$\chi(\mathbf{r}, \mathbf{v}; \mathbf{\Omega}, \nu) = \frac{B_{LU} \int_{\nu_c} \int_{\mathbf{\Omega}_c} I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') R_v(\nu', \mathbf{\Omega}'; \nu, \mathbf{\Omega}) d\nu' d\mathbf{\Omega}' + C_{LU} q(\gamma)}{\frac{B_{LU}}{4\pi} \int_{\nu_c} \int_{\mathbf{\Omega}_c} I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') q(\gamma') d\nu' d\mathbf{\Omega}' + C_{LU}}, \quad (12)$$

where the collisional rate of excitation C_{LU} is directly proportional to the electron density, the cross section for electron collision, and a factor involving the electron temperature (see e.g. Jefferies 1968). The first term in the numerator corresponds to the emission of photons following the excitation of atoms by a previous absorption of photons. The second term corresponds to collisional excitation. The denominator is essentially the normalization factor.

Before proceeding further, we first replace the rather cumbersome expressions for the profiles $\phi_\nu(\mathbf{r}, \mathbf{\Omega})$ and $\psi_\nu(\mathbf{r}, \mathbf{\Omega})$ by the following more convenient functionals in \mathbf{r} , $\mathbf{\Omega}$, and ν . Equation (10) may be written as

$$\phi_\nu(\mathbf{r}, \mathbf{\Omega}) = 4\pi \int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' \int d^3v F_L(\mathbf{r}, \mathbf{v}) R_v(\nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}') \quad (13)$$

$$= F_\phi(\mathbf{r}, \mathbf{\Omega}, \nu), \quad \mathbf{\Omega} \in \mathbf{\Omega}_c, \quad \nu \in \nu_c, \quad (14)$$

$$\equiv F_\phi \left(\int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' R_v(\nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}') \right),$$

while similarly, from equations (11) and (12),

$$\psi_\nu(\mathbf{r}, \mathbf{\Omega}) = F_\psi(\mathbf{r}, \mathbf{\Omega}, \nu) \quad (15)$$

$$\equiv F_\psi \left(\int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') R_v(\nu', \mathbf{\Omega}'; \nu, \mathbf{\Omega}); \int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') q(\gamma'); q(\gamma) \right), \quad \mathbf{\Omega} \in \mathbf{\Omega}_c, \quad \nu \in \nu_c.$$

The functionals F_ϕ and F_ψ have been introduced only for clarity in exposition, as will be obvious in Section III. Their functional form is given explicitly by equation (13) for F_ϕ and by equations (11) and (12) for F_ψ , whereas their arguments are given implicitly by equations (14) and (15).

We are now in a position to determine the ratio N_L/N_U . The equation of statistical equilibrium for a two-level atom is (see e.g. Thomas 1957)

$$\begin{aligned}
N_L \left(B_{LU} \int_{\nu_c} \int_{\Omega_c} I_\nu(\mathbf{r}, \boldsymbol{\Omega}) \phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \, d\nu \, d\boldsymbol{\Omega} + C_{LU} \right) \\
= N_U \left(B_{UL} \int_{\nu_c} \int_{\Omega_c} I_\nu(\mathbf{r}, \boldsymbol{\Omega}) \psi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \, d\nu \, d\boldsymbol{\Omega} + A_{UL} + C_{UL} \right). \quad (16)
\end{aligned}$$

At this stage in the analysis it is usual in the literature to make various simplifying assumptions concerning the profiles $\phi_\nu(\mathbf{r}, \boldsymbol{\Omega})$ and $\psi_\nu(\mathbf{r}, \boldsymbol{\Omega})$. For example, equating the two profiles with $j_\nu(\mathbf{r}, \boldsymbol{\Omega})$ at all \mathbf{r} , with $\boldsymbol{\Omega} \in \Omega_c$ and $\nu \in \nu_c$, enables the source function $S_\nu(\mathbf{r}, \boldsymbol{\Omega})$ given by equation (3) to be written in the well-known linear form

$$S_\nu(\mathbf{r}, \boldsymbol{\Omega}) = S(\mathbf{r}) = (1 - \epsilon) \int_{\nu_c} \int_{\Omega_c} \phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \bar{J}(\mathbf{r}) \, d\nu \, d\boldsymbol{\Omega} + \epsilon B_{\nu_0}(T_e), \quad (17)$$

where

$$\bar{J}(\mathbf{r}) = \frac{1}{4\pi} \int_{\nu_c} \int_{\Omega_c} I_\nu(\mathbf{r}, \boldsymbol{\Omega}) \, d\nu \, d\boldsymbol{\Omega}, \quad (18)$$

ϵ is essentially the ratio of collisional to radiative de-excitation, and $B_{\nu_0}(T_e)$ is the Planck function at the electron temperature T_e . Equation (17) gives the form of the "frequency-independent" source function used extensively by many authors under the assumption of complete redistribution. The corresponding equation of transfer has been solved by many different methods, two of which have been modified to treat rather idealized departures from complete redistribution (Hearn 1963, 1964; Hummer 1969). The Feautrier technique, however, has not, as yet, been further generalized to solve problems involving more complicated source functions than that given by equation (17) above.

In this paper we wish to retain the generality of $\phi_\nu(\mathbf{r}, \boldsymbol{\Omega})$ and $\psi_\nu(\mathbf{r}, \boldsymbol{\Omega})$ given by equations (14) and (15), and thence to investigate the possible applicability of the Feautrier technique in the most general case. To do this it is again convenient to use the functional notation so that the ratio N_L/N_U can be written in the form

$$\begin{aligned}
N_L/N_U &= F(\mathbf{r}) \\
&= F \left(\int_{\nu_c} \int_{\Omega_c} d\nu \, d\boldsymbol{\Omega} \, I_\nu(\mathbf{r}, \boldsymbol{\Omega}) F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu); \int_{\nu_c} \int_{\Omega_c} d\nu \, d\boldsymbol{\Omega} \, I_\nu(\mathbf{r}, \boldsymbol{\Omega}) F_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \right). \quad (19)
\end{aligned}$$

The equation of transfer (equation (1)), along with equations (2), (3), (14), (15), and (19), then becomes

$$\begin{aligned}
\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa(\mathbf{r})} \nabla \right) I_\nu(\mathbf{r}, \boldsymbol{\Omega}) &= - \left(1 - \frac{F_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)}{F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) F(\mathbf{r})} \right) \\
&\times \left(I_\nu(\mathbf{r}, \boldsymbol{\Omega}) - \frac{2h\nu_0^3/c^2}{\frac{F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) F(\mathbf{r}) B_{LU}}{F_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) B_{UL}} - 1} \right) F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu), \quad (20)
\end{aligned}$$

for $\Omega \in \Omega_c$ and $\nu \in \nu_c$. In equation (20)

$$\kappa(\mathbf{r}) = (\hbar\nu_0/4\pi)N_L(\mathbf{r}) \quad (21)$$

and we have taken

$$j_\nu(\mathbf{r}, \Omega) \equiv \psi_\nu(\mathbf{r}, \Omega), \quad \Omega \in \Omega_c, \quad \nu \in \nu_c, \quad (22)$$

for all \mathbf{r} . This latter identity has been proved by Oxenius (1965). We have also used the well-known relationship $A_{UL}/B_{UL} = 2\hbar\nu_0^3/c^2$. The equation of transfer given by (20) is now in the required form.

III. FEAUTRIER FORMULATION

The Feautrier technique requires the radiation field to be divided into the positive and negative components of Ω . Following this procedure therefore equation (20) becomes

$$\begin{aligned} \pm \left(\Omega \cdot \frac{1}{\kappa} \nabla \right) I_\nu(\mathbf{r}, \pm \Omega) = & - \left(1 - \frac{F_\psi(\mathbf{r}, \pm \Omega, \nu)}{F_\phi(\mathbf{r}, \pm \Omega, \nu) F(\mathbf{r})} \right) \\ & \times \left(I_\nu(\mathbf{r}, \pm \Omega) - \frac{(2\hbar\nu_0^3/c^2) F_\psi(\mathbf{r}, \pm \Omega, \nu)}{F_\phi(\mathbf{r}, \pm \Omega, \nu) F(\mathbf{r}) B_{LU}/B_{UL} - F_\psi(\mathbf{r}, \pm \Omega, \nu)} \right) F_\phi(\mathbf{r}, \pm \Omega, \nu), \end{aligned} \quad (23)$$

for all $\nu \in \nu_c$ and $\Omega \in \Omega_h$, where Ω is now taken over the positive half-range Ω_h .

To proceed further, certain symmetry conditions pertaining to the redistribution functionals F_ϕ and F_ψ need to be derived. From equations (8) and (13) we obtain

$$\begin{aligned} F_\phi(\mathbf{r}, \Omega, \nu) = & \int_{\nu_c'} \int_{\Omega_c'} d\nu' d\Omega' \int d^3v F_L(\mathbf{r}, v) \\ & \times g(\Omega \cdot \Omega') q(\nu - \Delta\Omega \cdot v) p(\nu - \Delta\Omega \cdot v, \nu' - \Delta\Omega' \cdot v), \end{aligned} \quad (24)$$

so that

$$\begin{aligned} F_\phi(\mathbf{r}, -\Omega, \nu) = & \int_{\nu_c'} \int_{\Omega_c'} d\nu' d\Omega' \int d^3v F_L(\mathbf{r}, v) \\ & \times g(-\Omega \cdot \Omega') q(\nu + \Delta\Omega \cdot v) p(\nu + \Delta\Omega \cdot v, \nu' - \Delta\Omega' \cdot v). \end{aligned} \quad (25)$$

We now assume the velocity distribution given by $F_L(\mathbf{r}, v)$ to be Maxwellian in form, i.e.

$$F_L(\mathbf{r}, v) \propto \exp\{-\beta(\mathbf{r}) \mathbf{v} \cdot \mathbf{v}\}, \quad (26)$$

where $\beta(\mathbf{r})$ is an explicit function of \mathbf{r} only. Obviously then

$$F_L(\mathbf{r}, -v) = F_L(\mathbf{r}, v), \quad (27)$$

and thus a change of variable from Ω' to $-\Omega'$ and v to $-v$ in equation (25) yields

$$F_\phi(\mathbf{r}, -\Omega, \nu) \equiv F_\phi(\mathbf{r}, \Omega, \nu), \quad \Omega \in \Omega_h, \quad \nu \in \nu_c. \quad (28)$$

This result also follows from equation (4).

The functional $F_\psi(\mathbf{r}, \mathbf{\Omega}, \nu)$ does not satisfy the same type of symmetry relationship because of the appearance of the radiation field $I_\nu(\mathbf{r}, \mathbf{\Omega})$. This may be more easily seen by combining equations (11), (12), and (15) to give

$$F_\psi(\mathbf{r}, \mathbf{\Omega}, \nu) = \int F_U(\mathbf{r}, \mathbf{v}) \chi(\mathbf{r}, \mathbf{v}; \mathbf{\Omega}, \nu) d^3\mathbf{v}, \quad (29)$$

where

$$\begin{aligned} \chi(\mathbf{r}, \mathbf{v}; \mathbf{\Omega}, \nu) = & \left(\frac{B_{LU}}{4\pi} \int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') g(\mathbf{\Omega} \cdot \mathbf{\Omega}') q(\nu' - \Delta\mathbf{\Omega}' \cdot \mathbf{v}) \right. \\ & \left. \times p(\nu' - \Delta\mathbf{\Omega}' \cdot \mathbf{v}, \nu - \Delta\mathbf{\Omega} \cdot \mathbf{v}) + C_{LU} q(\nu - \Delta\mathbf{\Omega} \cdot \mathbf{v}) \right) \\ & \div \left(\frac{B_{LU}}{4\pi} \int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' I_\nu(\mathbf{r}, \mathbf{\Omega}') q(\nu' - \Delta\mathbf{\Omega}' \cdot \mathbf{v}) + C_{LU} \right). \end{aligned} \quad (30)$$

Thus, proceeding in the same manner as described above for $F_\phi(\mathbf{r}, \mathbf{\Omega}, \nu)$, equation (30) gives

$$\begin{aligned} \chi(\mathbf{r}, -\mathbf{v}; -\mathbf{\Omega}, \nu) = & \left(\frac{B_{LU}}{4\pi} \int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' I_{\nu'}(\mathbf{r}, -\mathbf{\Omega}') g(\mathbf{\Omega} \cdot \mathbf{\Omega}') q(\nu' - \Delta\mathbf{\Omega}' \cdot \mathbf{v}) \right. \\ & \left. \times p(\nu' - \Delta\mathbf{\Omega}' \cdot \mathbf{v}, \nu - \Delta\mathbf{\Omega} \cdot \mathbf{v}) + C_{LU} q(\nu - \Delta\mathbf{\Omega} \cdot \mathbf{v}) \right) \\ & \div \left(\frac{B_{LU}}{4\pi} \int_{\nu_c} \int_{\mathbf{\Omega}_c} d\nu' d\mathbf{\Omega}' I_{-\nu'}(\mathbf{r}, \mathbf{\Omega}') q(-\nu' + \Delta\mathbf{\Omega}' \cdot \mathbf{v}) + C_{LU} \right). \end{aligned} \quad (31)$$

The integrand of the first term appearing in the denominator in equation (31) has been written in the form shown, rather than $I_{\nu'}(\mathbf{r}, -\mathbf{\Omega}') q(\nu' - \Delta\mathbf{\Omega}' \cdot \mathbf{v})$, because of the required mathematical operations on the function $\chi(\mathbf{r}, \mathbf{v}; \mathbf{\Omega}, \nu)$ to follow. This term may be written in a more convenient form if we assume symmetry of the scattering process in the rest frame of the atom about line centre, i.e.

$$g(-\gamma) p(-\gamma, -\gamma') = g(\gamma) p(\gamma, \gamma') \quad (32a)$$

and, in particular,

$$g(-\gamma) = g(\gamma). \quad (32b)$$

The above assumption is valid for all broadening mechanisms except that due to statistical broadening (see e.g. Kuhn 1962). However, although it therefore appears that the above symmetry relationship is a restriction of the Feautrier technique when applied to the "exact" line transfer equation, departures from symmetry, if they exist at all, will be very small and thus the application of the Feautrier technique should still be valid.

It is not difficult to show that equations (32) imply

$$I_{-\nu}(\mathbf{r}, \mathbf{\Omega}) \equiv I_\nu(\mathbf{r}, \mathbf{\Omega}), \quad \nu \in \nu_c, \quad \mathbf{\Omega} \in \mathbf{\Omega}_c, \quad (33)$$

so that, using the functional notation again,

$$F_{\psi}(\mathbf{r}, -\mathbf{\Omega}, \nu) = F_{\psi} \left(\int_{\nu_c} \int_{\mathbf{\Omega}'_c} d\nu' d\mathbf{\Omega}' I_{\nu'}(\mathbf{r}, -\mathbf{\Omega}') R_{\nu}(\nu', \mathbf{\Omega}'; \nu, \mathbf{\Omega}); \right. \\ \left. \int_{\nu'_c} \int_{\mathbf{\Omega}'_c} d\nu' d\mathbf{\Omega}' I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') q(\gamma'); q(\gamma) \right) \quad (34)$$

for all $\mathbf{\Omega} \in \mathbf{\Omega}_h$ and $\nu \in \nu_c$.

We are now in a position to study the transfer equations given by (23). These equations are obviously highly nonlinear in the unknown $I_{\nu}(\mathbf{r}, \mathbf{\Omega})$, and progress can only be made by recognizing certain features of the physics involved. Thus, to make the problem numerically tractable we assume that stimulated emission is small when compared with the absorption and spontaneous emission processes. This is a good approximation for many problems of astrophysical interest and is frequently made throughout the pertinent literature. Therefore we take

$$F_{\psi}(\mathbf{r}, \pm\mathbf{\Omega}, \nu) / F_{\phi}(\mathbf{r}, \pm\mathbf{\Omega}, \nu) F(\mathbf{r}) \ll 1, \quad (35)$$

so that

$$\{F_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu) - F_{\psi}(\mathbf{r}, -\mathbf{\Omega}, \nu)\} / F_{\phi}(\mathbf{r}, \pm\mathbf{\Omega}, \nu) F(\mathbf{r}) \approx 0. \quad (36)$$

A comparison of equations (15) and (34) shows that equation (36) is exact for isotropic $I_{\nu}(\mathbf{r}, \mathbf{\Omega})$. Note that we have not neglected stimulated emission but rather have essentially taken the difference between $F_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu)$ and $F_{\psi}(\mathbf{r}, -\mathbf{\Omega}, \nu)$, for $\mathbf{\Omega} \in \mathbf{\Omega}_h$, to be small compared with the product $F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) F(\mathbf{r})$. Equation (36) may therefore be written in the form

$$\frac{F_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu)}{F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) F(\mathbf{r})} \approx \frac{F_{\psi}(\mathbf{r}, -\mathbf{\Omega}, \nu)}{F_{\phi}(\mathbf{r}, -\mathbf{\Omega}, \nu) F(\mathbf{r})} \approx \frac{\bar{F}_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu)}{F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) F(\mathbf{r})}, \quad (37)$$

where

$$\bar{F}_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu) = \frac{1}{2}\{F_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu) + F_{\psi}(\mathbf{r}, -\mathbf{\Omega}, \nu)\}, \quad \mathbf{\Omega} \in \mathbf{\Omega}_h, \quad \nu \in \nu_c. \quad (38)$$

Thus, defining the two functions $\Phi(\mathbf{r}, \mathbf{\Omega}, \nu)$ and $\Psi(\mathbf{r}, \mathbf{\Omega}, \nu)$ by

$$\Phi(\mathbf{r}, \mathbf{\Omega}, \nu) = \frac{1}{2}\{I_{\nu}(\mathbf{r}, \mathbf{\Omega}) + I_{\nu}(\mathbf{r}, -\mathbf{\Omega})\} \quad (39)$$

and

$$\Psi(\mathbf{r}, \mathbf{\Omega}, \nu) = \frac{1}{2}\{I_{\nu}(\mathbf{r}, \mathbf{\Omega}) - I_{\nu}(\mathbf{r}, -\mathbf{\Omega})\} \quad (40)$$

for all $\mathbf{\Omega} \in \mathbf{\Omega}_h$ and $\nu \in \nu_c$, equation (23), along with equations (28) and (37), gives

$$\left(\mathbf{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \Phi(\mathbf{r}, \mathbf{\Omega}, \nu) = - \left(1 - \frac{\bar{F}_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu)}{F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) F(\mathbf{r})} \right) \Psi(\mathbf{r}, \mathbf{\Omega}, \nu) F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) \quad (41)$$

and

$$\left(\mathbf{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \Psi(\mathbf{r}, \mathbf{\Omega}, \nu) = - \left(1 - \frac{\bar{F}_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu)}{F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) F(\mathbf{r})} \right) \\ \times \left(\Phi(\mathbf{r}, \mathbf{\Omega}, \nu) - \frac{(2h\nu_0^3/c^2) \bar{F}_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu)}{F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) F(\mathbf{r}) B_{LU}/B_{UL} - \bar{F}_{\psi}(\mathbf{r}, \mathbf{\Omega}, \nu)} \right) F_{\phi}(\mathbf{r}, \mathbf{\Omega}, \nu) \quad (42)$$

for all $\mathbf{\Omega} \in \mathbf{\Omega}_h$ and $\nu \in \nu_c$.

Equations (41) and (42) form the basic set of Feautrier equations. They are, however, more readily recognizable in their second-order form. Before deriving this form, we first simplify the above expressions. For example, equations (15) and (34) give after rearrangement

$$\begin{aligned} \bar{F}_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) = F_\psi \left(\int_{\nu_c} \int_{\boldsymbol{\Omega}_h} d\nu' d\boldsymbol{\Omega}' \Phi(\mathbf{r}, \boldsymbol{\Omega}', \nu') \{R_v(\nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) + R_v(\nu', -\boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega})\}; \right. \\ \left. 2 \int_{\nu_c} \int_{\boldsymbol{\Omega}_h} d\nu' d\boldsymbol{\Omega}' \bar{\Phi}(\mathbf{r}, \boldsymbol{\Omega}', \nu') q(\gamma'); q(\gamma) \right). \end{aligned} \quad (43)$$

The function $F(\mathbf{r})$ specified by equation (19) may also be written, although in a somewhat more complicated form, as an explicit functional of $\Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$ and $\Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$. However, under the approximation giving rise to equations (36) and (37), we find the simpler relationship

$$\begin{aligned} F(\mathbf{r}) = F \left(2 \int_{\nu_c} \int_{\boldsymbol{\Omega}_h} d\nu d\boldsymbol{\Omega} \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu); \right. \\ \left. 2 \int_{\nu_c} \int_{\boldsymbol{\Omega}_h} d\nu d\boldsymbol{\Omega} \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \bar{F}_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \right). \end{aligned} \quad (44)$$

Equations (43) and (44) show that $\bar{F}_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$ and $F(\mathbf{r})$ may be written as explicit functionals of only $\Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$ such that, writing

$$Q_1(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Phi) = \kappa(\mathbf{r}) \left(1 - \frac{\bar{F}_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)}{F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) F(\mathbf{r})} \right) F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \quad (45)$$

and

$$Q_2(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Phi) = \frac{(2h\nu_0^3/c^2) \bar{F}_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)}{F_\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) F(\mathbf{r}) B_{LU}/B_{UL} - \bar{F}_\psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)} \quad (46)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$ and $\nu \in \nu_c$, equations (41) and (42) may be written as

$$(\boldsymbol{\Omega} \cdot \{Q_1(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Phi)\}^{-1} \nabla) \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) = -\Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \quad (47)$$

and

$$(\boldsymbol{\Omega} \cdot \{Q_1(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Phi)\}^{-1} \nabla) \Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) = -\Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) + Q_2(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Phi). \quad (48)$$

These may then be transformed into the well-known second-order Feautrier equation in $\Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$

$$(\boldsymbol{\Omega} \cdot Q_1^{-1} \nabla)(\boldsymbol{\Omega} \cdot Q_1^{-1} \nabla) \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) = \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) - Q_2(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Phi) \quad (49)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$ and $\nu \in \nu_c$. Equations of this form have been discussed in detail in Part I, and are considered further in the following section.

Finally, the boundary constraints are given by equations (47) and (48) in the usual manner together with the relationships at the boundary \mathbf{r}_s :

$$\Psi(\mathbf{r}_s^{(1)}, \boldsymbol{\Omega}, \nu) = \Phi(\mathbf{r}_s^{(1)}, \boldsymbol{\Omega}, \nu) - I_\nu(\mathbf{r}_s^{(1)}, -\boldsymbol{\Omega}) \quad (50)$$

and

$$\Psi(\mathbf{r}_s^{(2)}, \boldsymbol{\Omega}, \nu) = I_\nu(\mathbf{r}_s^{(2)}, \boldsymbol{\Omega}) - \Phi(\mathbf{r}_s^{(2)}, \boldsymbol{\Omega}, \nu) \quad (51)$$

for all $\Omega \in \Omega_h$ and $\nu \in \nu_c$. The problem can then be solved numerically when $I_\nu(r_s^{(1)}, -\Omega)$ is specified at the surface $r_s^{(1)}$ and, for example, $I_\nu(r_s^{(2)}, \Omega)$ is specified sufficiently deep within a semi-infinite atmosphere.

To proceed further, one can (1) solve equation (49) in its nonlinear form (this has not, as yet, been attempted by the Feautrier or any other method), or (2) make the usual assumptions and approximations to reduce the equation to a linear form, or (3) consider an alternative form of the transfer equation which still exhibits all the pertinent physical microscopic processes. In the next section we consider the third suggestion in detail.

IV. ALTERNATIVE FORM OF EQUATION OF TRANSFER

In the preceding two sections we have derived the "exact" equation of transfer for spectral line radiation and have thence developed the corresponding second-order Feautrier system of equations. All these equations are found to be highly nonlinear in the required unknown which thus suggests that the computation of their solutions could be rather prohibitive. It should be emphasized, however, that the nonlinearity is not due to the application of the Feautrier technique since such nonlinearity is inherent in the exact line transfer equation and would need to be overcome by *all* methods applied to the problem. Therefore, in this section we wish to consider an alternative form of the line transfer equation which does not exhibit the above-mentioned nonlinearities. This form has been discussed in some detail by Hummer (1969) and a very brief summary is given here.

The fundamental assumption requires stimulated emission to be treated as negative absorption. This necessitates the equality of the profiles for stimulated emission and absorption, i.e.

$$\phi_\nu(r, \Omega) \equiv \psi_\nu(r, \Omega) \quad (52)$$

for all $r, \Omega \in \Omega_c$, and $\nu \in \nu_c$. Thus, in much the same manner as described in Section II (but without the identity given by equation (22)), the required transfer equation may be written in the form

$$(\Omega \cdot \nabla) I_\nu(r, \Omega) = -\kappa_\nu(r, \Omega) \{I_\nu(r, \Omega) - S_\nu(r, \Omega)\}, \quad (53)$$

with

$$\kappa_\nu(r, \Omega) = (h\nu_0/4\pi) \{N_L(r) B_{LU} - N_U(r) B_{UL}\} \phi_\nu(r, \Omega) \quad (54)$$

and

$$S_\nu(r, \Omega) = \frac{4\pi(1-\epsilon)}{\phi_\nu(r, \Omega)} \int_{\nu_c} \int_{\Omega_c} I_{\nu'}(r, \Omega') R(r; \nu', \Omega'; \nu, \Omega) d\nu' d\Omega' + \epsilon B_{\nu_0}(T_e), \quad (55)$$

where

$$\epsilon = C_{UL} / [C_{UL} + A_{UL} \{1 - \exp(-h\nu_0/kT_e)\}^{-1}]. \quad (56)$$

All the terms appearing in equations (54) and (55) have the same meaning as specified in Sections II and III. Equations (53), (54), and (55) are exact under the assumption given by equation (52). The salient feature of these equations is the linearity of the source function in $I_\nu(r, \Omega)$. The resulting equation of transfer, however, is not linear due to the appearance of $I_\nu(r, \Omega)$, occurring via the ratio N_U/N_L , in $\kappa_\nu(r, \Omega)$. This

difficulty may be overcome by defining

$$\kappa_\nu(\mathbf{r}, \boldsymbol{\Omega}) = \kappa(\mathbf{r}) \phi_\nu(\mathbf{r}, \boldsymbol{\Omega}), \quad (57)$$

where

$$\kappa(\mathbf{r}) = (\hbar\nu_0/4\pi)(N_L B_{LU} - N_U B_{UL}). \quad (58)$$

Equation (53) is then solved as a function of $\kappa(\mathbf{r})$.

We are now in a position to derive the Feautrier system of equations in terms of the new unknowns $\Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$ and $\Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$ specified by equations (39) and (40) respectively. Thus, dividing the radiation field into its positive and negative components of $\boldsymbol{\Omega}$, we find

$$\begin{aligned} \pm \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa(\mathbf{r})} \nabla \right) I_\nu(\mathbf{r}, \pm \boldsymbol{\Omega}) &= -\phi_\nu(\mathbf{r}, \pm \boldsymbol{\Omega}) I_\nu(\mathbf{r}, \pm \boldsymbol{\Omega}) \\ &+ 4\pi(1-\epsilon) \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} I_{\nu'}(\mathbf{r}, \boldsymbol{\Omega}') R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \pm \boldsymbol{\Omega}) d\nu' d\boldsymbol{\Omega}' \\ &+ \epsilon \phi_\nu(\mathbf{r}, \pm \boldsymbol{\Omega}) B_{\nu_0}(T_e) \end{aligned} \quad (59)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$ and $\nu \in \nu_c$. Subtracting and adding these equations, and using the symmetry of $\phi_\nu(\mathbf{r}, \boldsymbol{\Omega})$ specified by equation (28), we find

$$\begin{aligned} \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) &= -\phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\ &+ 2\pi(1-\epsilon) \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} I_{\nu'}(\mathbf{r}, \boldsymbol{\Omega}') \{ R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \\ &\quad - R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, -\boldsymbol{\Omega}) \} d\nu' d\boldsymbol{\Omega}' \end{aligned} \quad (60)$$

and

$$\begin{aligned} \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) &= -\phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\ &+ 2\pi(1-\epsilon) \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} I_{\nu'}(\mathbf{r}, \boldsymbol{\Omega}') \{ R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \\ &\quad + R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, -\boldsymbol{\Omega}) \} d\nu' d\boldsymbol{\Omega}' \\ &+ \epsilon \phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) B_{\nu_0}(T_e) \end{aligned} \quad (61)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$ and $\nu \in \nu_c$.

The integral terms appearing in equations (60) and (61) may be simplified using certain *general* symmetry conditions satisfied by the redistribution function. Equations (8) and (9) show that

$$\begin{aligned} R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) &= \frac{1}{4\pi} \int F_L(\mathbf{r}, \mathbf{v}) g(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') q(\nu' - \Delta \boldsymbol{\Omega}' \cdot \mathbf{v}) \\ &\quad \times p(\nu' - \Delta \boldsymbol{\Omega}' \cdot \mathbf{v}, \nu - \Delta \boldsymbol{\Omega} \cdot \mathbf{v}) d^3 \mathbf{v}. \end{aligned} \quad (62)$$

Obviously, if we have a symmetry condition on $F_L(\mathbf{r}, \mathbf{v})$ of the form given by equation

(27), then

$$R(\mathbf{r}; \nu', -\boldsymbol{\Omega}'; \nu, -\boldsymbol{\Omega}) \equiv R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \quad (63)$$

and

$$R(\mathbf{r}; \nu', \pm \boldsymbol{\Omega}'; \nu, \mp \boldsymbol{\Omega}) \equiv R(\mathbf{r}; \nu', \mp \boldsymbol{\Omega}'; \nu, \pm \boldsymbol{\Omega}) \quad (64)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_c$, $\boldsymbol{\Omega}' \in \boldsymbol{\Omega}'_c$, $\nu \in \nu_c$, and $\nu' \in \nu'_c$. Equations (60) and (61) after rearrangement then become

$$\begin{aligned} \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) &= -\phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\ &+ (1-\epsilon) \phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \int_{\nu'_c} \int_{\boldsymbol{\Omega}'_h} \Psi(\mathbf{r}, \boldsymbol{\Omega}', \nu') Q_3^-(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \, d\nu' \, d\boldsymbol{\Omega}' \end{aligned} \quad (65)$$

and

$$\begin{aligned} \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) &= -\phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\ &+ (1-\epsilon) \phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) \int_{\nu'_c} \int_{\boldsymbol{\Omega}'_h} \Phi(\mathbf{r}, \boldsymbol{\Omega}', \nu') Q_3^+(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \, d\nu' \, d\boldsymbol{\Omega}' \\ &+ \epsilon \phi_\nu(\mathbf{r}, \boldsymbol{\Omega}) B_{\nu_0}(T_e), \end{aligned} \quad (66)$$

where

$$Q_3^\pm(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) = \frac{4\pi}{\phi_\nu(\mathbf{r}, \boldsymbol{\Omega})} \left(R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \pm R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, -\boldsymbol{\Omega}) \right) \quad (67)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$, $\nu \in \nu_c$, $\boldsymbol{\Omega}' \in \boldsymbol{\Omega}'_h$, and $\nu' \in \nu'_c$.

Equations (65) and (66) are now linear in the required unknowns $\Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$ and $\Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$. Clearly, they may be reduced to the form

$$\begin{aligned} &\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi_\nu(\mathbf{r}, \boldsymbol{\Omega})} \nabla \right) \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi_\nu(\mathbf{r}, \boldsymbol{\Omega})} \nabla \right) \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\ &= \Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) - \frac{8\pi(1-\epsilon)}{\phi_\nu(\mathbf{r}, \boldsymbol{\Omega})} \int_{\nu'_c} \int_{\boldsymbol{\Omega}'_h} \Phi(\mathbf{r}, \boldsymbol{\Omega}', \nu') R(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \, d\nu' \, d\boldsymbol{\Omega}' \\ &- Q(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Psi), \end{aligned} \quad (68)$$

where

$$\begin{aligned} Q(\mathbf{r}, \boldsymbol{\Omega}, \nu; \Psi) &= \epsilon B_{\nu_0}(T_e) + \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi_\nu(\mathbf{r}, \boldsymbol{\Omega})} \nabla \right) \\ &\times \left((1-\epsilon) \int_{\nu'_c} \int_{\boldsymbol{\Omega}'_h} \Psi(\mathbf{r}, \boldsymbol{\Omega}', \nu') Q_3^-(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \, d\nu' \, d\boldsymbol{\Omega}' \right) \end{aligned} \quad (69)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$ and $\nu \in \nu_c$. A similar equation may be obtained in $\Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu)$.

These second-order differential equations are analogous to the ordinary Feautrier equations, and differ only in that they exhibit coupling terms in Φ and Ψ . However, this apparent difficulty may be easily overcome. The coupling terms are linear in Φ and Ψ and this, together with the use of the finite difference approach, enables a system of linear matrix equations in Φ and Ψ to be specified. The resulting expressions are then readily uncoupled (see Cannon 1970a) and thence the solution proceeds

in exactly the same manner as in the ordinary situation (Feautrier 1964) using the boundary conditions given by equations (65) and (66) together with equations (50) and (51). This point has been discussed in detail in Part I.

It has been shown by Fox (1957) and Wachspress (1960), and stressed by Cuny (1967), that expressions of the form given by equation (68), for example, are stable under numerical reduction using the above technique provided that

$$\frac{1}{\Phi(\mathbf{r}, \mathbf{\Omega}, \nu)} \left(\frac{8\pi(1-\epsilon)}{\phi_\nu(\mathbf{r}, \mathbf{\Omega})} \int_{\nu_c} \int_{\mathbf{\Omega}_h} \Phi(\mathbf{r}, \mathbf{\Omega}', \nu') R(\mathbf{r}; \nu', \mathbf{\Omega}'; \nu, \mathbf{\Omega}) d\nu' d\mathbf{\Omega}' \right) \lesssim 1, \quad (70)$$

at all \mathbf{r} . This will ensure that the diagonal elements of the matrices obtained are dominant, and therefore that the required matrix inversions are stable. The "approximately equal to" sign in expression (70) allows for the possibility of the left-hand side being slightly greater than unity for *some* values of ν and $\mathbf{\Omega}$. An inspection of equations (10) and (56) (noting also that $R(\mathbf{r}; \nu', \mathbf{\Omega}'; \nu, \mathbf{\Omega})$ is integrated over only the positive half-space $\mathbf{\Omega}'_h$) shows that expression (70) indeed holds, and thus the Feautrier equations obtained using the alternative form of the transfer equation satisfy the required stability criteria. The coupling terms do not affect the stability of the resulting equations.

A simpler form of the above equations may be derived by dividing the required redistribution function into its even and odd components of $\mathbf{\Omega}$. This is done in the Appendix. In fact, it can be shown that the coupling terms disappear altogether in the one-dimensional situation if the redistribution function is independent of position \mathbf{r} .

Further simple forms of equation (68) may be obtained under certain physical idealizations. For example, Hummer (1962) has shown that, for (1) zero natural line width and (2) radiation and collisional damping with complete redistribution in the rest frame of the atom, the symmetry condition

$$R(\mathbf{r}; -\nu', -\mathbf{\Omega}'; \nu, \mathbf{\Omega}) \equiv R(\mathbf{r}; \nu', \mathbf{\Omega}'; \nu, \mathbf{\Omega}) \quad (71)$$

for all $\mathbf{\Omega}' \in \mathbf{\Omega}'_c$, $\nu' \in \nu'_c$, $\mathbf{\Omega} \in \mathbf{\Omega}_c$, and $\nu \in \nu_c$ is automatically satisfied. This implies symmetry in the scattering process about line centre in the rest frame of the atom, so that, using equation (71), the integral term involving Q_3^- in equation (65) disappears. The resulting equation in $\Phi(\mathbf{r}, \mathbf{\Omega}, \nu)$ is then of the standard form

$$\begin{aligned} & \left(\mathbf{\Omega} \cdot \frac{1}{\kappa \phi_\nu(\mathbf{r}, \mathbf{\Omega})} \nabla \right) \left(\mathbf{\Omega} \cdot \frac{1}{\kappa \phi_\nu(\mathbf{r}, \mathbf{\Omega})} \nabla \right) \Phi(\mathbf{r}, \mathbf{\Omega}, \nu) \\ &= \Phi(\mathbf{r}, \mathbf{\Omega}, \nu) - \frac{8\pi(1-\epsilon)}{\phi_\nu(\mathbf{r}, \mathbf{\Omega})} \int_{\nu_c} \int_{\mathbf{\Omega}'_h} \Phi(\mathbf{r}, \mathbf{\Omega}', \nu') R(\mathbf{r}; \nu', \mathbf{\Omega}'; \nu, \mathbf{\Omega}) d\nu' d\mathbf{\Omega}' \\ & - \epsilon B_{\nu_0}(T_e). \end{aligned} \quad (72)$$

The determination of $\Phi(\mathbf{r}, \mathbf{\Omega}, \nu)$ in this particular case would completely determine the radiation field, that is, $\Psi(\mathbf{r}, \mathbf{\Omega}, \nu)$ would not need to be evaluated.

V. INCLUSION OF VELOCITY FIELD

The previous sections have been concerned with the equation of transfer for spectral line radiation in an atmosphere exhibiting zero mass motion. Here we wish to further generalize the Feautrier technique given not only a *general* redistribution law but a *general* multidimensional macroscopic differential velocity field. The method described applies to both the "exact" transfer equation discussed in Sections II and III and the alternative form given in Section IV. For clarity in exposition, however, and because of computational convenience, we consider only the alternative form. Identical arguments can be used to derive the Feautrier system in the so-called exact case.

It is not difficult to show that the required equation of transfer then has the form (see e.g. Hummer 1968)

$$\begin{aligned} \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa(\mathbf{r})} \nabla \right) I_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) = & -\phi_{\bar{\nu}}(\mathbf{r}, \boldsymbol{\Omega}) I_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) \\ & + 4\pi(1-\epsilon) \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} I_{\nu'}(\mathbf{r}, \boldsymbol{\Omega}') R(\mathbf{r}; \bar{\nu}', \boldsymbol{\Omega}'; \bar{\nu}, \boldsymbol{\Omega}) d\nu' d\boldsymbol{\Omega}' \\ & + \epsilon \phi_{\bar{\nu}}(\mathbf{r}, \boldsymbol{\Omega}) B_{\nu_0}(T_e) \end{aligned} \quad (73)$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_c$ and $\nu \in \nu_c$, where

$$\bar{\nu} \equiv \nu - \boldsymbol{\Delta} \boldsymbol{\Omega} \cdot \mathbf{V}(\mathbf{r}), \quad \bar{\nu}' \equiv \nu' - \boldsymbol{\Delta} \boldsymbol{\Omega}' \cdot \mathbf{V}(\mathbf{r}), \quad (74)$$

and $\mathbf{V}(\mathbf{r})$ is a general macroscopic velocity field as a function of position \mathbf{r} . In equation (73)

$$\begin{aligned} \phi_{\bar{\nu}}(\mathbf{r}, \boldsymbol{\Omega}) = & 4\pi \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} d\nu' d\boldsymbol{\Omega}' \int d^3v F_L(\mathbf{r}, \mathbf{v}) R_v(\bar{\nu}, \boldsymbol{\Omega}; \bar{\nu}', \boldsymbol{\Omega}') \\ = & \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} d\nu' d\boldsymbol{\Omega}' \int d^3v F_L(\mathbf{r}, \mathbf{v}) g(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') q(\nu - \boldsymbol{\Delta} \boldsymbol{\Omega} \cdot (\mathbf{v} + \mathbf{V})) \\ & \times p(\nu - \boldsymbol{\Delta} \boldsymbol{\Omega} \cdot (\mathbf{v} + \mathbf{V}), \nu' - \boldsymbol{\Delta} \boldsymbol{\Omega}' \cdot (\mathbf{v} + \mathbf{V})), \\ \equiv & \phi(\mathbf{r}, \boldsymbol{\Omega}, \nu, \mathbf{V}), \quad \boldsymbol{\Omega} \in \boldsymbol{\Omega}_c, \nu \in \nu_c, \end{aligned} \quad (75)$$

where we have used equations (8), (9), and (10). We can obviously write

$$\begin{aligned} \phi(\mathbf{r}, -\boldsymbol{\Omega}, -\nu, \mathbf{V}) = & \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} d\nu' d\boldsymbol{\Omega}' \int d^3v F_L(\mathbf{r}, \mathbf{v}) g(-\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') q(-\nu + \boldsymbol{\Delta} \boldsymbol{\Omega} \cdot (\mathbf{v} + \mathbf{V})) \\ & \times p(-\nu + \boldsymbol{\Delta} \boldsymbol{\Omega} \cdot (\mathbf{v} + \mathbf{V}), \nu' - \boldsymbol{\Delta} \boldsymbol{\Omega}' \cdot (\mathbf{v} + \mathbf{V})). \end{aligned} \quad (76)$$

To proceed further we assume that the scattering process is symmetric about line centre in the rest frame of the atom, i.e.

$$q(-\gamma) p(-\gamma, -\gamma') = q(\gamma) p(\gamma, \gamma'), \quad (77)$$

$$q(-\gamma) p(-\gamma, -\gamma') = q(\gamma) p(\gamma, \gamma'), \quad (78)$$

so that, with the usual symmetry condition on the Maxwellian distribution $F_L(\mathbf{r}, \mathbf{v})$, equation (77) gives

$$\phi(\mathbf{r}, -\mathbf{\Omega}, -\nu, V) \equiv \phi(\mathbf{r}, \mathbf{\Omega}, \nu, V), \quad \mathbf{\Omega} \in \mathbf{\Omega}_c, \quad \nu \in \nu_c. \quad (79)$$

Similarly, we have

$$\phi(\mathbf{r}, \pm\mathbf{\Omega}, \mp\nu, V) \equiv \phi(\mathbf{r}, \mp\mathbf{\Omega}, \pm\nu, V), \quad \mathbf{\Omega} \in \mathbf{\Omega}_c, \quad \nu \in \nu_c. \quad (80)$$

Returning our attention to equation (73), if the radiation field is divided into positive and negative components of $\mathbf{\Omega}$, we find

$$\begin{aligned} \pm \left(\mathbf{\Omega} \cdot \frac{1}{\kappa(\mathbf{r})} \nabla \right) I_{\pm\nu}(\mathbf{r}, \pm\mathbf{\Omega}) &= -\phi(\mathbf{r}, \pm\mathbf{\Omega}, \pm\nu, V) I_{\pm\nu}(\mathbf{r}, \pm\mathbf{\Omega}) \\ &+ 4\pi(1-\epsilon) \int_{\nu_c} \int_{\mathbf{\Omega}_c} I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') R(\mathbf{r}; \bar{\nu}', \mathbf{\Omega}'; \pm\bar{\nu}, \pm\mathbf{\Omega}) d\nu' d\mathbf{\Omega}' \\ &+ \epsilon\phi(\mathbf{r}, \pm\mathbf{\Omega}, \pm\nu, V) B_{\nu_0}(T_e) \end{aligned} \quad (81)$$

for all $\mathbf{\Omega} \in \mathbf{\Omega}_h$ and $\nu \in \nu_c$. It is important to note that, although $\mathbf{\Omega}$ has now been restricted to the positive half-space $\mathbf{\Omega}_h$, ν still ranges over the complete line profile of frequency ν_c .

Thus, defining the new unknowns $\bar{\Phi}$ and $\bar{\Psi}$ by

$$\bar{\Phi}(\mathbf{r}, \mathbf{\Omega}, \nu) = \frac{1}{2}\{I_{\nu}(\mathbf{r}, \mathbf{\Omega}) + I_{-\nu}(\mathbf{r}, -\mathbf{\Omega})\} \quad (82)$$

and

$$\bar{\Psi}(\mathbf{r}, \mathbf{\Omega}, \nu) = \frac{1}{2}\{I_{\nu}(\mathbf{r}, \mathbf{\Omega}) - I_{-\nu}(\mathbf{r}, -\mathbf{\Omega})\} \quad (83)$$

for all $\mathbf{\Omega} \in \mathbf{\Omega}_h$ and $\nu \in \nu_c$, equation (81), together with equations (79) and (80), yields

$$\begin{aligned} \left(\mathbf{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \bar{\Phi}(\mathbf{r}, \mathbf{\Omega}, \nu) &= -\phi(\mathbf{r}, \mathbf{\Omega}, \nu, V) \bar{\Psi}(\mathbf{r}, \mathbf{\Omega}, \nu) \\ &+ 2\pi(1-\epsilon) \int_{\nu_c} \int_{\mathbf{\Omega}_c} I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') \{R(\mathbf{r}; \bar{\nu}', \mathbf{\Omega}'; \bar{\nu}, \mathbf{\Omega}) \\ &\quad - R(\mathbf{r}; \bar{\nu}', \mathbf{\Omega}'; -\bar{\nu}, -\mathbf{\Omega})\} d\nu' d\mathbf{\Omega}' \end{aligned} \quad (84)$$

and

$$\begin{aligned} \left(\mathbf{\Omega} \cdot \frac{1}{\kappa} \nabla \right) \bar{\Psi}(\mathbf{r}, \mathbf{\Omega}, \nu) &= -\phi(\mathbf{r}, \mathbf{\Omega}, \nu, V) \bar{\Phi}(\mathbf{r}, \mathbf{\Omega}, \nu) \\ &+ 2\pi(1-\epsilon) \int_{\nu_c} \int_{\mathbf{\Omega}_c} I_{\nu'}(\mathbf{r}, \mathbf{\Omega}') \{R(\mathbf{r}; \bar{\nu}', \mathbf{\Omega}'; \bar{\nu}, \mathbf{\Omega}) \\ &\quad + R(\mathbf{r}; \bar{\nu}', \mathbf{\Omega}'; -\bar{\nu}, -\mathbf{\Omega})\} d\nu' d\mathbf{\Omega}' \\ &+ \epsilon\phi(\mathbf{r}, \mathbf{\Omega}, \nu, V) B_{\nu_0}(T_e) \end{aligned} \quad (85)$$

for all $\mathbf{\Omega} \in \mathbf{\Omega}_h$ and $\nu \in \nu_c$.

The integrals appearing in equations (84) and (85) may be simplified using the *general* symmetry conditions satisfied by the redistribution function and given by equations (63) and (64). The resulting equations become, after a rather lengthy rearrangement,

$$\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla\right) \bar{\Phi}(\mathbf{r}, \boldsymbol{\Omega}, \nu) = -\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu, V) \bar{\Psi}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\ + (1-\epsilon)\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu, V) \int_{\nu_c} \int_{\boldsymbol{\Omega}'_h} \bar{\Psi}(\mathbf{r}, \boldsymbol{\Omega}', \nu') \bar{Q}_3^-(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) d\nu' d\boldsymbol{\Omega}' \quad (86)$$

and

$$\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla\right) \bar{\Psi}(\mathbf{r}, \boldsymbol{\Omega}, \nu) = -\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu, V) \bar{\Phi}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\ + (1-\epsilon)\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu, V) \int_{\nu_c} \int_{\boldsymbol{\Omega}'_h} \bar{\Phi}(\mathbf{r}, \boldsymbol{\Omega}', \nu') \bar{Q}_3^+(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) d\nu' d\boldsymbol{\Omega}' \quad (87)$$

$$+ \epsilon\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu, V) B_{\nu_0}(T_e),$$

where

$$\bar{Q}_3^\pm(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) = \frac{4\pi}{\phi(\mathbf{r}, \boldsymbol{\Omega}, \nu, V)} \left(R(\mathbf{r}; \nu' - \boldsymbol{\Omega}' \cdot (\mathbf{v} + \mathbf{V}), \boldsymbol{\Omega}'; \nu - \Delta\boldsymbol{\Omega} \cdot (\mathbf{v} + \mathbf{V}), \boldsymbol{\Omega}) \right. \\ \left. \pm R(\mathbf{r}; \nu' - \boldsymbol{\Omega}' \cdot (\mathbf{v} + \mathbf{V}), \boldsymbol{\Omega}'; -\nu + \Delta\boldsymbol{\Omega} \cdot (\mathbf{v} + \mathbf{V}), -\boldsymbol{\Omega}) \right), \quad (88)$$

for all $\boldsymbol{\Omega}' \in \boldsymbol{\Omega}'_h$, $\nu' \in \nu'_c$, $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$, and $\nu \in \nu_c$.

Equations (86), (87), and (88) are identical in structure to equations (65), (66), and (67) respectively, which are for the case of zero mass motion. One may therefore proceed to develop the Feautrier system of second-order differential equations in exactly the same manner as indicated in Section IV. In fact, a computer program written to solve the transfer equation for zero mass motion may be readily modified to consider the velocity-dependent case by redefining the Φ and Ψ in the form $\bar{\Phi}$ and $\bar{\Psi}$ and replacing Q_3^\pm by \bar{Q}_3^\pm . This has been done, for example, for the case of complete redistribution in a two-dimensional atmosphere by Cannon and Rees (1971) using the technique developed by Cannon (1970a) for stationary media. The stability criteria (expression (70)) are still satisfied in the velocity-dependent problem.

VI. CONCLUSIONS

It has been shown that the equation of transfer for spectral line radiation given a *general* redistribution law may be written as a Feautrier system of second-order differential equations with two-point boundary conditions. These equations are highly nonlinear in the unknown to be determined. However, as this nonlinearity is not dependent upon the actual Feautrier technique but rather is inherent in the exact specification of the line transfer problem, a well-known "alternative" form of the transfer equation more amenable to numerical solution has been studied. It has been shown that this equation may also be solved by the Feautrier technique. The

resulting second-order differential equations are linear in the required unknown and satisfy the necessary stability criteria, and thus are exceedingly stable and efficient when solved numerically.

Multidimensional macroscopic differential velocity fields have also been included in the analysis. The equations so obtained using the Feautrier technique have an identical functional form with that mentioned above for the stationary situation, and thus may be solved by the same well-developed numerical methods.

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APPENDIX

Equation (68) in Section IV may be derived in a simpler form. This is done by separating the general redistribution function into its even and odd components of Ω , that is,

$$R(\mathbf{r}; \nu', \Omega'; \nu, \Omega) = R^{(1)}(\mathbf{r}; \nu', \Omega'; \nu, \Omega) + R^{(2)}(\mathbf{r}; \nu', \Omega'; \nu, \Omega), \quad (\text{A1})$$

where

$$R^{(1)}(\mathbf{r}; \nu', \Omega'; \nu, -\Omega) = R^{(1)}(\mathbf{r}; \nu', \Omega'; \nu, \Omega) \quad (\text{A2})$$

and

$$R^{(2)}(\mathbf{r}; \nu', \Omega'; \nu, -\Omega) = -R^{(2)}(\mathbf{r}; \nu', \Omega'; \nu, \Omega) \quad (\text{A3})$$

for all $\Omega' \in \Omega'_c$, $\nu' \in \nu'_c$, $\Omega \in \Omega_c$, and $\nu \in \nu_c$. The new redistribution functions $R^{(1)}$ and $R^{(2)}$ both satisfy the symmetry conditions given by equations (63) and (64).

In a multidimensional situation we may proceed to solve the problem by rewriting equations (53), (54), and (55) in the form

$$\begin{aligned}
\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla\right) I_{\nu}^{(k)}(\mathbf{r}, \boldsymbol{\Omega}) &= -\phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) I_{\nu}^{(k)}(\mathbf{r}, \boldsymbol{\Omega}) \\
&+ 4\pi(1-\epsilon) \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} I_{\nu'}(\mathbf{r}, \boldsymbol{\Omega}') R^{(k)}(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) d\nu' d\boldsymbol{\Omega}' \\
&+ \epsilon \phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) B_{\nu_0}(T_e), \quad k = 1, 2, \quad \boldsymbol{\Omega} \in \boldsymbol{\Omega}_c, \quad \nu \in \nu_c,
\end{aligned} \tag{A4}$$

where

$$I_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) = I_{\nu}^{(1)}(\mathbf{r}, \boldsymbol{\Omega}) + I_{\nu}^{(2)}(\mathbf{r}, \boldsymbol{\Omega}). \tag{A5}$$

Thus, proceeding as in Sections III and IV, equation (A4) may be reduced to

$$\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla\right) \Phi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) = -\phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) \Psi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu), \tag{A6}$$

$$\begin{aligned}
\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla\right) \Psi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) &= -\phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) \Phi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\
&+ 8\pi(1-\epsilon) \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} \Phi(\mathbf{r}, \boldsymbol{\Omega}', \nu') R^{(1)}(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) d\nu' d\boldsymbol{\Omega}' \\
&+ \epsilon \phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) B_{\nu_0}(T_e),
\end{aligned} \tag{A7}$$

$$\begin{aligned}
\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla\right) \Phi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) &= -\phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) \Psi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\
&+ 8\pi(1-\epsilon) \int_{\nu_c} \int_{\boldsymbol{\Omega}_c} \Psi(\mathbf{r}, \boldsymbol{\Omega}', \nu') R^{(2)}(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) d\nu' d\boldsymbol{\Omega}',
\end{aligned} \tag{A8}$$

and

$$\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa} \nabla\right) \Psi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) = -\phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega}) \Phi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu), \tag{A9}$$

where, with obvious notation,

$$\Phi(\mathbf{r}, \boldsymbol{\Omega}, \nu) = \Phi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) + \Phi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \tag{A10}$$

and

$$\Psi(\mathbf{r}, \boldsymbol{\Omega}, \nu) = \Psi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) + \Psi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \tag{A11}$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$ and $\nu \in \nu_c$, and where we have used the general redistribution symmetry relationships given by equations (63) and (64).

The above equations then reduce to two coupled equations in $\Phi^{(1)}$ and $\Psi^{(2)}$ of the form

$$\begin{aligned}
&\left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi} \nabla\right) \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi} \nabla\right) \Phi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\
&= \Phi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) - \frac{8\pi(1-\epsilon)}{\phi_{\nu}(\mathbf{r}, \boldsymbol{\Omega})} \int_{\nu_c} \int_{\boldsymbol{\Omega}_h} R^{(1)}(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \\
&\quad \times \left\{ \Phi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}', \nu') - \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi_{\nu'}(\mathbf{r}, \boldsymbol{\Omega}')} \nabla\right) \Psi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}', \nu') \right\} d\nu' d\boldsymbol{\Omega}' \\
&\quad - \epsilon B_{\nu_0}(T_e)
\end{aligned} \tag{A12}$$

and

$$\begin{aligned}
 & \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi} \nabla \right) \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi} \nabla \right) \Psi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) \\
 &= \Psi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}, \nu) - \frac{8\pi(1-\epsilon)}{\phi_\nu(\mathbf{r}, \boldsymbol{\Omega})} \int_{\nu_c} \int_{\boldsymbol{\Omega}_h} R^{(2)}(\mathbf{r}; \nu', \boldsymbol{\Omega}'; \nu, \boldsymbol{\Omega}) \\
 & \quad \times \left\{ \Psi^{(2)}(\mathbf{r}, \boldsymbol{\Omega}', \nu') - \left(\boldsymbol{\Omega} \cdot \frac{1}{\kappa \phi_{\nu'}(\mathbf{r}, \boldsymbol{\Omega}')} \nabla \right) \Phi^{(1)}(\mathbf{r}, \boldsymbol{\Omega}', \nu') \right\} d\nu' d\boldsymbol{\Omega}' \quad (\text{A13})
 \end{aligned}$$

for all $\boldsymbol{\Omega} \in \boldsymbol{\Omega}_h$ and $\nu \in \nu_c$.

Equations (A12) and (A13) are now of a coupled form much simpler than that given by equation (68) in Section IV. The numerical technique described briefly in Section IV may be applied to the above equations without loss in generality. The solution is completed by determining $\Phi^{(2)}$ and $\Psi^{(1)}$ from equations (A9) and (A6) respectively.

